Prof. Dr. Peter Arbenz SS 2018

Numerical Methods for Solving Large Scale Eigenvalue Problems

Assignment 1

1. Consider the wave equation for the linearized model of a guitar string. The partial differential equation is given by equation (1.8) in the lecture notes. We choose $\rho(x) = \rho$[kg/m] (constant weight), and $p(x) = T$[N] (constant tension). We assume there are no exterior forces, i.e., $q(x) = 0$. Separation of variables, $u(x,t) = v(t)w(x)$, leads to differential equations for $v(t)$ and $w(x)$, depending on an additional parameter λ (see equation (1.12) in the lecture notes). For $v(t)$, the solution at a specific λ is

$$v(t) = a\cos(\sqrt{\lambda}t) + b\sin(\sqrt{\lambda}t),$$

where a and b are still to be determined. This corresponds to the frequency $\frac{\sqrt{\lambda}}{2\pi}$. The differential equation for $w(x)$ remains to be solved.

(a) Approximate $w(x)$ using the finite difference method. Define $N+2$ equidistant nodes $0 = x_0 < \cdots < x_{N+1} = \ell$ on the interval $[0,\ell]$, where $N = 100$, and define $w_k = w(x_k)$. Use

$$\frac{\partial^2 w}{\partial x^2}(x_k) \approx \frac{1}{h^2}\left(w_{k+1} - 2w_k + w_{k-1}\right), \tag{1}$$

to approximate the second derivative, where $h = \frac{\ell}{N+1}$. The resulting system is an eigenvalue problem:

$$A\mathbf{w} = \lambda\mathbf{w}.$$

In MATLAB, use `eigs` to calculate the ground state (the eigenvector corresponding to the smallest eigenvalue), and the next two states (eigenvectors corresponding to 2nd and 3rd smallest eigenvalues), and plot them.

Also calculate the frequency of the resulting waves.

Choose $N = 100$, $T = 10\,\mathrm{N}$ and $\rho = 0.001\,\mathrm{kg/m}$, $\ell = 0.5\,\mathrm{m}$.

(b) **Bonus:** We want the string to play an "A", i.e., the fundamental frequency ($\hat{=}$ smallest eigenvalue) is 440 Hz. Determine the tension T that achieves this fundamental frequency using the algorithm from **(a)**.

2. **Poisson equation**. The following eigenvalue problem is given

$$-\Delta u(x,y) = \lambda u(x,y), \qquad 0 < x < 5, \quad 0 < y < 4, \tag{2}$$

with Neumann boundary conditions at the left and the right

$$\frac{\partial u}{\partial n}(0,y) = \frac{\partial u}{\partial n}(5,y) = 0, \quad 0 < y < 4,$$

as well as Dirichlet and Cauchy boundary conditions at the top and the bottom

$$u(x,0) = u(x,4) + \frac{\partial u}{\partial n}(x,4) = 0, \quad 0 < x < 5,$$

(a) Solve this problem exactly with separation of variables.

(b) How does the matrix eigenvalue problem look like if discretized on a rectangular grid.

(c) Is the derived matrix symmetric or symmetrizable?

3. Calculate the ground state of

$$\begin{aligned} -\Delta u &= \lambda u \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

using the MATLAB PDE Toolbox. The domain Ω should be chosen as the unit circle, or as the L-shaped domain. Use the following commands in MATLAB:

```
% initializes a triangular mesh on the unit circle
[p,e,t]=initmesh('circleg');

                % refines the mesh
[p,e,t]=refinemesh('circleg',p,e,t);

                % Calculates eigenvectors and eigenvalues of the system
                % (some constants have been inserted)
[v,l]=pdeeig('circleb1',p,e,t,1,0,1,[0 10]);

                % Plot the approximation of the first eigenfunction
pdesurf(p,t,v(:,1))
```

For the L-shaped domain, replace `circleg` and `circleb1` by `lshapeg` and `lshapeb`.

We are interested in the behavior of the error in the computed eigenvalue when the mesh is refined. Successively refine the mesh, and plot the relative error for the two domains. What is the convergence rate?

Hints: Use the computed eigenvalue for the finest mesh as the "correct" eigenvalue for calculating the error.

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Hand in: Mar. 5th, 2018.
Please print out your MATLAB m-files and plots and hand them in at the next exercise session, or send them to arbenz@inf.ethz.ch. Please use as subject LSEVP18.