Parallel MIC(0) preconditioning of 3D nonconforming FEM systems

P. Arbenz, S. Margenov

Institute of Computational Science, ETH, Zurich
Institute for Parallel Processing, Bulgarian Academy of Sciences, Sofia

1 Introduction

The recent efforts in development of efficient solution methods for non-conforming finite element (FE) systems are focused on robustness and/or special algorithmic features. Consider the elliptic equation

$$\begin{aligned} -\nabla \cdot (a(x)\nabla u(x)) &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ (a(x)\nabla u(x)) \cdot n &= 0 \quad \text{on } \Gamma_N. \end{aligned} \tag{1}$$

We assume that Ω is a convex polyhedral domain in $\mathbb{R}^3$ and $a(x) = [a_{ij}(x)]_{i,j=1}^3$ is a symmetric positive definite matrix where the $a_{ij}(x)$ are piecewise smooth functions on $\bar{\Omega}$ satisfying the uniform positive definiteness condition of the matrix $a(x)$. The problem (1) can be discretized in various ways. Among the most popular and frequently used methods of approximation are the finite volume method, the Galerkin finite element method (FEM), and the mixed FEM. Many engineering problems need very accurate velocity (flux) determination in the presence of heterogeneities and large jumps in the coefficient matrix $a(x)$. This can be achieved through the mixed FEM. However, the technique of the mixed FEM leads generally to an algebraic saddle point problem that is more difficult and more expensive to solve. An important discovery of Arnold and Brezzi [1] is that the Schur system for the Lagrange multipliers can be obtained also as a discretization of (1) by a Galerkin method using nonconforming elements. The application of rotated trilinear quadrilateral FEs is studied in this article.

There are two general approaches to construct parallel preconditioners, based respectively on (a) domain decomposition, or (b) (block) incomplete factorization. The second approach does not lead to optimal preconditioners in terms of the problem size, but produces highly parallel and efficient preconditioners. Here, we first modify the stiffness matrix A locally to obtain an auxiliary matrix B. Then we apply a point-wise modified incomplete Cholesky factorization (MIC(0)) to the matrix B. This is a 3D extension of the preconditioning technique first proposed in 2D for skewed five point stencil discretization (see e.g. [5]). Related methods for nonconforming Crouzeix-Raviart and for rotated bilinear FE systems were recently introduced in [6] and in [3], respectively.

2 Local construction and condition number analysis

Let us assume that Ω is partitioned using quadrilaterals $e \in \omega_h$. The rotated trilinear nonconforming finite elements on quadrilaterals are implemented for the numerical solution of (1) leading to the FEM space V_h (see [7] for more details). Then the finite element formulation is: find a function $u_h \in V_h$, satisfying

$$\mathcal{A}_h(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_h, \tag{2}$$

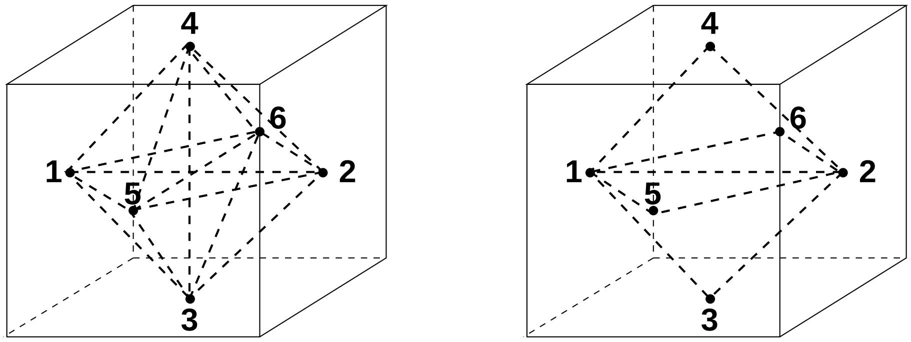

Figure 1: Node numbering of a rotated trilinear quadrilateral element e: connectivity pattern of A_e (left) and of B_e (right).

where

$$\mathcal{A}_h(u_h, v_h) = \sum_{e\in\omega_h} \int_e a(e)\nabla u_h \cdot \nabla v_h dx, \quad a(e) = \frac{1}{|e|}\int_e a(x)dx. \tag{3}$$

The standard computational procedure leads to the linear system of equations $A\mathbf{u} = \mathbf{f}$.

The idea of our algorithm is to apply MIC(0) factorization (see, e.g., **19.04** [2, 4]) to a modified sparse matrix whose special block structure allows for a scalable parallel implementation. This is based on the experience in developing such kind of algorithms for 2-D problems (see [3, 5, 6]). Following the standard FEM assembling procedure we write A in the form

$$A = \sum_{e\in\omega_h} L_e^T A_e L_e, \tag{4}$$

where A_e is the element stiffness matrix, L_e stands for the restriction mapping of the global vector of unknowns to the local one corresponding to the actual element e. Let us consider the following approximation B_e of A_e:

$$A_e = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}, \quad B_e = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & b_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & b_{44} & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & b_{55} & 0 \\ a_{61} & a_{62} & 0 & 0 & 0 & b_{66} \end{bmatrix},$$

where the b_{kk} are determined such that A_e and B_e have equal row sums. Assembling the locally defined matrices B_e we get the global matrix B by

$$B = \sum_{e\in\omega_h} L_e^T B_e L_e, \tag{5}$$

The definition of B_e corresponds to the local numbering shown in Fig. 1. Here the dashed lines represent the connectivity pattern of (a) the dense element stiffness matrix A_e and (b) the locally modified sparse approximation B_e. At this point we introduce the preconditioner $\mathcal{C}$ of A which is defined as the MIC(0) factorization of B, that is, $\mathcal{C} = \mathcal{C}_{\mathrm{MIC}(0)}(B)$. This of course requires that B permits a stable MIC(0) factorization.

Now, let us consider the model problem associated with the bilinear form

$$a_h(u, v) = \sum_{e\in\omega_h} \int_e a(e) \sum_{i=1}^{3} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx\,, \tag{6}$$

where ω_h consists of cubic elements congruent to $(0,h)^3$, and the coefficient $a(e)$ is piecewise constant on each $e \in \omega_h$. Then, the element matrices read as

$$A_e = \frac{4a(e)h}{9}\begin{bmatrix} 17 & -1 & -4 & -4 & -4 & -4 \\ -1 & 17 & -4 & -4 & -4 & -4 \\ -4 & -4 & 17 & -1 & -4 & -4 \\ -4 & -4 & -1 & 17 & -4 & -4 \\ -4 & -4 & -4 & -4 & 17 & -1 \\ -4 & -4 & -4 & -4 & -1 & 17 \end{bmatrix}, \quad B_e = \frac{4a(e)h}{9}\begin{bmatrix} 17 & -1 & -4 & -4 & -4 & -4 \\ -1 & 17 & -4 & -4 & -4 & -4 \\ -4 & -4 & 8 & 0 & 0 & 0 \\ -4 & -4 & 0 & 8 & 0 & 0 \\ -4 & -4 & 0 & 0 & 8 & 0 \\ -4 & -4 & 0 & 0 & 0 & 8 \end{bmatrix}.$$

Now, a local spectral analysis is applied to derive the uniform estimate

$$\kappa(B^{-1}A) \leq 3.$$

In the model case, it also follows immediately that B is an M-matrix satisfying the conditions for stable MIC(0) factorization which ensures the correctness of the presented algorithm.

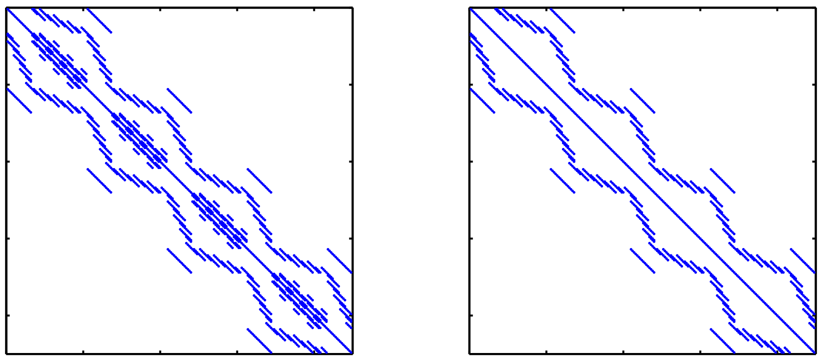

Figure 2: Sparsity pattern of matrix A (left) and of its approximation B (right).

3 Numerical tests

The presented numerical tests illustrate the convergence rate of the proposed preconditioning algorithm, solving the system $A\mathbf{u} = \mathbf{f}$ for the model problem $-\Delta u = f$. PCG is considered converged if the relative stopping criterion

$$(\mathcal{C}^{-1}\mathbf{r}^{n_{it}}, \mathbf{r}^{n_{it}}) < 10^{-12}(\mathcal{C}^{-1}\mathbf{r}^0, \mathbf{r}^0)$$

was satisfied by the residual $\mathbf{r}^i$ of the i-th iteration step. Here, $(\cdot,\cdot)$ is the standard Euclidean inner product. The computational domain is the unit cube $\Omega = (0,1)^3$ where homogeneous Dirichlet boundary conditions are imposed at the bottom side. A uniform mesh is used, where $h = 1/n$, and the size of the discrete problem is $N = n(n+1)(3n-2)$. The number of CG iterations (without preconditioning) are given in the first row of the table. The second row contains the PCG iterations if a MIC(0) factorization is applied to the original stiffness matrix A. The last row corresponds to the case studied in this article where MIC(0) factorization is applied to the locally introduced auxiliary matrix B. The sparsity pattern of A in (4) and of B in (5) are displayed in Fig. 2. The qualitative analysis of the results shows that

- the number of iterations in both PCG cases is $O(n^{1/2}) = O(N^{1/6})$, namely, it grows proportionally to $\sqrt{n}$ in agreement with the properties of the MIC(0) factorization of A;

Table 1: CG iterations and PCG iterations for MIC(0) factorization of A and B.

Method	n=7	n=15	n=31	n=63
CG	37	73	145	278
$\mathcal{C}_{\mathrm{MIC}(0)}(A)$	23	35	51	73
$\mathcal{C}_{\mathrm{MIC}(0)}(B)$	27	42	61	87

- the number of iterations for $\mathcal{C}_{\mathrm{MIC}(0)}(A)$ and $\mathcal{C}_{\mathrm{MIC}(0)}(B)$ are very close which is more optimistic than what could be predicted from the estimate $\kappa(B^{-1}A) \leq 3$.

4 Concluding remarks

A new preconditioner for 3D nonconforming FEM elliptic systems is proposed. The reported results could be viewed as a substantial next step in the development of highly parallelizable and efficient preconditioners based on MIC(0) factorization. The key moment is that the developed auxiliary matrix B has a special block structure where the diagonal blocks are diagonal matrices. These blocks correspond to plains in 3D case, which allows for a better balance between computations and communications, compared to some recently reported results concerning 2D cases (see for more details in [3, 5, 6]). Let us stress the attention on the anisotropic sparsity structure of the element matrices B_e. This should be carefully taken into account, if problems with mesh or/and coefficient anisotropy are considered.

Acknowledgment: This work has been supported by the Bulgarian NSF grant IO-01/03 and by the Swiss NSF grant 7IP65642 within the SCOPES program.

References

[1] D. N. Arnold, F. Brezzi: *Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates*, RAIRO Model. Math. Anal. Numer. 19 (1985), pp. 7–32.

[2] R. Blaheta: *Displacement decomposition – incomplete factorization preconditioning techniques for linear elasticity problems*, Numer. Linear Algebra Appl. 1 (1994), pp. 107–126.

[3] G. Bencheva, S. Margenov: *Performance analysis of parallel MIC(0) preconditioning of rotated bilinear nonconforming FEM systems*, Math. Balkanica 18 (2004), pp. 319–336.

[4] I. Gustafsson: *Modified incomplete Cholesky (MIC) factorization*, In: D.J. Evans (ed.): Preconditioning Methods; Theory and Applications, Gordon and Breach, 1983, pp. 265–293.

[5] I. Gustafsson, G. Lindskog: *On parallel solution of linear elasticity problems: Part I: Theory*, Numer. Linear Algebra Appl. 5(2) (1998), pp. 123–139.

[6] R. D. Lazarov, S. D. Margenov: *On a two-level parallel MIC(0) preconditioning of Crouzeix-Raviart non-conforming FEM systems*, In: I. Dimov, I. Lirkov, S. Margenov, Z. Zlatev (eds.): Numerical Methods and Applications, Springer LNCS 2542, 2003, pp. 192–201.

[7] R. Rannacher, S. Turek: *Simple nonconforming quadrilateral Stokes Element*, Numer. Methods Partial Differential Equations 8(2) (1992), pp. 97–112.