Assignment 3

1. **Quadratic eigenvalue problem.** The numbers λ (=eigenvalues) are sought, for which

$$A\mathbf{x} + \lambda B\mathbf{x} + \lambda^2 C\mathbf{x} = \mathbf{0}$$

has nontrivial solutions. Derive a linear eigenvalue problem (which means an eigenvalue problem in which λ appears only linearly), that has the same eigenvalues. What is the relationship among the eigenvectors?

Assume that all matrices A, B, and C are symmetric. Can you construct a symmetric linearized eigenvalue problem. Does this mean that all eigenvalues are real?

Hint: Use an auxiliary vector $\mathbf{y} = \lambda\mathbf{x}$.

Note: This method has similarities with the method that transforms ODEs of high order to ODE systems of first order.

2. Write a MATLAB implementation of Hyman's method to find a root of $f(\lambda) = \det(A - \lambda I)$.

(a) Write a function `hyman.m` that inputs a Hessenberg matrix H and a scalar λ and returns two scalars $f(\lambda)$ and $f'(\lambda)$.

(b) Generate a random Hessenberg matrix:

```
n = ...
A = rand(n);
[P,H] = hess(A);
```

Use Newton's method to find an eigenvalue of H.

(Make sure that your procedure finds complex eigenvalues as well.)

(c) Verify quadratic convergence.

(d) Modify Newton's method to find a second eigenvalue.

3. For the eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$ with real symmetric A let us define a residual for eigenpairs, $\mathbf{r}(\mathbf{x}, \lambda) := A\mathbf{x} - \lambda\mathbf{x}$. Show that

$$\|\mathbf{r}(\mathbf{x}, \rho(\mathbf{x}))\|_2 \leq \|\mathbf{r}(\mathbf{x}, \lambda)\|_2 \quad \text{for all } \lambda \tag{1}$$

where $\rho(\mathbf{x})$ is the Rayleigh quotient.

For the generalized eigenvalue problem $A\mathbf{x} = \lambda M\mathbf{x}$ with real symmetric A and SPD M the Rayleigh quotient is defined by

$$\rho(\mathbf{x}) = \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T M\mathbf{x}}, \qquad \mathbf{x} \neq \mathbf{0}.$$

What inequality corresponds to (1) for the generalized eigenvalue problem?

Hint: Transform the problem in a special eigenvalue problem.

Please submit your solution via e-mail to Peter Arbenz (`arbenz@inf.ethz.ch`) by March 13, 2018. (12:00). Please specify the tag **EWP18-3** in the subject field.