A Simple Parallel/Distributed Sampling Technique:
Local Glauber Dynamics

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joint work with
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Sampling Proper Colorings
Sampling a Proper $q$-Coloring
Sampling a Proper $q$-Coloring
Sampling a Proper $q$-Coloring

Markov chain
- over set of proper colorings
- uniform distribution as unique stationary distribution
- rapidly mixing
Sampling a Proper $q$-Coloring

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Sampling a Proper $q$-Coloring

Markov chain
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$t = 1$
Sampling a Proper $q$-Coloring

Markov chain
- over set of proper colorings
- uniform distribution as unique stationary distribution
- rapidly mixing

$t = 2$
Sampling a Proper $q$-Coloring

Markov chain
- over set of proper colorings
- uniform distribution as unique stationary distribution
- rapidly mixing

$t = 3$
Sampling a Proper $q$-Coloring

Markov chain
- over set of proper colorings
- uniform distribution as unique stationary distribution
- rapidly mixing

$t = 4$
Sampling a Proper $q$-Coloring

Markov chain
- over set of proper colorings
- uniform distribution as unique stationary distribution
- rapidly mixing

$t \geq t_{mix}$
Single-Site Glauber Dynamics
Single-Site Glauber Dynamics

update color of a random node to a random color, if proper
Single-Site Glauber Dynamics

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update color of a random node to a random color, if proper
pick node \( v \) uniformly at random
Single-Site Glauber Dynamics

update color of a random node to a random color, if proper
pick node $v$ uniformly at random
pick color $c$ uniformly at random
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Single-Site Glauber Dynamics

update color of a random node to a random color, if proper
pick node $v$ uniformly at random
pick color $c$ uniformly at random
if none of $v$'s neighbors has color $c$, update $v$'s color to $c$
Single-Site Glauber Dynamics

update color of a random node to a random color, if proper
pick node v uniformly at random
pick color c uniformly at random
if none of v’s neighbors has color c, update v’s color to c

![Diagram showing node update from t to t+1]
Single-Site Glauber Dynamics

update color of a random node to a random color, if proper

pick node $v$ uniformly at random
pick color $c$ uniformly at random
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$t$
$t + 1$
Single-Site Glauber Dynamics

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  pick color $c$ uniformly at random
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$t$

$t + 1$

$t + 2$
Single-Site Glauber Dynamics

update color of a **random node** to a **random color**, if proper

pick node \( v \) uniformly at random
pick color \( c \) uniformly at random
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Centralized

[Jerrum 1995]

**Single-Site Glauber**

\( O(n \log n) \) steps \( q \geq 2\Delta + 1 \)
Centralized

[Jerrum 1995]

**Single-Site Glauber**  
\( O(n \log n) \) steps \( q \geq 2\Delta + 1 \)

Decentralized
Centralized

[Jerrum 1995]
Single-Site Glauber \( O(n \log n) \) steps \( q \geq 2\Delta + 1 \)

Decentralized

[Feng, Sun, Yi 2017]

**What can be sampled locally?**

local/decentralized sampling techniques?
local/decentralized transition rules for Markov chain?
Centralized

[Jerrum 1995]

Single-Site Glauber $O(n \log n)$ steps $q \geq 2\Delta + 1$

Decentralized

[Feng, Sun, Yi 2017]

LubyGlauber $O(\Delta \log n)$ steps $q \geq \alpha \Delta$ for $\alpha > 2$
Centralized

[Jerrum 1995]

update a single node

\(O(n \log n)\) steps \(q \geq 2\Delta + 1\)

Decentralized

[Feng, Sun, Yi 2017]

LubyGlauber

\(O(\Delta \log n)\) steps \(q \geq \alpha\Delta\) for \(\alpha > 2\)
Centralized

[Jerrum 1995]

update a single node

$O(n \log n)$ steps

$q \geq 2\Delta + 1$

Decentralized

[Feng, Sun, Yi 2017]

update an independent set

$O(\Delta \log n)$ steps

$q \geq \alpha \Delta$ for $\alpha > 2$
Centralized

[Jerrum 1995]
update a single node  \( O(n \log n) \) steps  \( q \geq 2\Delta + 1 \)

Decentralized

[Feng, Sun, Yi 2017]
update an independent set  \( O(\Delta \log n) \) steps  \( q \geq \alpha\Delta \) for \( \alpha > 2 \)

[Feng, Sun, Yi 2017]
LocalMetropolis  \( O(\log n) \) steps  \( q \geq \alpha\Delta \) for \( \alpha > 2 + \sqrt{2} \)
Centralized

[Jerrum 1995]

update a single node

$O(n \log n)$ steps $q \geq 2\Delta + 1$

Decentralized

[Feng, Sun, Yi 2017]

update an independent set

$O(\Delta \log n)$ steps $q \geq \alpha \Delta$ for $\alpha > 2$

[Feng, Sun, Yi 2017]

update all nodes

$O(\log n)$ steps $q \geq \alpha \Delta$ for $\alpha > 2 + \sqrt{2}$
**Centralized**

[Jerrum 1995]
update a single node \( O(n \log n) \) steps \( q \geq 2\Delta + 1 \)

**Decentralized**

[Feng, Sun, Yi 2017]
update an independent set \( O(\Delta \log n) \) steps \( q \geq \alpha\Delta \) for \( \alpha > 2 \)

[Feng, Sun, Yi 2017]
update all nodes \( O(\log n) \) steps \( q \geq \alpha\Delta \) for \( \alpha > 2 + \sqrt{2} \)

[F., Ghaffari 2018]
Local Glauber \( O(\log n) \) steps \( q \geq \alpha\Delta \) for \( \alpha > 2 \)
### Centralized

[Jerrum 1995]

**update a single node**  \( O(n \log n) \) steps  \( q \geq 2\Delta + 1 \)

### Decentralized

[Feng, Sun, Yi 2017]

**update an independent set**  \( O(\Delta \log n) \) steps  \( q \geq \alpha\Delta \) for \( \alpha > 2 \)

[Feng, Sun, Yi 2017]

**update all nodes**  \( O(\log n) \) steps  \( q \geq \alpha\Delta \) for \( \alpha > 2 + \sqrt{2} \)

[F., Ghaffari 2018]

**update an almost independent set**  \( O(\log n) \) steps  \( q \geq \alpha\Delta \) for \( \alpha > 2 \)
Local Glauber Dynamics
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Local Glauber Dynamics

update colors of a small-degree node set to random colors, if still proper
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update colors of a small-degree node set to random colors, if still proper

each node $v$

independently marks itself with probability $\gamma \in (0,1)$
Local Glauber Dynamics

update colors of a **small-degree node set** to **random colors**, if still proper

each node \(v\)

independently marks itself with probability \(\gamma \in (0,1)\)
Local Glauber Dynamics

update colors of a small-degree node set to random colors, if still proper

each node $v$
  independently marks itself with probability $\gamma \in (0,1)$
  if marked, picks a proposal $c_v$ uniformly at random
Local Glauber Dynamics

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- if $c_v$ does not lead to (potential) conflicts, updates color
Local Glauber Dynamics

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$t$ $t+1$
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$t$

$t + 1$
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[F., Ghaffari 2018]

**Theorem**

Local Glauber converges to uniform distribution over proper $q$-colorings in $O(\log n)$ steps if $q \geq \alpha \Delta$ for $\alpha > 2$. 
Proof Sketch
Path Coupling Lemma (simplified & informal)
If there is a coupling so that for all pairs of adjacent states the two Markov chains started in those two states come closer together in expectation, then the Markov chain is rapidly mixing.
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Reviewing
Path Coupling for
Single-Site Glauber Dynamics
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update color of a random node to a random color, if proper
Reviewing Path Coupling for Single-Site Glauber Dynamics

update color of a random node to a random color, if proper
pick node v uniformly at random
pick color c uniformly at random
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Naïve Path Coupling for Single-Site Glauber Dynamics
Naïve Path Coupling for Single-Site Glauber Dynamics

\[ v_0 \]
Naïve Path Coupling for Single-Site Glauber Dynamics
Naïve Path Coupling for Single-Site Glauber Dynamics

naïve coupling:
- same vertex
- same color
Naïve Path Coupling for Single-Site Glauber Dynamics

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expected number of differing nodes:
Naïve Path Coupling for Single-Site Glauber Dynamics

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expected number of differing nodes:
Naïve Path Coupling for Single-Site Glauber Dynamics

naïve coupling:
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expected number of differing nodes:

\[ 1 - \frac{1}{n} \left( 1 - \frac{\Delta}{q} \right) \]  

[node \( v_0 \)]
Naïve Path Coupling for Single-Site Glauber Dynamics

naïve coupling:
- same vertex
- same color

expected number of differing nodes:

$$1 - \frac{1}{n} \left( 1 - \frac{\Delta}{q} \right)$$

[node $v_0$]
Naïve Path Coupling for Single-Site Glauber Dynamics

naïve coupling:
- same vertex
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expected number of differing nodes:

$$1 - \frac{1}{n} \left(1 - \frac{\Delta}{q}\right) \quad [node \, v_0]$$
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expected number of differing nodes:

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[node \( v_0 \)]
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expected number of differing nodes:

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[node \( v_0 \)]
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[node $v_0$]
Naïve Path Coupling for Single-Site Glauber Dynamics

naïve coupling:
- same vertex
- same color

expected number of differing nodes:

\[
1 - \frac{1}{n} \left( 1 - \frac{\Delta}{q} \right) + \Delta \cdot \frac{1}{n} \cdot \frac{2}{q}
\]

[node \(v_0\)]

[neighbors of \(v_0\)]
Naïve Path Coupling for Single-Site Glauber Dynamics

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[neighbors of $v_0$]
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[node $v_0$]

[neighbors of $v_0$]
Naïve Path Coupling for Single-Site Glauber Dynamics

naïve coupling:
- same vertex
- same color

expected number of differing nodes:

\[ 1 - \frac{1}{n} \left( 1 - \Delta \frac{\Delta}{q} \right) + \Delta \cdot \frac{1}{n} \cdot \frac{2}{q} \]

[node \( v_0 \)]

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[node \( v_0 \)]  
[neighbors of \( v_0 \)]
Naïve Path Coupling for Single-Site Glauber Dynamics

naïve coupling:
- same vertex
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expected number of differing nodes:

$$1 - \frac{1}{n} \left( 1 - \frac{\Delta}{q} \right)$$  

[node $v_0$]

$$+ \Delta \cdot \frac{1}{n} \cdot \frac{2}{q}$$  

[neighbors of $v_0$]
Naïve Path Coupling for Single-Site Glauber Dynamics

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expected number of differing nodes:

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\[ + \Delta \cdot \frac{1}{n} \cdot \frac{2}{q} \]

[node \( v_0 \)]

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[node \(v_0\)]

[neighbors of \(v_0\)]
Naïve Path Coupling for Single-Site Glauber Dynamics

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expected number of differing nodes:

\[ 1 - \frac{1}{n} \left(1 - \frac{\Delta}{q}\right) \]

\[ + \Delta \cdot \frac{1}{n} \cdot \frac{2}{q} \]

[node \( v_0 \)]

[neighbors of \( v_0 \)]
Naïve Path Coupling for Single-Site Glauber Dynamics

naïve coupling:
- same vertex
- same color

expected number of differing nodes:

\[
1 - \frac{1}{n} \left( 1 - \frac{\Delta}{q} \right) + \Delta \cdot \frac{1}{n} \cdot \frac{2}{q} = 1 - \frac{1}{n} \left( 1 - \frac{3\Delta}{q} \right)
\]
Improved Path Coupling for Single-Site Glauber Dynamics
Improved Path Coupling for Single-Site Glauber Dynamics
Improved Path Coupling for Single-Site Glauber Dynamics

\[ v_0 \]
Improved Path Coupling for Single-Site Glauber Dynamics

\[ v_0 \]
Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
- same node $v$
- same color for $v \notin N(v_0)$
- flipped color for $v \in N(v_0)$
Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
- same node \( v \)
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Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
- same node \( v \)
- same color for \( v \notin N(v_0) \)
- flipped color for \( v \in N(v_0) \)

expected number of differing nodes:

\[
1 - 1^n - 2\Delta q\]

\[
+ \Delta q \cdot 1^n \cdot 1^q \text{ neighbors of } v_0\]
Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
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expected number of differing nodes:
Improved Path Coupling for Single-Site Glauber Dynamics

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Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
- same node $v$
- same color for $v \notin N(v_0)$
- flipped color for $v \in N(v_0)$

expected number of differing nodes:

$$1 - \frac{1}{n} \left(1 - \frac{\Delta}{q}\right)$$  \hspace{1cm} [node $v_0$]
Improved Path Coupling for Single-Site Glauber Dynamics

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- same node $v$
- same color for $v \notin N(v_0)$
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expected number of differing nodes:

$$1 - \frac{1}{n} \left( 1 - \frac{\Delta}{q} \right)$$

[node $v_0$]
Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
- same node $v$
- same color for $v \notin N(v_0)$
- flipped color for $v \in N(v_0)$

expected number of differing nodes:

$$1 - \frac{1}{n} \left(1 - \frac{\Delta}{q}\right) \quad [\text{node } v_0]$$
Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
- same node $v$
- same color for $v \notin N(v_0)$
- flipped color for $v \in N(v_0)$

expected number of differing nodes:

$$1 - \frac{1}{n} \left( 1 - \frac{\Delta}{q} \right)$$  

[node $v_0$]
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[node $v_0$]
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- same node \( v \)
- same color for \( v \notin N(v_0) \)
- flipped color for \( v \in N(v_0) \)

expected number of differing nodes:

\[
1 - \frac{1}{n} \left( 1 - \frac{\Delta}{q} \right) + \Delta \cdot \frac{1}{n} \cdot \frac{1}{q}
\]

[node \( v_0 \)]

[neighbors of \( v_0 \)]
Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
- same node $v$
- same color for $v \notin N(v_0)$
- flipped color for $v \in N(v_0)$

expected number of differing nodes:

$$1 - \frac{1}{n} \left( 1 - \frac{\Delta}{q} \right)$$ [node $v_0$]

$$+ \Delta \cdot \frac{1}{n} \cdot \frac{1}{q}$$ [neighbors of $v_0$]
Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
- same node \( v \)
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[node \( v_0 \)]

[neighbors of \( v_0 \)]
Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
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- same color for \( v \notin N(v_0) \)
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[node \( v_0 \)]

[neighbors of \( v_0 \)]
Improved Path Coupling for Single-Site Glauber Dynamics

flipped coupling:
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- same color for \( v \notin N(v_0) \)
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\]

[node \( v_0 \)]

[neighbors of \( v_0 \)]
Sketch of Path Coupling for Local Glauber Dynamics
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update colors of a small-degree node set to random colors, if still proper
Sketch of Path Coupling for Local Glauber Dynamics

update colors of a small-degree node set to random colors, if still proper  
  each node $v$  
    independently marks itself with probability $\gamma \in (0,1)$  
    if marked, picks a proposal $c_v$ uniformly at random  
    if $c_v$ does not lead to (potential) conflicts, updates color
Path Coupling for Local Glauber Dynamics
Path Coupling for Local Glauber Dynamics
Path Coupling for Local Glauber Dynamics

flipped coupling:
- mark same nodes
- flipped colors iff there is a neighbor with flipped colors
Path Coupling for Local Glauber Dynamics

flipped coupling:
- mark same nodes
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[node \( v_0 \)]
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< 1
## Centralized

[Jerrum 1995]

**Single-Site Glauber**

\[ O(n \log n) \text{ steps} \quad q \geq 2\Delta + 1 \]

## Decentralized

[Feng, Sun, Yi 2017]

**LubyGlauber**

\[ O(\Delta \log n) \text{ steps} \quad q \geq \alpha\Delta \text{ for } \alpha > 2 \]

[Feng, Sun, Yi 2017]

**LocalMetropolis**

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| Feng, Sun, Yi 2017 | Luby Glauber | $O(\Delta \log n)$ steps | $q \geq \alpha\Delta$ for $\alpha > 2$
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| Feng, Sun, Yi 2017 | Local Metropolis | $O(\log n)$ steps | $q \geq \alpha\Delta$ for $\alpha > 2 + \sqrt{2}$
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### Centralized

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Thank you!