Assignment 1

1. Consider the wave equation for the linearized model of a guitar string. The partial differential equation is given by equation (1.8) in the lecture notes. We choose $\rho(x) = \rho[\text{kg/m}]$ (constant weight), and $p(x) = T[\text{N}]$ (constant tension). We assume there are no exterior forces, i.e., $q(x) = 0$. Separation of variables, $u(x,t) = v(t)w(x)$, leads to differential equations for $v(t)$ and $w(x)$, depending on an additional parameter λ (see equation (1.12) in the lecture notes). For $v(t)$, the solution at a specific λ is

$$v(t) = a\cos(\sqrt{\lambda}t) + b\sin(\sqrt{\lambda}t),$$

where a and b are still to be determined. This corresponds to the frequency $\frac{\sqrt{\lambda}}{2\pi}$. The differential equation for $w(x)$ remains to be solved.

(a) Approximate $w(x)$ using the finite difference method. Define $N+2$ equidistant nodes $0 = x_0 < \cdots < x_{N+1} = \ell$ on the interval $[0,\ell]$, where $N = 100$, and define $w_k = w(x_k)$. Use

$$\frac{\partial^2 w}{\partial x^2}(x_k) \approx \frac{1}{h^2}\left(w_{k+1} - 2w_k + w_{k-1}\right), \tag{1}$$

to approximate the second derivative, where $h = \frac{\ell}{N+1}$. The resulting system is an eigenvalue problem:

$$A\boldsymbol{w} = \lambda\boldsymbol{w}.$$

In Matlab, use `eigs` to calculate the ground state (the eigenvector corresponding to the smallest eigenvalue), and the next two states (eigenvectors corresponding to 2nd and 3rd smallest eigenvalues), and plot them.

Also calculate the frequency of the resulting waves.

Choose $N = 100$, $T = 10\,\text{N}$ and $\rho = 0.001\,\text{kg/m}$, $\ell = 0.5\,\text{m}$.

Solution: We start with the equation (1.8) from the lecture notes

$$-\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial}{\partial x}\left(p(x)\frac{\partial u(x,t)}{\partial x}\right) + q(x)u(x,t) = 0,$$

with the boundary conditions $u(0,t) = u(\ell,t) = 0$. Using the assumptions $\rho(x) = \rho$, $p(x) = T$, and $q(x) = 0$, we obtain

$$-\rho\frac{\partial^2 u(x,t)}{\partial t^2} + T\frac{\partial^2 u(x,t)}{\partial x^2} = 0.$$

We apply the method of separation of variables $u(x,t) = w(x)v(t)$ with the ansatz $v(t) = a\cos(\sqrt{(}\lambda t) + b\sin(\sqrt{\lambda}t))$. We arrive at

$$\rho\lambda w(x)v(t) + Tw''(x)v(t) = 0.$$

This has to be satisfied for all $v(t)$, so we get

$$-\frac{T}{\rho}w''(x) = \lambda w(x).$$

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With the discretization scheme (1), we get the following linear eigenvalue problem

$$-\frac{T}{\rho h^2}\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix}\begin{bmatrix} w_1 \\ \vdots \\ \vdots \\ w_N \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ \vdots \\ \vdots \\ w_N \end{bmatrix}.$$

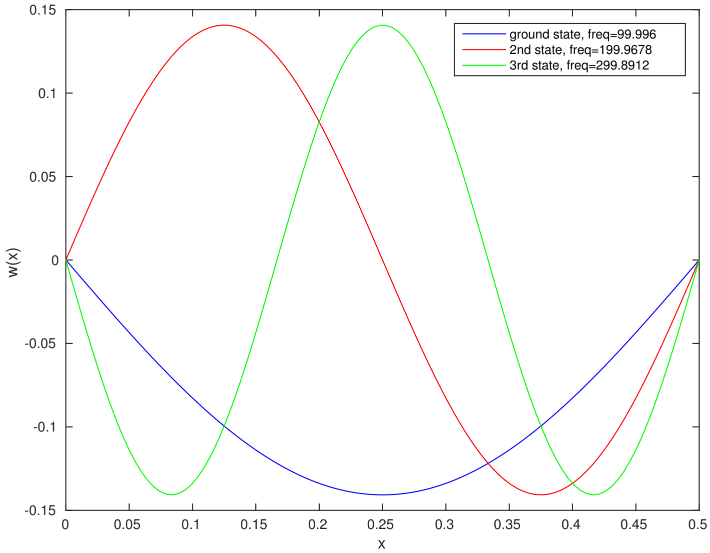

Figure 1: Eigen vectors of the first three smallest eigenvalues.

(b) Bonus: We want the string to play an "A", i.e., the fundamental frequency ($\hat{=}$ smallest eigenvalue) is 440 Hz. Determine the tension T that achieves this fundamental frequency using the algorithm from **(a)**.

Solution: This could be done reformulating the eigenvalue problem in the following form

$$-\frac{\rho}{(880\pi)^2}w''(x) = Tw(x),$$

and solving again for T. Or scale the problem and us the previous computed eigenvalue

$$-\underbrace{\frac{\rho}{\alpha^{-1}T}}_{T_{\text{new}}}w''(x) = \underbrace{\alpha\lambda}_{=(880\pi)^2} w(x) \quad \Rightarrow T_{\text{new}} = \frac{(880\pi)^2}{\lambda}T$$

Therefore the new tension is 193.6.

2. Poisson equation. The following eigenvalue problem is given

$$-\Delta u(x,y) = \lambda u(x,y), \qquad 0 < x < 5, \quad 0 < y < 4, \tag{2}$$

with Neumann boundary conditions at the left and the right

$$\frac{\partial u}{\partial n}(0,y) = \frac{\partial u}{\partial n}(5,y) = 0, \quad 0 < y < 4,$$

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as well as Dirichlet and Cauchy boundary conditions at the top and the bottom

$$u(x,0) = u(x,4) + \frac{\partial u}{\partial n}(x,4) = 0, \quad 0 < x < 5,$$

(a) Solve this problem exactly with separation of variables.

Solution: Using separation of variables $u(x,y) = v(x)w(y)$ and dividing with u, we obtain

$$\underbrace{-\frac{v''(x)}{v(x)}}_{=\mu} \underbrace{-\frac{w''(y)}{w(y)}}_{=\nu} = \lambda, \quad 0 < x < 5, \quad 0 < y < 4.$$

To satisfy this equation the quotients v''/v and w''/w have to be constant. We use μ and ν as these constants. The corresponding eigenvalue for the original problem is then given by $\lambda = \nu + \mu$.

The Neumann boundary conditions at the left and right for the separated variables become

$$v'(0) = v'(5) = 0.$$

The Dirichlet and Cauchy boundary conditions become

$$w(0) = w(4) + w'(4) = 0.$$

First we solve in x-direction

$$-v''(x) = \mu v(x), \quad 0 < x < 5$$

From that we can use the ansatz $v(x) = \alpha\cos(\sqrt{\mu}x) + \beta\sin(\sqrt{\mu}x)$. From the boundary conditions we get that $\beta = 0$ and $\mu_k = (\frac{\pi}{5}k)^2$ for $k \in \mathbb{Z}$. Note that the coefficient α can then be chosen arbitrary (we choose $\alpha = 1$).

Now, we solve in y-direction

$$-w''(y) = \nu w(y), \quad 0 < y < 4$$

From that we can assume the ansatz $v(x) = \gamma\cos(\sqrt{\nu}y) + \delta\sin(\sqrt{\nu}y)$. From $w(0) = 0$ it follows that $\gamma = 0$. Similar to the x-direction we can choose $\delta = 1$. Using the top boundary conditions, we get

$$\sin(\sqrt{\nu}4) = -\sqrt{\nu}\cos(\sqrt{\nu}4).$$

This can be solved numerically, we get $l \in \mathbb{Z}_{\neq 0}$ different values for ν. The first three values are

$$\nu_{\pm 1} = \pm 0.642\ldots^2,$$
$$\nu_{\pm 2} = \pm 1.348\ldots^2,$$
$$\nu_{\pm 3} = \pm 2.075\ldots^2.$$

The solution of the original problem is given by

$$\begin{aligned} \lambda_{kl} &= \mu_k + \nu_l \\ u_{kl}(x,y) &= \cos\left(\sqrt{|\mu_k|}x\right)\sin\left(\operatorname{sgn}(k)\sqrt{|\nu_k|}y\right) \end{aligned} \quad \text{for } k \in \mathbb{Z}, l \in \mathbb{Z}_{\neq 0}.$$

(b) How does the matrix eigenvalue problem look like if discretized on a rectangular grid.

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Solution: We begin discretizing the domain in each direction separately. In each direction we consider N_x and N_y grid points. We write the grid coordinates as $x_i = h_x i$ and $y_j = h_y j$ with $h_x = \frac{5}{N_x}$ and $h_y = \frac{4}{N_y}$, respectively. We are looking for the values $u_{ij} = u(x_i, y_j)$ for $0 \le i \le N_x$ and $0 \le j \le N_y$. The boundary conditions are discretized as

$$\left.\begin{aligned} \frac{u_{1,j} - u_{-1,j}}{2h_x} &= 0, \\ \frac{u_{N_x+1,j} - u_{N_x-1,j}}{2h_x} &= 0, \end{aligned}\right. \qquad 0 \le j \le N_y$$

$$\begin{aligned} u_{i,0} = 0, \quad 0 \le i \le N_x&, \\ u_{i,N_y} + \frac{u_{i,N_y+1} - u_{i,N_y-1}}{2h_y} = 0&, \qquad 0 \le i \le N_x \end{aligned}$$

For the inner points we get

$$-\frac{1}{h_x^2}u_{i-1,j} - \frac{1}{h_y^2}u_{i,j-1} + \left(\frac{2}{h_x^2} + \frac{2}{h_y^2}\right)u_{i,j} - \frac{1}{h_x^2}u_{i+1,j} - \frac{1}{h_y^2}u_{i,j+1} = \lambda u_{i,j}, \text{for} \quad 1 \le i \le N_x - 1, \quad 2 \le j \le$$

Note that we have Dirichlet boundary conditions on the bottom, so we get there N_x less equations there. For $j = 1$ we get

$$-\frac{1}{h_x^2}u_{i-1,1} - \frac{1}{h_y^2}\underbrace{u_{i,0}}_{=0} + \left(\frac{2}{h_x^2} + \frac{2}{h_y^2}\right)u_{i,1} - \frac{1}{h_x^2}u_{i+1,1} - \frac{1}{h_y^2}u_{i,2} = \lambda u_{i,1}, \text{for} \quad 1 \le i \le N_x - 1.$$

For the Neumann and Cauchy boundary conditions, we plug in in the equation from boundary conditions for the unknown outside of the domain. For the top, we arrive with scaling the equation by one half at

$$-\frac{1}{2h_x^2}u_{i-1,N_y} - \frac{1}{h_y^2}u_{i,N_y-1} + \left(\frac{1}{h_x^2} + \frac{1+h_y}{h_y^2}\right)u_{i,N_y} - \frac{1}{2h_x^2}u_{i+1,N_y} = \frac{1}{2}\lambda u_{i,N_y}, \text{for} \quad 1 \le i \le N_x - 1,$$

The same procedure gives for the left boundary

$$-\frac{1}{2h_y^2}u_{0,j-1} + \left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right)u_{0,j} - \frac{1}{h_x^2}u_{1,j} - \frac{1}{2h_y^2}u_{0,j+1} = \frac{1}{2}\lambda u_{0,j}, \text{for } 2 \le j \le N_y - 1,$$

and for right boundary it gives

$$-\frac{1}{h_x^2}u_{N_x-1,j} - \frac{1}{2h_y^2}u_{N_x,j-1} + \left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right)u_{N_x,j} - \frac{1}{2h_y^2}u_{N_x,j+1} = \frac{1}{2}\lambda u_{N_x,j}, \text{for } 2 \le j \le N_y - 1.$$

Also the four corners (x_0, y_1), (x_0, y_{N_y}), (x_{N_x}, y_1), and (x_{N_x}, y_{N_y}) have to be considered separately, but are skipped here. We can construct now the matrix $A \in \mathbb{R}^{((N_x+1)N_y)\times((N_x+1)N_y)}$.

(c) Is the derived matrix symmetric or symmetrizable?

Solution: The way the matrix has been constructed it is already symmetric. Otherwise we would have to scale the matrix such that it is symmetric.

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3. Calculate the ground state of

$$-\Delta u = \lambda u \qquad \text{in } \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega$$

using the MATLAB PDE Toolbox. The domain Ω should be chosen as the unit circle, or as the L-shaped domain. Use the following commands in MATLAB:

```
% initializes a triangular mesh on the unit circle
[p,e,t]=initmesh('circleg');

                % refines the mesh
[p,e,t]=refinemesh('circleg',p,e,t);

                % Calculates eigenvectors and eigenvalues of the system
                % (some constants have been inserted)
[v,l]=pdeeig('circleb1',p,e,t,1,0,1,[0 10]);

                % Plot the approximation of the first eigenfunction
pdesurf(p,t,v(:,1))
```

For the L-shaped domain, replace `circleg` and `circleb1` by `lshapeg` and `lshapeb`.

We are interested in the behavior of the error in the computed eigenvalue when the mesh is refined. Successively refine the mesh, and plot the relative error for the two domains. What is the convergence rate?

Hints: Use the computed eigenvalue for the finest mesh as the "correct" eigenvalue for calculating the error.

Solution: The approximated convergence rate is two, c.f. Figure 2.

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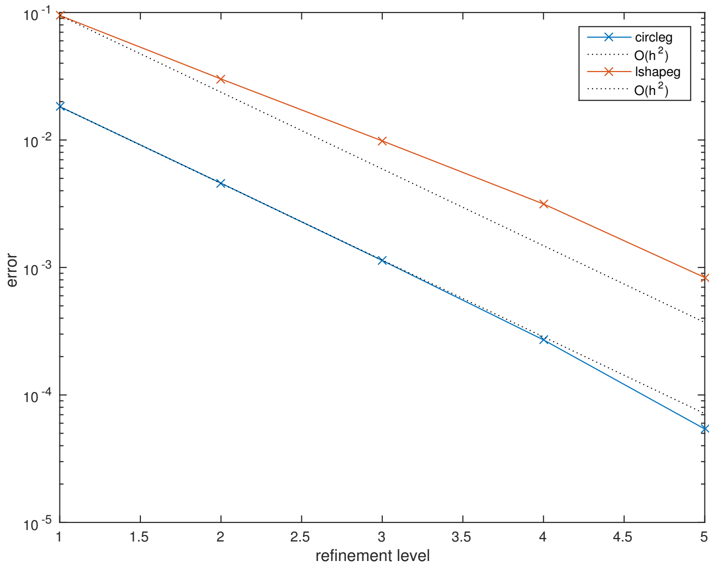

Figure 2: Convergence.

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