A TOPOGRAPHY OF LABELLED MODAL LOGICS

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Abstract. Labelled Deductive Systems provide a general method for representing logics in a modular and transparent way. A Labelled Deductive System consists of two parts, a base logic and a labelling algebra, which interact through a fixed interface. The labelling algebra can be viewed as an independent parameter: the base logic stays fixed for a given class of related logics from which we can generate the one we want by plugging in the appropriate algebra. Our work identifies an important property of the structured presentation of logics, their combination, and extension. Namely, there is tension between modularity and extensibility: a narrow interface between the base logic and labelling algebra can limit the degree to which we can make use of extensions to the labelling algebra. We illustrate this in the case of modal logics and apply simple results from proof theory to give examples.

1. Introduction

The idea of a Labelled Deductive System (LDS) has been proposed as a general technique for representing logics and their consequence relations. In the LDS approach, instead of defining a consequence relation over formulae $(\ldots A \vdash B \ldots)$, we define it over pairs consisting of a label and a formula $(\ldots x : A \vdash y : B \ldots)$. The labels then allow us to track the information we need in order to formalize the more subtle aspects of the relation. For modal logic, for instance, we might want to distinguish between local and global consequence, so the label could keep track of which world (in the Kripke sense) the formula lives in. Or for a substructural logic, where the consequence relation should be sensitive to operations like weakening and contraction, the labels could track resources and their use [4].

Labelled formulae have been implicitly used in proof systems (e.g. [1, 5]) for some time. More recently, however, people have begun to study Labelled Deductive Systems systematically as a general tool, in which the labelling is explicitly considered as part of the logic itself [6, 7, 15]. Along with the 'fine control' that explicit labelling offers it also supports modular theory development. Let us fix some terminology which will be useful for discussing this. We view a proof system for an LDS as consisting of two parts: a *base logic* for manipulating labelled formulae, and a separate *labelling algebra* for reasoning about properties of the labels. In the base logic (and possibly also the labelling algebra) there may be rules which combine formulae from the two parts; these rules form an *interface* for communication between the otherwise separate parts. This construction supports *modularity* in that we can view the labelling algebra as an independent parameter: the base stays fixed for a given class of related logics from which we can choose the one we want by plugging in the appropriate labelling algebra. D'Agostino and Gabbay [4, p.244] explain such modularity as one of the important features of the LDS approach:

> The labelling algebra represents this metalevel information as a *separate* component of a standard derivation system and can be treated as an independent parameter. In the LDS approach, logical systems are not studied statically, in isolation, but dynamically, observing the process of their generation and their interaction (via modifications of the labelling algebras) on the basis of a fixed proof-theoretical hard core (the underlying system of deduction). [their emphasis]

We give a concrete example of this kind of modular development in [2], where we formalize a class of propositional modal logics. Our base logic there is a labelled natural deduction presentation of propositional calculus extended with introduction and elimination rules for $\Box$ (i.e. the logic K). Our labelling algebras are *relational theories* comprised of Horn clause axioms expressing accessibility of worlds in Kripke frames. That is, labels are possible worlds in the frame and the labelling theory provides a binary relation R, modelling the accessibility relation. Then the labelled formula $x : A$ specifies that A holds at the world indicated by its label x, and A is a theorem if it holds at an arbitrary world. The interface is unidirectional, from the labelling algebra to the base theory, and is given by the rules for $\Box$, e.g. the elimination rule says how we can combine a labelled formula $x : \Box A$ and a relation $x\ R\ y$ to conclude a new labelled formula $y : A$. This combination is enough to formalize, among others, the large and well-known Geach hierarchy, simply by varying the behavior of R.

In our work we used a generic proof system (Isabelle [11]) supporting a metalogic to axiomatize the rules of the base and labelling theories. Our interface between the two parts was strictly enforced. Unlike other approaches based on semantic embedding (e.g. [10]), in which both parts are combined in a single first-order theory, we used different judgements in the metalogic (as in [9]) to separate the two theories and control interaction between them. We showed that our formalization yields a modular presentation of modal logics that is correct (sound and complete with respect to the intended Kripke semantics) for modal consequence for any logic where R can be axiomatized using Horn clause axioms.

This work raises an interesting question. Given that separation between parts of an LDS is a philosophically attractive way to study classes of logics, is there any limit to which classes of modal logics we can capture? In [2] we show correctness for modal logics where the labelling algebra is axiomatized using Horn clause rules, but why stop there? After all, there are modal logics whose relational theories require full first-order, or even higher-order, axiomatizations [16, 17]. Shouldn't it be possible to represent them soundly and completely simply by axiomatizing their corresponding labelling algebras?

Our contribution in this paper is to show, perhaps surprisingly, that the issue is not that simple. There are several natural candidates for the 'hard core' of the base logic, which differ in their interfaces, essentially in the way that the rules for *falsum*, $\perp$, make use of labelling information. We show that the way the two LDS parts are separated has not only theoretical but also practical significance.

local base *falsum*: If we enforce a strong separation, by using only the $\Box$ rules as an interface and making the rules for $\perp$ purely 'local' (so labels in labelled formulae can *only* be manipulated with the $\Box$ rules), then we get a class of logics that does not correspond to any modal logic, although it has independently interesting paraconsistent properties.

global base *falsum*: If we allow slightly more communication, adopting 'global' rules for $\perp$ and again retaining the $\Box$ rules as our interface, then we get the system we describe in [2], which allows us to formalize modal logics with Horn clause axiomatizable Kripke semantics.

universal *falsum*: In order to formalize all modal logics with first-order axiomatizable frames, we must allow even a wider interface between the two parts; in particular we must identify *falsum* in the base logic and labelling algebra by adding rules to propagate $\perp$ between the parts. The result is, effectively, a loss of separation similar to the approaches based on semantic embedding.

In the remainder of this paper we investigate these three possibilities and give a proof theoretic analysis and supporting examples to illustrate how tradeoffs arise. In Section 2 we present a labelled version of K and extensions which build on the second possibility above. After, in Section 3, we present proof normalization results for these theories. These are based on standard results for natural deduction proofs, e.g. [12, 13, 18], which we adapt to our natural deduction LDS presentations. These results are interesting in their own right (e.g. for automated theorem proving based on search for normal-form proofs) but our main application of them is to show tradeoffs in the interface of an LDS and which class of logics it captures. In particular, in Section 4, we show that a narrow interface, while leading to a clean separation of the base logic and labelling algebra, strongly restricts the class of representable logics. In the final section we draw conclusions and compare and apply our results to related work on labelled deduction and modal logics.

2. Labelled Modal Logics

We begin by introducing a labelled natural deduction (ND) system for the base modal logic K and its labelling algebra. Our primary focus is on this system, corresponding to 'global base *falsum*' above, and we will later consider changes of the base logic corresponding to the other possibilities listed above. This system was introduced in [2] and we briefly review relevant parts. We assume the reader is familiar with modal logics (see, e.g., [3]) and natural deduction (e.g. major and minor premises in an inference, discharged and open assumptions, see [12, 13, 18]).

2.1. LABELLED K

Let W be a set of *labels* and R a binary relation over W. If x and y are labels, then $x\ R\ y$ is a *relational formula* (*rwff*). We associate labels with formulae: if A is a propositional modal formula built from $\bot$, $\rightarrow$, $\Box$, then $x:A$ is a *labelled formula* (*lwff*). Hence, if p is a sentence letter, and A, B are propositional modal formulae, then $x:p$, $x:\bot$, $x:A \rightarrow B$, $x:\Box A$ are all lwffs. Lwffs containing other propositional and modal connectives (e.g. $\neg$, $\wedge$, $\vee$, $\Diamond$) can be defined in the usual manner, e.g. $x:\Diamond A \equiv x:(\Box(A \rightarrow \bot) \rightarrow \bot)$.

We henceforth assume that the possibly subscripted variables x, y, z range over labels, the possibly subscripted variables A, B, C range over propositional modal formulae, and $\Gamma = \{x_1 : A_1, \ldots, x_n : A_n\}$ and $\Delta = \{x_1\ R\ y_1, \ldots, x_m\ R\ y_m\}$ are arbitrary sets of lwffs and rwffs.

The rules given in Figure 1 determine K, the base ND system which formalizes a labelled version of the modal logic K. We also give there the two derived rules for possibility, since we will explicitly use them in the latter. Notice that $\bot E$ is what we call a 'global' rule, and its restriction is inessential but technically convenient [18, p.194]. We discuss this rule and its 'global nature' in detail in Section 4.

	$\begin{array}{c}[x:A]\\ \vdots\\ \dfrac{x:B}{x:A\to B}\to I\end{array}$	$\begin{array}{c}[x\;R\;y]\\ \vdots\\ \dfrac{y:A}{x:\Box A}\,\Box I\end{array}$	$\dfrac{y:A \quad x\;R\;y}{x:\Diamond A}\,\Diamond I$
$\begin{array}{c}[x:A\to\bot]\\ \vdots\\ \dfrac{y:\bot}{x:A}\,\bot E\end{array}$	$\dfrac{x:A\to B \quad x:A}{x:B}\to E$	$\dfrac{x:\Box A \quad x\;R\;y}{y:A}\,\Box E$	$\begin{array}{c}\qquad [y:A]\;[x\;R\;y]\\ \qquad\vdots\\ \dfrac{x:\Diamond A \quad z:B}{z:B}\,\Diamond E\end{array}$

Where, in $\bot E$, A is atomic and different from $\bot$; in $\Box I$ (respectively $\Diamond E$), y is different from x (respectively x and z) and does not occur in those assumptions on which $y:A$ (respectively $z:B$) depends, except those of the form $x\;R\;y$ (respectively $y:A$ and $x\;R\;y$), which are discharged by the inference.

Figure 1. The rules of K (with derived rules for $\Diamond$)

2.2. EXTENSIONS OF K: RELATIONAL THEORIES

In K no assumptions are made on the relations between possible worlds in the Kripke frame $\langle W, R\rangle$, hence there are no assumptions about R in the labelling algebra. Other modal logics are obtained from K by placing conditions on the accessibility relation in the frame; e.g. we get the well-known logic T from K by adding that R is reflexive, and then $S4$ from T by further adding transitivity. In [2] we show that a large subclass of the modal logics with first-order axiomatizable frames, including those usually of actual interest (K, D, T, B, $S4$, $S4.2$, $KD45$, $S5$,...) can be captured by extending K with Horn relational theories.

But there are properties of R which fall outside the Horn fragment, since they can only be expressed in a full first-order or higher-order setting. We briefly review Horn relational theories, and then introduce extensions of K by first-order theories.

2.3. HORN RELATIONAL THEORIES

A *Horn formula (over the binary relation R)* is a closed formula of the form $\forall x_1 \ldots \forall x_n((t_1\;R\;s_1 \wedge \ldots \wedge t_m\;R\;s_m) \to t_0\;R\;s_0)$, where the t_i and s_i are terms built from the labels $x_1, \ldots, x_n$ and function symbols. Corresponding to each such Horn formula is a *Horn rule*

$$\frac{t_1\;R\;s_1 \quad \ldots \quad t_m\;R\;s_m}{t_0\;R\;s_0}$$

A *Horn (relational) theory* $\mathcal{T}_H$ is then a theory generated by a set of such Horn rules. Since the addition of a Horn formula to a theory is equivalent (in first-order theories) to adding the corresponding rule, we will talk about additions based on either formulae or rules as is convenient.

Let i, j, m, and n be natural numbers, and let $\Box^n$ (respectively $\Diamond^n$) stand for a sequence of n consecutive $\Box$s (respectively $\Diamond$s); for example $\Diamond^2\Box^3\Diamond^0 A$ is $\Diamond\Diamond\Box\Box\Box A$. Among all the properties of the accessibility relation that can be expressed as Horn rules in the theory of one binary predicate R, it is possible to identify the subclass of *restricted* (i, j, m, n) *convergency axioms*, as the class of properties of the accessibility relation which correspond to instances of the *generalized Geach axiom schema* $\Diamond^i\Box^m A \rightarrow \Box^j\Diamond^n A$ (see [3]), and yield, among others, the logics K, D, T, B, $S4$, $S4.2$, $KD45$ and $S5$. Restricted (i, j, m, n) convergency axioms are closed formulae of the form

$$\forall x \forall y \forall z((x\ R^i\ y \wedge x\ R^j\ z) \rightarrow \exists u(y\ R^m\ u \wedge z\ R^n\ u))\ , \tag{1}$$

where $x\ R^0\ y$ means $x = y$ and $x\ R^{i+1}\ y$ means $\exists v(x\ R\ v \wedge v\ R^i\ y)$, and $m = n = 0$ implies $i = j = 0$. Although there are instances of (1) which explicitly require identity, the restriction that $m = n = 0$ implies $i = j = 0$ is a necessary and sufficient condition for identity to be inessential. In this paper we consider only theories without identity.

We point out in [2, Prop. 3] that for any theory defined by a collection of instances of (1) there is a Horn relational theory conservatively extending it. As an example, consider the properties given in Figure 2, all of which are instances of (1); e.g. transitivity and convergency are given by $(0, 2, 1, 0)$ and $(1, 1, 1, 1)$. We also present there the corresponding Horn rules and characteristic modal axioms.

Various combinations of Horn rules define labelled equivalents of standard modal logics: the logic $L = K + \mathcal{T}_H$ is obtained by extending K with a given Horn relational theory $\mathcal{T}_H$.

We adopt the convention of naming the logic $K + \mathcal{T}$ as KAx, where $\mathcal{T}$ is a relational theory and Ax is a string consisting of the standard names of the characteristic axioms corresponding to the relational rules generating $\mathcal{T}$; e.g. $KT4$ identifies the logic also known as $S4$. Moreover, we distinguish between relational theories generated by Horn or first-order rules by appending subscripts H and F to $\mathcal{T}$, e.g. $KT4_H$, $KT4_F$. For example, $KT4_H$ is obtained by extending K with the Horn rules R_refl and R_trans, or alternatively by extending either KT_H with R_trans or $K4_H$ with R_refl.

2.4. FIRST-ORDER RELATIONAL THEORIES

In addition to Horn relational theories, we also consider a full first-order labelling language, i.e. we extend the previous definition of rwffs as follows: if x, y are labels and ρ_1, ρ_2 are rwffs, then $x\ R\ y$, $\emptyset$ (relational *falsum*),

Seriality, $D: \Box A \to \Diamond A$ $\dfrac{}{x\ R\ f(x)}\ R\_ser$	Reflexivity, $T: \Box A \to A$ $\dfrac{}{x\ R\ x}\ R\_refl$
Transitivity, $4: \Box A \to \Box\Box A$ $\dfrac{x\ R\ y \quad y\ R\ z}{x\ R\ z}\ R\_trans$	Euclideaness, $5: \Diamond A \to \Box\Diamond A$ $\dfrac{x\ R\ y \quad x\ R\ z}{y\ R\ z}\ R\_eucl$
Convergency, $2: \Diamond\Box A \to \Box\Diamond A$ $\dfrac{x\ R\ y \quad x\ R\ z}{y\ R\ g(x,y,z)}\ R\_conv1 \quad \dfrac{x\ R\ y \quad x\ R\ z}{z\ R\ g(x,y,z)}\ R\_conv2$	

Where $f: W \to W$ and $g: (W \times W \times W) \to W$ are function constants.

Figure 2. Some properties of R, characteristic axioms, and Horn rules

$\rho_1 \supset \rho_2$ (implication), $\forall x.\rho_1$ are all rwffs. Notice that we use algebraic symbols ($\emptyset$, $\supset$) to avoid confusion with the symbols in the base logic. As notation, we henceforth assume that the possibly subscripted variable ρ ranges over rwffs, and we will call $x\ R\ y$ a *basic rwff.*

The rules of the first-order ND system of the accessibility relation, ND_R, are given in Figure 3. We also give there the three derived rules for disjunction ($\cup$); rwffs over other connectives (e.g. negation $\sim$, conjunction $\cap$, $\exists$) and corresponding rules are defined as usual, and we will explicitly use them in the following.

The axioms representing first-order properties of R can now be added directly in their full form, and the relational theory $\mathcal{T}_F$ (and hence the logic $K + \mathcal{T}_F$) is obtained by extending ND_R with a collection of such rules. As an example, for instances of restricted (i, j, m, n) convergency and for irreflexivity we add the rules ($rconv$ is schematic):

$$\dfrac{}{\forall x.y.z.\ (x\ R^i\ y \cap x\ R^j\ z) \supset \exists u(y\ R^m\ u \cap z\ R^n\ u)}\ rconv$$

$$\dfrac{}{\forall x.\ \sim (x\ R\ x)}\ irrefl$$

2.5. DERIVATIONS

Consider an arbitrary modal logic $L = K + \mathcal{T}$, where $\mathcal{T}$ is a Horn or first-order relational theory, and let φ be either an lwff or an rwff. An *L-derivation* of φ from a set of lwffs Γ and a set of rwffs Δ is a tree formed using the rules in L, ending with φ and depending only on Γ, Δ. We write $\Gamma, \Delta \vdash_L \varphi$ when φ can be so derived. φ is an *L-theorem*, $\vdash_L \varphi$, if it is

$$\dfrac{\begin{matrix}[\rho \supset \emptyset]\\ \vdots \\ \emptyset\end{matrix}}{\rho}\ \emptyset E \qquad \dfrac{\begin{matrix}[\rho_1]\\ \vdots \\ \rho_2\end{matrix}}{\rho_1 \supset \rho_2}\ {\supset} I \qquad \dfrac{\rho_1 \supset \rho_2 \quad \rho_1}{\rho_2}\ {\supset} E \qquad \dfrac{\rho}{\forall x.\ \rho}\ \forall I \qquad \dfrac{\forall x.\ \rho}{\rho[t/x]}\ \forall E$$

$$\dfrac{\rho_2}{\rho_1 \cup \rho_2}\ {\cup} Il \qquad \dfrac{\rho_1}{\rho_1 \cup \rho_2}\ {\cup} Ir \qquad \dfrac{\rho_1 \cup \rho_2 \quad \begin{matrix}[\rho_1]\\ \vdots \\ \rho\end{matrix} \quad \begin{matrix}[\rho_2]\\ \vdots \\ \rho\end{matrix}}{\rho}\ {\cup} E$$

Where, in $\forall I$, x must not occur free in any open assumption on which ρ depends.

Figure 3. The rules of ND_R (with derived rules for $\cup$)

L-derivable from empty Γ and Δ. In the following we will often omit the 'L'. We systematically use Π, with or without indices, to range over derivations, and we write $\substack{\Pi \\ \varphi}$ to specify that the formula φ is the conclusion of the derivation Π. Similarly, we write $\substack{\varphi \\ \Pi}$ (respectively $\substack{[\varphi] \\ \Pi}$) to distinguish a possibly empty set of occurrences of the open (respectively discharged) assumption φ in Π. Furthermore, we write $\Pi[z/y]$ for the systematic substitution of z for y in Π, with a suitable renaming of the variables to avoid clashes. We use superscripts to associate discharged assumptions with rule applications. Also we combine multiple applications of the same rule into one, and mark the rule with an asterisk.

As an example, consider the following $K4$-derivation of $x:\Diamond\Diamond B$ from the assumptions $x:\Box A$, $y:\Diamond(A \to B)$, and $x\ R\ y$:

$$\dfrac{\dfrac{y:\Diamond(A \to B) \qquad \dfrac{\dfrac{[z:A \to B]^1 \qquad \dfrac{x:\Box A \qquad \begin{matrix} x\ R\ y \quad [y\ R\ z]^1 \\ \Pi \\ x\ R\ z \end{matrix}}{z:A}\ \Box E}{z:B}\ {\to} E \qquad [y\ R\ z]^1}{y:\Diamond B}\ \Diamond I}{y:\Diamond B}\ \Diamond E^1 \qquad x\ R\ y}{x:\Diamond\Diamond B}\ \Diamond I$$

The subderivation Π depends on the relational theory: for $K4_H$ we simply apply R_trans; for $K4_F$ we replace Π with:

$$\dfrac{\dfrac{\dfrac{}{\forall x.y.z.\ (x\ R\ y \cap y\ R\ z) \supset x\ R\ z}\ trans}{(x\ R\ y \cap y\ R\ z) \supset x\ R\ z}\ \forall E^* \qquad \dfrac{x\ R\ y \quad [y\ R\ z]^1}{x\ R\ y \cap y\ R\ z}\ {\cap} I}{x\ R\ z}\ {\supset} E$$

3. Normalization

Our goal in this paper is to show that there are design decisions in modularizing LDS presentations: the interface between the labelling algebra and the base logic effects which class of logics it correctly captures. To show this we need to show that with particular base logics (e.g. K) certain desired formulae (e.g. characteristic modal axioms) are not provable. We will establish this using syntactic methods based on adapting standard results of ND [12, 13, 18] to our labelled systems. To begin with, we show that each L-derivation can be transformed into a normal form which does not contain unnecessary detours or redundancies. We first discuss normalization for K, introducing the necessary terminology and pointing out the modifications with respect to standard results. After, we consider extensions with relational theories.

3.1. NORMALIZATION FOR K

Normalization need not be considered for rwffs in K, since rwffs only appear as assumptions, or as results of 'trivial' derivations, i.e.

Fact 1 *$\Gamma, \Delta \vdash_K x\ R\ y$ iff $\Delta \vdash_K x\ R\ y$ iff $x\ R\ y \in \Delta$.*

An immediate consequence is that rwffs appear only in the fringes of K-derivations of lwffs. This important characteristic of the shape of derivations is preserved in extensions of the base modal logic K with relational theories, where rwffs can be derived and not simply assumed (c.f. Lemma 6 below).

An occurrence of an lwff in a derivation that stands at the same time as the conclusion of an introduction rule and as the major premise of an elimination rule is called a *maximum lwff*. A maximum lwff is a detour in a derivation which can be removed to obtain a simpler derivation. This is achieved by applying the two following *proper reductions*, corresponding to the two possible forms of a maximum lwff (one for $\to$ and one for $\Box$). We only show the part of the derivation where the reduction actually takes place; the missing parts remain unchanged. As notation, we write $\Pi \succ_1 \Pi'$ when Π (properly) reduces to Π', and we write $\succ$ (respectively $\succeq$) for the transitive (respectively transitive-reflexive) closure of $\succ_1$.

$$\dfrac{\dfrac{\begin{matrix}[x:A]\\ \Pi_2\\ x:B\end{matrix}}{x:A\to B}\to I \qquad \begin{matrix}\Pi_1\\ x:A\end{matrix}}{x:B}\to E \quad \succ_1 \quad \begin{matrix}\Pi_1\\ x:A\\ \Pi_2\\ x:B\end{matrix} \qquad\qquad \dfrac{\dfrac{\begin{matrix}[x\ R\ y]\\ \Pi\\ y:A\end{matrix}}{x:\Box A}\Box I \qquad x\ R\ z}{z:A}\Box E \quad \succ_1 \quad \begin{matrix}x\ R\ z\\ \Pi[z/y]\\ z:A\end{matrix}$$

A derivation Π is *irreducible* if there is no Π' such that Π reduces to Π'. If Π is irreducible, then we say that Π is a *normal derivation*, or, equivalently, that Π is in *normal form*, and if $\Pi \succeq \Pi'$ where Π' is normal, then we say that Π *normalizes* to Π'.

Theorem 2 *All K-derivations normalize.*

The theorem follows by a straightforward adaptation of the proof for standard ND systems, where it is shown that proper reductions always reduce a well-founded measure on derivations.

3.2. NORMALIZATION FOR $L = K + \mathcal{T}$

Fact 1 generalizes to:

Fact 3 $\Gamma, \Delta \vdash_{K+\mathcal{T}} \rho$ *iff* $\Delta \vdash_{\mathcal{T}} \rho$.

We therefore distinguish between $\mathcal{T}$-derivations of an rwff ρ from a set of rwffs Δ, and L-derivations of an lwff $x : A$ from sets Γ, Δ.

All the rwffs in a $\mathcal{T}_H$-derivation are basic rwffs, and $\mathcal{T}_H$ does not contain introduction or elimination rules. Hence, every $\mathcal{T}_H$-derivation is already normal. Moreover, definitions and results for K-derivations generalize to $K+\mathcal{T}_H$-derivations under some minor modifications reflecting the fact that in the latter basic rwffs can be also derived and not only assumed (e.g. substitute $\genfrac{}{}{0pt}{}{\Pi}{xRz}$ for the assumption $x\ R\ z$ in the proper reduction for $\Box$). From Theorem 2 we immediately have that:

Theorem 4 *All $K + \mathcal{T}_H$-derivations normalize.*

Let us now turn to first-order relational theories. Since $\mathcal{T}_F$ is a first-order ND system, it is straightforward to show that it possesses all the standard normalization properties for these systems (c.f. [18]). In particular, we can define normal $\mathcal{T}_F$-derivations and show that all $\mathcal{T}_F$-derivations normalize, and then combine the results for K and $\mathcal{T}_F$ to prove that:

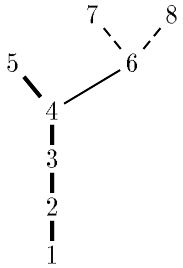
Theorem 5 *All $K + \mathcal{T}_F$-derivations normalize.*

Normal derivations have a number of convenient properties, which follow from their structure. To discuss them, we further adapt some standard terminology.

A *path* in an L-derivation of an lwff $x : A$ is a sequence of formulae $\varphi_0, \ldots, \varphi_n, x : A$, such that φ_0 is an hypothesis, and φ_i is a premise immediately above φ_{i+1}, for $0 \leq i \leq n-1$. Each φ_i can be either an lwff or an rwff, depending on L. An *rpath* in an L-derivation of an rwff ρ_n is a sequence of rwffs $\rho_0, \ldots, \rho_n$, such that ρ_0 is an hypothesis, and ρ_i is a premise immediately above ρ_{i+1}, for $0 \leq i \leq n-1$. A *track* is an initial part of a path which stops at the first minor premise of an elimination rule of K or at the conclusion. In other words, a track can only pass through the major premises of elimination rules of K (we do not consider singular tracks consisting of one single formula, nor do we define tracks for rwffs). A track that is also a path, and thus ends at the conclusion of the derivation, is called a *main track*.

As an example, consider the following $K4_H$-derivation, together with its underlying tree (with numbers substituted for the formulae): the rpaths (dashed lines) are $(7,6)$ and $(8,6)$; the only, and thus main, track (highlighted in bold) is $(5,4,3,2,1)$.

$$\cfrac{\cfrac{\cfrac{\cfrac{[x:\Box A]^3 \qquad \cfrac{[x\;R\;y]^2 \quad [y\;R\;z]^1}{x\;R\;z}\;R\_trans}{z:A}\;\Box E}{y:\Box A}\;\Box I^1}{x:\Box\Box A}\;\Box I^2}{x:\Box A \to \Box\Box A}\;\to I^3$$



Lemma 6 *Let* Π *be a normal* $K+\mathcal{T}$*-derivation of an lwff. As shown in Figure 4, a track in* Π *is divided into two parts: a central subderivation in the base logic, and a* $\mathcal{T}$ *part. The central* K*-subderivation is divided into (at most) three subparts, in which only rules of* K *are applied: an elimination part, in which only elimination rules are applied; a* $\perp$*-part, in which* $\perp E$ *is applied; an introduction part, in which only introduction rules are applied. The* $\mathcal{T}$ *part is attached to the (elimination part of the) central* K*-subderivation by instances of* $\Box E$*, in which only basic rwffs appear. No lwffs appear in the* $\mathcal{T}$ *part, which consists of a fringe of subderivations purely in the labelling algebra. In particular, for* K *(Figure 4.a), it consists only of assumed basic rwffs; for* $K+\mathcal{T}_H$ *(Figure 4.b), it consists of* $\mathcal{T}_H$*-derivations of basic rwffs; for* $K+\mathcal{T}_F$ *(Figure 4.c), it consists of normal* $\mathcal{T}_F$*-derivations of basic rwffs (hence, the last steps in normal* $\mathcal{T}_F$*-derivations in a normal* $K+\mathcal{T}_F$*-derivation are either eliminations or applications of* $\emptyset E$*).*

The lemma follows by a straightforward adaptation of [18, Fact 6.2.10]. Notice that the observations about the $\mathcal{T}$ part (its content and the way it is connected to the central K-subderivation) hold for all, not just normal $K+\mathcal{T}$-derivations. Moreover, if $\Diamond I$ is explicitly added, then $\mathcal{T}$-derivations of basic rwffs appear also at the fringes of the introduction part (and the lemma can be suitably modified).

Finally, it is possible to introduce a suitable notion of subformula for lwffs (by the standard syntactic analysis of the propositional modal formula), and derive a subformula principle. The notion of *sublwff* is defined inductively by: $y:A$ is a sublwff of $x:A$ for all x,y; if $y:B \to C$ is a sublwff of $y:A$, then so are $y:B$ and $y:C$; if $y:\Box B$ is a sublwff of $y:A$, then so is $y:B$.

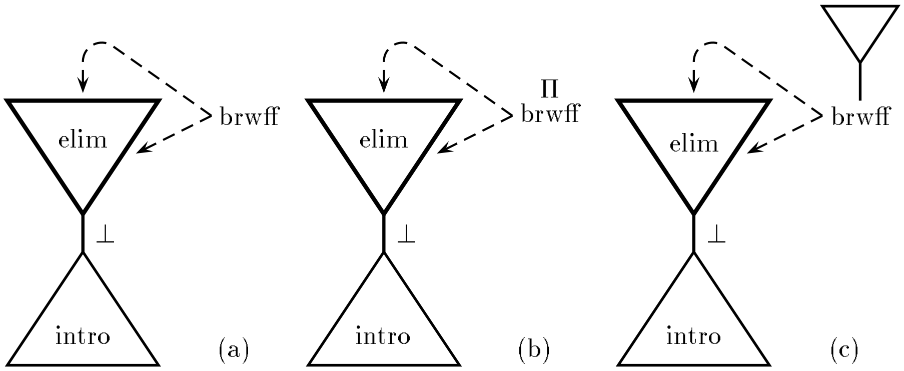


Figure 4. The form of tracks in a normal derivation of an lwff in K, $K + \mathcal{T}_H$, $K + \mathcal{T}_F$ (brwff stands for basic rwff)

Lemma 7 (Sublwff principle for $K + \mathcal{T}$) *Every lwff in a normal $K + \mathcal{T}$-derivation of $x : A$ from Γ, Δ is either a sublwff of $x : A$ or of some lwff of Γ, or is an assumption $y : B \rightarrow \perp$ discharged by an application of $\perp E$ and $y : B$ is a sublwff of the kind mentioned.*

Although we cannot define a notion of subformula for basic rwffs in the logics $K + \mathcal{T}_H$, we remark that in some Horn relational rules with non-empty premises, the labels in the conclusion already occur in the premises (in the case of functions, the arguments of the function already occur in the premises, but not the function itself). On the other hand, a subformula principle for rwffs in $\mathcal{T}_F$ can be easily shown by adapting standard argumentations for first-order logic.

4. The Role of *falsum* and Theory Separation

The technical results developed above allow us to analyze modularity in our LDS presentations of modal logics. We describe and compare the three candidates for a base logic given in the introduction. We start with the second, corresponding to the base logic K described in Section 2.1.

4.1. GLOBAL BASE *FALSUM*

We have indicated in Section 2.3 that K is an expressive enough base logic to capture, using Horn theories, the modal logics usually of interest. Also, we have shown that we have a precisely defined and restricted interaction between the base logic and any algebra with which we extend it: any track in a normal derivation of an lwff consists of a central subderivation purely in the base logic and a fringe of subderivations purely in the labelling algebra, attached to the central subderivation by instances of $\Box E$ (Lemma 6). This is in the philosophical spirit of LDSs, and it is also pragmatically useful: since derivations of rwffs use only the resources of the labelling algebra, we may be able to employ theory specific reasoners successfully to automate proof construction.

However, clearly not all first-order Kripke frames correspond to some algebra $\mathcal{T}_H$. Thus we have to extend the range of possible algebras we allow to all those definable with first-order logic ($\mathcal{T}_F$). In this case we can again apply previous results (Lemma 6) to conclude that the same kind of controlled interaction we observed between K and $\mathcal{T}_H$ holds between K and $\mathcal{T}_F$. Unfortunately with the same interface we loose completeness.

Theorem 8 *There are extensions of K with first-order relational theories which are incomplete with respect to their intended Kripke semantics.*

Proof Consider the logic $L = K + \mathcal{T}_F$, where $\mathcal{T}_F$ is obtained by adding to ND_R the rule

$$\frac{}{\forall x.y.z.\ (x\ R\ y \cap x\ R\ z) \supset (y\ R\ z \cup z\ R\ y)}\ conn\,.$$

This rule corresponds to the modal axiom $\Box(\Box A \to B) \vee \Box(\Box B \to A)$ [17, p.104], which is equivalent to $\Diamond(\Box A \wedge \neg B) \to \Box(\Box B \to A)$. We show[1] that if A and B are different sentence letters, then $\not\vdash_L x : \Diamond(\Box A \wedge \neg B) \to \Box(\Box B \to A)$. It follows immediately that since the characteristic axiom is not derivable, the modal logic L is incomplete. To show non-provability, assume $\vdash_L x : \Diamond(\Box A \wedge \neg B) \to \Box(\Box B \to A)$ where A and B are different sentence letters. By Lemma 6, there exists a normal derivation where the last inferences are introductions, and, starting from the goal, we build the subderivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}[x : \Diamond(\Box A \wedge \neg B)]^1\ [x\ R\ z]^2\ [z : \Box B]^3 \\ \Pi_1 \\ z : A\end{array}}{z : \Box B \to A} \to I^3}{x : \Box(\Box B \to A)} \Box I^2}{x : \Diamond(\Box A \wedge \neg B) \to \Box(\Box B \to A)} \to I^1$$

Since A is a sentence letter, by Lemma 6, $z : A$ is the result of an application of an elimination rule; by permutative reductions (c.f. Footnote 1), we can assume it to be an application of $\Diamond E$:

[1] We omit many tedious details: the proof is obtained by considering all the possible inferences with respect to normalization and the sublwff principle. To improve readability, we explicitly use derived rules for K and $\mathcal{T}_F$. Strictly speaking, by analogy with the disjunction and existential elimination rules in first-order intuitionistic logic, the explicit presence of $\Diamond E$ and $\cup E$ requires us to consider other reductions (the so-called *permutative reductions*), which can be easily adapted from the ones presented in [18].

$$
\dfrac{\dfrac{\dfrac{\dfrac{[x:\Diamond(\Box A\wedge\neg B)]^1 \qquad \begin{array}{c}[x\ R\ z]^2\ [z:\Box B]^3\ [y:\Box A\wedge\neg B]^4\ [x\ R\ y]^4\\ \Pi_2\\ z:A\end{array}}{z:A}\,\Diamond E^4}{z:\Box B\rightarrow A}\rightarrow I^3}{x:\Box(\Box B\rightarrow A)}\,\Box I^2}{x:\Diamond(\Box A\wedge\neg B)\rightarrow\Box(\Box B\rightarrow A)}\rightarrow I^1
$$

Now $z:A$ can only be the result of an application of $\bot E$ or of $\Box E$, since we can exclude the other elimination rules by Lemmata 6 and 7. Suppose that we apply $\bot E$. Then we can replace Π_2 with either one of two similar derivations; we consider only one, and point out that analogous reasoning applies for the other:

$$
\dfrac{\dfrac{\dfrac{[y:\Box A\wedge\neg B]^4}{y:\neg B}\wedge Er \qquad \dfrac{[z:\Box B]^3 \qquad \begin{array}{c}[x\ R\ z]^2\ [x\ R\ y]^4\ [z:\neg A]^5\\ \Pi_3\\ z\ R\ y\end{array}}{y:B}\,\Box E}{y:\bot}\neg E}{z:A}\,\bot E^5
$$

But, given *conn* and $x\ R\ z$ and $x\ R\ y$, we can only derive $y\ R\ z\cup z\ R\ y$:

$$
\dfrac{\dfrac{\dfrac{}{\forall x.y.z.\ (x\ R\ y\cap x\ R\ z)\supset(y\ R\ z\cup z\ R\ y)}\,conn}{(x\ R\ y\cap x\ R\ z)\supset(y\ R\ z\cup z\ R\ y)}\,\forall E^* \qquad \dfrac{[x\ R\ y]^4 \quad [x\ R\ z]^2}{x\ R\ y\cap x\ R\ z}\cap I}{y\ R\ z\cup z\ R\ y}\supset E
$$

and $z\ R\ y$ cannot be a consequence of $y\ R\ z\cup z\ R\ y$.[2] Hence, $z:A$ cannot be the result of an application of $\bot E$. A similar reasoning applies in the case of $\Box E$, and we conclude that $\not\vdash_L x:\Diamond(\Box A\wedge\neg B)\rightarrow\Box(\Box B\rightarrow A)$.

Analogous problems arise when trying to derive other modal formulae. However, the above is the simplest counter-example to the completeness of general $K+\mathcal{T}_F$ logics that we have been able to identify. Moreover, the provability of the corresponding modal axiom is philosophically the first requirement to be fulfilled by the addition of a relational rule. Other counter-examples can be devised in a similar way; e.g. it is possible to show that $x:\Box A\rightarrow\Diamond A$ does not follow from the rule for seriality, $\forall x.\exists y.\ x\ R\ y$.

[2] This follows by a straightforward semantic argument, with the observation that there are no further (relational) assumptions. Note that if R were symmetric, then $z\ R\ y$ would follow from $y\ R\ z\cup z\ R\ y$ by $\cup E$. Hence, this particular counter-example does not hold for extensions of the logic KB, for which, however, other counter-examples can be devised in a similar way.

4.2. UNIVERSAL *FALSUM*

We can isolate the reason for incompleteness in the way that $\perp$ is treated in K: $\perp E$ allows *falsum* to propagate across worlds, but in fact we have to allow more than this. *falsum* needs to be able to propagate also between the base logic and the algebra; i.e. we have to add rules equivalent to:

$$\frac{x:\perp}{\emptyset} \qquad \frac{\emptyset}{x:\perp}$$

Effectively, we identify the constants for *falsum* together in the two theories. In practice, perhaps the most sensible way to do this is to redefine the base logic and the labelling algebra by replacing the rule $\perp E$ with the rules $\perp E_L$ and $\perp E_R$, and the rule $\emptyset E$ with the rules $\emptyset E_L$ and $\emptyset E_R$:

$$\begin{array}{c}[x:A\rightarrow\perp]\\ \vdots\\ \dfrac{y:\perp}{x:A}\ \perp E_L\end{array} \qquad \begin{array}{c}[\rho\supset\emptyset]\\ \vdots\\ \dfrac{y:\perp}{\rho}\ \perp E_R\end{array} \qquad \begin{array}{c}[x:A\rightarrow\perp]\\ \vdots\\ \dfrac{\emptyset}{x:A}\ \emptyset E_L\end{array} \qquad \begin{array}{c}[\rho\supset\emptyset]\\ \vdots\\ \dfrac{\emptyset}{\rho}\ \emptyset E_R\end{array}$$

Deriving the characteristic axiom for *conn* given these rules is now a simple exercise. However, with this wider interface we loose the controlled interaction between the base logic and the labelling algebra in derivations: Lemma 6 does not hold in these systems, since we can now derive lwffs from rwffs and rwffs from lwffs. For example, we can infer an arbitrary rwff from a modal contradiction, or an arbitrary lwff from a contradiction in the relational theory. As indicated in the introduction, by doing this we move much closer to a traditional semantic embedding in which there is only one theory and we cannot distinguish between their connectives and, in particular, *falsum*.

4.3. LOCAL BASE *FALSUM*

We have shown that K is not enough to formalize all the modal logics with first order semantics, but we have not yet considered what might be beneath it. And in fact there is a further obvious narrowing of the base logic, and we should ask what is, or is not, possible there.

We can easily formalize the consequence relation for the logic of $\rightarrow$ and $\perp$ without labels; it is only when we introduce $\Box$ that we have a problem, because this involves (in the usual semantics) a notion of truth relative to a location. We might, therefore expect that we need to make use of the labels *only* in formalizing the rules for $\Box$: after all we interpret $\rightarrow$ and $\perp$ purely locally in the Kripke semantics. And indeed in K the rules we give for $\rightarrow$ are local; $\perp E$, however, is not, since it relates two worlds (x and y): if we can derive *falsum* at an arbitrary y from $x : A \rightarrow \perp$, then we are able to derive A at x. A natural question to ask is what happens if we insist that x and y be the same; i.e.

$$\dfrac{\begin{array}{c}[x:A \to \bot]\\ \vdots \\ x:\bot\end{array}}{x:A}\ \bot E$$

It turns out that this base logic is not quite suitable, but the reasons are subtle. We lack space to go into the details, so we will confine ourselves to a simple example, showing that it is not suitable for any serial modal logic, since by the same style of analysis as in the proof of Theorem 8 we can show that:

$$x\ R\ y \nvdash_K x:\Box A \to \neg\Box\neg A$$

However the systems we can obtain from local base *falsum* do have interesting paraconsistent properties: $\bot$ can only be propagated 'forwards' from a world to an accessible one, i.e. we have that:[3]

$$\dfrac{\dfrac{x:\bot}{x:\Box\bot}\ \bot E \qquad x\ R\ y}{y:\bot}\ \Box E$$

In fact, this is not enough to capture the consequence relation of any standard modal logic, since we loose the duality between $\Box$ and $\Diamond$.

5. Discussion and Comparison

Our work identifies an important property of the structured presentation of logics, their combination, and extension. Namely, there is tension between modularity and extensibility: a narrow interface between the base logic and labelling algebra can limit the degree to which we can make use of extensions to the labelling algebra. As a consequence, there are important design decisions in implementing LDSs whose resolution requires predicting the range of possible applications.

This problem is a practical one which arose in our own work in encoding propositional modal logics while trying to take the idea of modular development seriously. Our approach has been inspired by the LDS methodology proposed by Gabbay, and further developed for modal logics, in parallel with our work, by Russo [14]. However, our work differs from theirs in several important respects. First, Gabbay's proposed LDS methodology is very general and is based on a notion of diagrams and logic data-bases,

[3] Clearly, when R is symmetric we can propagate $\bot$ also 'backwards'.

manipulated by rules with multiple premises and conclusions. For example, in [6, p.57] the rule for $\Diamond E$ is presented as:

$$\frac{s : \Diamond B}{\textit{create } r,\ s < r \textit{ and } r : B}$$

the application of which updates a modal data-base with the two new conclusions (a rule to the same effect is given in [14]). The formal details are different from the kind of pure natural deduction encoding that we have presented, e.g. Figure 1. Second, although we have not provided details here, in our work we use a metalogic in which different judgements (e.g. in the sense of [9]) serve to separate the logic of the base core and the labelling algebra. In this paper, this difference is reflected by using different syntax for these theories. This separation is critical: it is only when we attempt to modularize and separate these two theories formally and define a precise interface between them that we see that only limited modularity (i.e. there are limits to the relational theories) is actually possible.

Of course, in implementing particular LDSs, Gabbay and Russo can similarly separate theories. The precise nature of this would be reflected in the rules they choose for propagating results between data-bases. It should be the case that if their rules enforce a similar separation, then they will encounter similar limitations to those reported here. That is, the problems we identify have some generality and should appear in other frameworks where theories are separated and results are communicated in a limited way between them.

The kind of labelled natural deduction encoding we employ is closest to the work of Simpson [15]. However his focus, proof techniques, and applications are based on using LDSs to investigate intuitionistic versions of modal logics, and his correctness considerations are quite different. Moreover, his relations have no independent theory with which one can work.

Notice that the universal *falsum* approach is adopted explicitly in [14]. Simpson's approach is different, and difficult to compare: he treats rwffs only as assumptions in inferences of lwffs via his 'geometric' rules, which are derivable in our systems. An example of an approach in which, like with local *falsum*, local inconsistency does not imply global inconsistency, is the work of Giunchiglia and Serafini [8], who show that particular 'multicontext systems', where (indexed) formulae are translated between contexts using 'bridge rules', define the same classes of provable formulae as certain standard modal logics. However their approach is, in general, radically different from ours, and not comparable.

In contrast, theorem provers for non-classical logic based on semantic embedding in first-order logic, e.g. [10], also manipulate labelled formula and have no limitations on their labelling algebra. But then, there is no separation either, i.e. there is precisely one $\perp$ from which one can conclude arbitrary relational or labelled formulae.

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