Assignment 2

1. Consider Example 1.1 from the script, and compute the resulting clustering yourself in MATLAB.

Start with the starting points indicated in the script, and construct the graph Laplacian. Plot the eigenvectors corresponding to the 4 smallest eigenvalues. Now, apply *k-means clustering* to the matrix V containing these eigenvectors.

Hint: You can download the file `kmeans.m` from the course homepage. It defines the function

```
function c_index = kmeans(V),
```

which constructs the *k-means clustering* of matrix V.

2. Let

$$A = \begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix}.$$

Find an unitary 2×2 matrix Q such that

$$Q^* A Q = \begin{pmatrix} \lambda_2 & \beta \\ 0 & \lambda_1 \end{pmatrix}.$$

Hint: Like in the proof of the Schur decomposition, the first column of Q must be the (normalized) eigenvector of A with eigenvalue λ_2.

3. Consider the matrix

$$A = \begin{pmatrix} 0 & & & -\alpha_0 \\ 1 & \ddots & & -\alpha_1 \\ & \ddots & 0 & \vdots \\ & & 1 & -\alpha_{n-1} \end{pmatrix}.$$

Proof that the following equality holds:

$$\det(xI - A) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} + x^n.$$

4. **(a)** A matrix of the from

$$E = I - \sigma \boldsymbol{u}\boldsymbol{v}^*, \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^n,$$

is called an *elementary* matrix. What is the form of E^{-1}? (Hint: Set $E^{-1} = I - \tau \boldsymbol{u}\boldsymbol{v}^*$ and then determine τ.) What are the eigenvalues and eigenvectors of E? What is the determinant of E?

(b) Let $\|\boldsymbol{u}\|_2 = 1$, and $H = I - 2\boldsymbol{u}\boldsymbol{u}^*$ be a so-called *Householder reflector*. Prove that $H^2 = I$ and $H^* = H^{-1}$. Interpret the action of H geometrically.

(c) Construct a Householder reflector, that maps a given vector $\boldsymbol{x}$ on a multiple of $\boldsymbol{e}_1$. (See Golub & van Loan: Matrix Computations)

(d) Construct an unitary matrix for which a predetermined first column $\boldsymbol{v}$ is given.

Please submit your solution via e-mail to Peter Arbenz (`arbenz@inf.ethz.ch`) by March 6, 2018. (12:00). Please specify the tag **EWP18-2** in the subject field.