A Simple Boosting Framework for Transshipment

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We want to solve the single-source shortest path problem (SSSP).

- Given an undirected graph where edges have weights. Compute shortest path from source to all other nodes.
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One of the oldest problems in computer science. Sequential setting .. easy! Dijkstra’s famous $\tilde{O}(m + n)$-time algorithm is optimal (modulo log).

What about parallel or distributed settings? The problem seems much harder: Dijkstra fails miserably.
In these settings, significant progress has been made on the 
\((1 + \varepsilon)\)-approximate shortest path problem.

Main ideas:
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Main ideas:

- **Hopset**: Add a small number of edges to a graph such that original shortest paths are \((1 + \varepsilon)\)-approximated with new paths with small number of hops. (Figure taken from Cohen’00)
In these settings, significant progress has been made on the \((1 + \varepsilon)\)-approximate shortest path problem.

Main ideas:

- **Continuous optimization:** (Today) Generalize the shortest path to transshipment. Find an bad approximate solution and boost it to an \((1 + \varepsilon)\)-approximate.
Recent wins of the continuous-optimization approach

Recent results based on **continuous optimization**.

**Parallel.**

$(1 + \varepsilon)$-apx with $\tilde{O}(1)$ depth and $\tilde{O}(m)$ work.

$(1 + \varepsilon)$-apx **deterministic** with $\tilde{O}(1)$ depth and $\tilde{O}(m)$ work.

**Distributed.**

[Li; 2020]

[ASZ; 2020]

[RGHZL; 2022]
Recent wins of the continuous-optimization approach

Recent results based on continuous optimization.

Parallel.
$(1 + \varepsilon)$-apx with $\tilde{O}(1)$ depth and $\tilde{O}(m)$ work.
$(1 + \varepsilon)$-apx deterministic with $\tilde{O}(1)$ depth and $\tilde{O}(m)$ work.  

Distributed.
$(1 + \varepsilon)$-apx in $OPT(G) \cdot n^{o(1)}$ rounds.  

[Li; 2020]
[ASZ; 2020]
[RGHZL; 2022]

[ZGYHS’22].
Introduction

Main Aspects of the Solution
- Idea 1: Transshipment generalizes shortest path
- Notation: Consider the LP primal-dual formulation
- Idea 2: Transshipment boosting with duals
- Idea 3: Multiplicative weights
- Idea 4: Self-reduction of transshipment and consequences

Conclusion
Idea 1: Transshipment generalizes shortest path

Transshipment.
Idea 1: Transshipment generalizes shortest path

Transshipment. Given a graph $G = (V, E)$ and a demand vector $d \in \mathbb{R}^V$ satisfying $\sum_v d(v) = 0$. Find a flow of minimum cost that satisfies the demands.
Idea 1: Transshipment generalizes shortest path

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Also known as: uncapacitated min-cost flow, earth mover’s distance, Wasserstein metric, optimal transport, transshipment.
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Note. Generalizes $(s - t)$ shortest path. (Also generalizes SSSP.)
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3 Conclusion
Notation: Consider the LP primal-dual formulation

Write the graph $G = (V, E)$ using the **node-edge incidence matrix** $B$.
Steps: (1) Orient edges arbitrarily. (2) For each arc, add column to $B$.

$$
\begin{align*}
B &= \begin{bmatrix}
e_1 & e_2 & e_3 & \cdots \\
v_1 & +1 & \cdots & \\
v_2 & & +1 & \cdots \\
v_3 & & & -1 & \cdots \\
v_4 & & & & +1 & \cdots \\
v_5 & -1 & -1 & & & \cdots 
\end{bmatrix}
\end{align*}
$$

Primal. Dual.
Notation: Consider the LP primal-dual formulation

Write the graph $G = (V, E)$ using the node-edge incidence matrix $B$.

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Primal.

$$\min_f \|f\|_1 : Bf = d$$

Dual.

$$B = \begin{bmatrix}
e_1 & e_2 & e_3 & \ldots \\
v_1 & +1 & & \ldots \\
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  v_5 & & & & & -1 & -1 & \cdots
\end{bmatrix}$$

**Primal.**

$$\min_f \|f\|_1 : Bf = d$$

- $f_e = 0$ if no flow along $e$
- $f_e > 0$ if flow in same direction as $e$
- $f_e < 0$ if flow in opposite direction

**Dual.**

\[\max \sum_{e \in E} f_e \] subject to:

1. $\sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = d_v$ for all $v \in V$
2. $f_e = 0$ for all $e \in E$

where $\delta^+(v)$ is the set of outgoing edges from $v$ and $\delta^-(v)$ is the set of incoming edges to $v$.
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![Graph](image)

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- $(Bf)_v = 0$ if flow conserved at $v$

**Dual.**

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B = \begin{bmatrix}
  e_1 & e_2 & e_3 & \cdots \\
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![Graph Diagram]

$B = \begin{bmatrix}
e_1 & e_2 & e_3 & \ldots \\
v_1 & +1 & & & \\
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**Primal.**

$$\min_{f} \|f\|_1 : Bf = d$$

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SP .. $f^* =$ shortest path from $s$ to $t$

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Dual.

\[ \max_{\phi} \langle d, \phi \rangle : \|B^T\phi\|_\infty \leq 1. \]
Notation: Consider the LP primal-dual formulation

Write the graph \( G = (V, E) \) using the node-edge incidence matrix \( B \).
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\end{bmatrix} \]

**Primal.**

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\begin{align*}
\min_f & \quad \|f\|_1 : Bf = d \\
f_e &= 0 \text{ if no flow along } e \\
f_e &= 0 \text{ if flow in same direction as } e \\
f_e &= 0 \text{ if flow in opposite direction} \\
(Bf)_v &= 0 \text{ if flow conserved at } v \\
\text{SP} & \quad f^* = \text{shortest path from } s \text{ to } t
\end{align*}
\]

**Dual.**

\[
\begin{align*}
\max_{\phi} & \quad \langle d, \phi \rangle : \|B^T \phi\|_\infty \leq 1. \\
\phi_v &= \text{potential (height) of } v
\end{align*}
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Notation: Consider the LP primal-dual formulation

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$$\max_{\phi} \langle d, \phi \rangle : \|B^T\phi\|_\infty \leq 1.$$ 

- $\phi_v =$ potential (height) of $v$
- $(B^T\phi)_e = \phi_a - \phi_b$ is height difference
Notation: Consider the LP primal-dual formulation

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- $(B^T \phi)_e = \phi_a - \phi_b$ is height difference
- $\|B^T \phi\|_\infty \leq 1$ height diff must be small
Notation: Consider the LP primal-dual formulation

Write the graph $G = (V, E)$ using the **node-edge incidence matrix** $B$.

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SP .. $\phi^*_v =$ distance of $v$ from source
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Idea 2: Transshipment boosting with duals

Why transshipment? Isn’t it harder?

Amazing property: we can boost a bad approximation to a good approximation.
Idea 2: Transshipment boosting with duals

Why transshipment? Isn’t it harder?

Amazing property: we can **boost** a bad approximation to a good approximation.

Primal.
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\min_f \|f\|_1 : Bf = d
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Dual.
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\max_\phi \langle d, \phi \rangle : \|B^T \phi\|_\infty \leq 1.
\]
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Amazing property: we can boost a bad approximation to a good approximation.

Primal.
\[ \min_{f} \|f\|_1 : Bf = d \]

Dual.
\[ \max_{\phi} \langle d, \phi \rangle : \|B^T \phi\|_{\infty} \leq 1. \]

Theorem ([Sherman; 2013], [BFKL; 2016], [Zuzic; unpublished])

Fix \( G \). Suppose we are given an oracle \( O_G(\cdot) \) which, given a demand \( d \), outputs an \( \alpha \)-approximate feasible dual \( O_G(d) \). There is an algorithm that produces a \( (1 + \varepsilon) \)-approximate feasible dual by calling \( O_G(\cdot) \) at most \( \text{poly}(\alpha, \varepsilon^{-1}, \log n) \) times.
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Corollary

Given such a (dual) \(n^{o(1)}\)-approximation oracle, we can solve \((1 + \frac{1}{n^{o(1)}})\)-approximate transshipment in \(n^{o(1)}\) oracle calls.
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Conclusion
**Feasibility task**
\[ \exists \phi \quad ||A\phi||_{\infty} + \langle b, \phi \rangle \leq \gamma \]
Feasibility task

\[ \exists \phi \quad \|A\phi\|_\infty + \langle b, \phi \rangle \leq \gamma \]

Multiplicative Weights

maintain sol. \( \phi_1, \phi_2, \ldots \)
Idea 3: Multiplicative weights

Feasibility task
\[ \exists \phi \; ||A\phi||_\infty + \langle b, \phi \rangle \leq \gamma \]

Multiplicative Weights

Given \( p, ||p||_1 \leq 1 \):
Find \( \phi \) such that
\[ \langle p, A\phi \rangle + \langle b, \phi \rangle \leq \gamma - \varepsilon \]
equiv:
\[ \langle A^T p + b, \phi \rangle \leq \gamma - \varepsilon \]

Oracle: linearized task

Question: How many oracle calls?
Idea 3: Multiplicative weights

Feasibility task
\[ \exists \phi \quad \|A\phi\|_\infty + \langle b, \phi \rangle \leq \gamma \]

Multiplicative Weights
maintain sol. \( \phi_1, \phi_2, \ldots \)

query(p)

Oracle: linearized task
Given \( p, \|p\|_1 \leq 1 \):
Find \( \phi \) such that
\[ \langle p, A\phi \rangle + \langle b, \phi \rangle \leq \gamma - \varepsilon \]
equiv
\[ \langle A^T p + b, \phi \rangle \leq \gamma - \varepsilon \]

Question: How many oracle calls?
The oracle will be queried \( \text{poly}(\varepsilon^{-1}, \log n, \rho) \) times.
Here, \( \rho \geq \|A\phi\|_\infty \) called width of the oracle.
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Idea 4: Self-reduction of transshipment and consequences

Dual: \( \max_{\phi} \langle d, \phi \rangle \) such that \( \| B^T \phi \|_{\infty} \leq 1 \)
Idea 4: Self-reduction of transshipment and consequences

Dual: \( \max_{\phi} \langle d, \phi \rangle \) such that \( \| B^T \phi \|_{\infty} \leq 1 \)

Binary search \( g \): \( \exists \phi, \langle d, \phi \rangle \geq g, \| B^T \phi \|_{\infty} \leq 1 \)
Idea 4: Self-reduction of transshipment and consequences

Dual: \( \max_{\phi} \langle d, \phi \rangle \) such that \( \|B^T \phi\|_\infty \leq 1 \)

Binary search \( g \):
\[ \exists \phi, \quad \langle d, \phi \rangle \geq g, \quad \|B^T \phi\|_\infty \leq 1 \]

Rewrite:
\[ \exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq 1 \geq \|B^T \phi\|_\infty \]
Idea 4: Self-reduction of transshipment and consequences

Dual: \( \max_{\phi} \langle d, \phi \rangle \) such that \( \| B^T \phi \|_\infty \leq 1 \)

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Rewrite: \( \exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq 1 \geq \| B^T \phi \|_\infty \)

Eliminate middle: \( \exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq \| B^T \phi \|_\infty \)
Idea 4: Self-reduction of transshipment and consequences

Dual: \( \max_{\phi} \langle d, \phi \rangle \) such that \( \| B^T \phi \|_\infty \leq 1 \)

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Multiplicative weights: \( \exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq \langle p, B^T \phi \rangle + \varepsilon \)
Idea 4: Self-reduction of transshipment and consequences

Dual: $\max_{\phi} \langle d, \phi \rangle$ such that $\|B^T \phi\|_{\infty} \leq 1$

Binary search $g$: $\exists \phi, \langle d, \phi \rangle \geq g, \|B^T \phi\|_{\infty} \leq 1$

Rewrite: $\exists \phi, \frac{1}{g} \langle d, \phi \rangle \geq 1 \geq \|B^T \phi\|_{\infty}$

Eliminate middle: $\exists \phi, \frac{1}{g} \langle d, \phi \rangle \geq \|B^T \phi\|_{\infty}$

Multiplicative weights: $\exists \phi, \frac{1}{g} \langle d, \phi \rangle \geq \langle p, B^T \phi \rangle + \varepsilon$

Rewrite: $\exists \phi, \left\langle \frac{1}{g}d - Bp, \phi \right\rangle \geq \varepsilon_{\text{residual}}$
Idea 4: Self-reduction of transshipment and consequences

Dual: \[ \max_\phi \langle d, \phi \rangle \text{ such that } \left\| B^T \phi \right\|_\infty \leq 1 \]

Binary search \( g \): \[ \exists \phi, \quad \langle d, \phi \rangle \geq g, \quad \left\| B^T \phi \right\|_\infty \leq 1 \]

Rewrite: \[ \exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq 1 \geq \left\| B^T \phi \right\|_\infty \]

Eliminate middle: \[ \exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq \left\| B^T \phi \right\|_\infty \]

Multiplicative weights: \[ \exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq \langle p, B^T \phi \rangle + \varepsilon \]

Rewrite: \[ \exists \phi, \quad \left\langle \frac{1}{g} d - Bp, \phi \right\rangle \geq \varepsilon \]

Oracle: \( \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon \). Note: oracle width \( \rho \geq \left\| B^T \phi \right\|_\infty \)
Idea 4: Self-reduction of transshipment and consequences

**Dual:** \( \max_{\phi} \langle d, \phi \rangle \) such that \( \| B^T \phi \|_\infty \leq 1 \)

**Binary search** \( g \): \( \exists \phi, \quad \langle d, \phi \rangle \geq g, \quad \| B^T \phi \|_\infty \leq 1 \)

**Rewrite:** \( \exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq 1 \geq \| B^T \phi \|_\infty \)

**Eliminate middle:** \( \exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq \| B^T \phi \|_\infty \)

**Multiplicative weights:** \( \exists \phi, \quad \frac{1}{g} \langle d, \phi \rangle \geq \langle p, B^T \phi \rangle + \varepsilon \)

**Rewrite:** \( \exists \phi, \quad \left\langle \frac{1}{g} d - Bp, \phi \right\rangle \geq \varepsilon \)

**Oracle:** \( \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon. \) Note: oracle width \( \rho \geq \| B^T \phi \|_\infty \)

**Equiv:** \( \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / \rho, \quad \| B^T \phi \|_\infty \leq 1 \)
Self-reduction consequences

Oracle: \( \exists \phi, \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / \rho, \| B^T \phi \|_\infty \leq 1 \)

Notes:

- **Q:** Why does this make progress at all?
Self-reduction consequences

Oracle: \( \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / \rho, \quad \| B^T \phi \|_\infty \leq 1 \)

Notes:

Q: Why does this make progress at all? Ans: Note that we can increase \( \rho \), the final answer is still \((1 + \varepsilon)\)-approximate but runtime increases.

Q: What if there is no solution to the new problem? Ans: Look at primal and prove the original problem has no solution. Computationally, we start with the dual LP and end with a dual LP. Result: Any \( \alpha \)-approximate dual solver can be boosted to \((1 + \varepsilon)\)-approximate dual solver. Runtime: \( \text{poly}(\alpha, \varepsilon^{-1}, \log n) \) query calls.
Self-reduction consequences

Oracle: \( \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / \rho, \quad \| B^T \phi \|_\infty \leq 1 \)

Notes:

- **Q:** Why does this make progress at all? **Ans:** Note that we can increase \( \rho \), the final answer is still \((1 + \varepsilon)\)-approximate but runtime increases. Also: we can now handle approximations in the answer. An \( \alpha \)-approx changes the width of the oracle. **Approximation \( \rightarrow \) runtime.**

\[
\exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / (\alpha \cdot \rho), \quad \| B^T \phi \|_\infty \leq 1
\]
Self-reduction consequences

Oracle: \( \exists \phi, \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / \rho, \quad \| B^T \phi \|_\infty \leq 1 \)

Notes:

- **Q:** Why does this make progress at all? **Ans:** Note that we can increase \( \rho \), the final answer is still \((1 + \varepsilon)\)-approximate but runtime increases. Also: we can now handle approximations in the answer. An \( \alpha \)-approx changes the width of the oracle. 

Approximation \( \rightarrow \) runtime.

\[ \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / (\alpha \cdot \rho), \quad \| B^T \phi \|_\infty \leq 1 \]

- **Q:** What if there is no solution to the new problem?

\[ \text{Ans: Look at primal and prove the original problem has no solution.} \]

Computationally, we start with the dual LP and end with a dual LP. Result: Any \( \alpha \)-approximate dual solver can be boosted to \((1 + \varepsilon)\)-approximate dual solver. Runtime: \(\text{poly}(\alpha, \varepsilon^{-1}, \log n)\) query calls.
Self-reduction consequences

Oracle: \[ \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / \rho, \quad \| B^T \phi \|_\infty \leq 1 \]

Notes:

- **Q:** Why does this make progress at all?  
  **Ans:** Note that we can increase \( \rho \), the final answer is still \((1 + \varepsilon)\)-approximate but runtime increases. Also, we can now handle approximations in the answer. An \( \alpha \)-approx changes the width of the oracle.  

Approximation \( \rightarrow \) runtime.

\[ \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / (\alpha \cdot \rho), \quad \| B^T \phi \|_\infty \leq 1 \]

- **Q:** What if there is no solution to the new problem?  
  **Ans:** Look at primal and prove the original problem has no solution.
Self-reduction consequences

**Oracle:** \[ \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / \rho, \quad \| B^T \phi \|_\infty \leq 1 \]

Notes:

- **Q:** Why does this make progress at all? **Ans:** Note that we can increase \( \rho \), the final answer is still \((1 + \varepsilon)\)-approximate but runtime increases. Also: we can now handle approximations in the answer. An \( \alpha \)-approx changes the width of the oracle. Approximation \( \rightarrow \) runtime.

\[ \exists \phi, \quad \langle d_{\text{residual}}, \phi \rangle \geq \varepsilon / (\alpha \cdot \rho), \quad \| B^T \phi \|_\infty \leq 1 \]

- **Q:** What if there is no solution to the new problem? **Ans:** Look at primal and prove the original problem has no solution.

- Computationally, we start with the dual LP and end with a dual LP. **Result:** Any \( \alpha \)-approximate dual solver can be boosted to \((1 + \varepsilon)\)-approximate dual solver. **Runtime:** \( \text{poly}(\alpha, \varepsilon^{-1}, \log n) \) query calls.
1 Introduction

2 Main Aspects of the Solution
   • Idea 1: Transshipment generalizes shortest path
   • Notation: Consider the LP primal-dual formulation
   • Idea 2: Transshipment boosting with duals
   • Idea 3: Multiplicative weights
   • Idea 4: Self-reduction of transshipment and consequences

3 Conclusion

Thank you!