Assignment 7

1. Let A be an n-by-n matrix and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ be linearly independent n-vectors. Is the subspace $\mathcal{V} := \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ a Krylov space, i.e. is there a vector $\boldsymbol{q} \in \mathcal{V}$ such that $\mathcal{V} = \mathcal{K}_k(A, \boldsymbol{q})$? The following holds and is to be proved.

$\mathcal{V}$ is a Krylov space if there is a k-by-k matrix M such that

$$R := AV - VM, \qquad V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k], \tag{1}$$

has rank one. So, given

$$AV = VM + \boldsymbol{v}\boldsymbol{w}^T,$$

construct an Arnoldi relation

$$AQ = QH + \gamma \boldsymbol{q}\boldsymbol{e}_k^T,$$

with $\mathcal{R}(Q) = \mathcal{R}(V)$.

2. **(a)** Implement the Lanczos algorithm with full reorthogonalization for $A\boldsymbol{x} = \lambda M\boldsymbol{x}$ in a Matlab function. It is assumed that A and M are *real, symmetric* n-by-n matrices, and M is *positive definite*.

The Lanczos iteration for this eigenproblem has the form

$$\beta_j \boldsymbol{q}_{j+1} = \boldsymbol{r}_j = (A - \sigma M)^{-1} M \boldsymbol{q}_j - \alpha_j \boldsymbol{q}_j - \beta_{j-1}\boldsymbol{q}_{j-1}, \qquad \boldsymbol{q}_i M \boldsymbol{q}_k = \delta_{i,k}.$$

Here, σ is a shift. Note that the matrix $(A - \sigma M)^{-1} M$ is symmetric (self-adjoint) w.r.t. the M-inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_M = \boldsymbol{y}^* M \boldsymbol{x}$.

Input:

- Matrices A and M.
- p eigenpairs to be computed.
- Starting vector $\boldsymbol{x}$.
- Shift σ.
- Tolerance τ for the stopping criterion. An eigenpair is considered found if $\beta_j |s_{j,*}^{(j)}| < \tau$ (in j-th iteration).

Output:

- p eigenpairs.
- The estimates $\beta_j |s_{j,*}^{(j)}|$.

Note: Use the Matlab routine `eig` to compute the last row of S_j where $T_j = S_j \Theta S_j^T$.

(b) As in the previous Assignment 6 question 2 (b), calculate the ten smallest eigenvalues and corresponding eigenvectors of the eigenvalue problem “vibrations in the car interior”, which was discussed in the lecture notes. Choose $\sigma = -0.01$ and $\tau = 1 \cdot 10^{-6}$.

The matrices are constructed by the following Matlab script:

Please submit your solution via e-mail to Peter Arbenz (`arbenz@inf.ethz.ch`) by May 2, 2018. (12:00). Please specify the tag **EWP18-7** in the subject field.

```
[p,e,t]=initmesh('auto');
[p,e,t]=refinemesh('auto',p,e,t);
[p,e,t]=refinemesh('auto',p,e,t);
p=jigglemesh(p,e,t);
[A,M]=assema(p,t,1,1,0);
```

Here `auto` defines the geometry of the car.

The eigenfunctions are stored in a $n \times 10$ matrix `U`, and they can be plotted, e.g., the 4th eigenfunction is plotted as follows

```
pdeplot(p,e,t,'xydata',U(:,4),'mesh','on','colorbar','off')
axis([-1 26 -1 15])
axis equal
axis off
colormap(hsv)
```

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