Asymptotic analysis - several variables

O, 2 variables:

$$O(g(m,n)) = \{ f(m,n) \mid \text{there is } c > 0,\ n_0 \in \mathbb{N} \text{ such that } |f(m,n)| \le c\,|g(m,n)| \text{ for } m,n \ge n_0 \}$$

Ω, Θ, more variables analogously

Examples:

- MMM: $k \times m$ matrix times $m \times n$ matrix is $O(kmn)$ — by definition
- Multiplying 2 polynomials of degrees m and n is $O(mn)$

Note: $O(mn+n)$ can be simplified to $O(mn)$
$O(m+n)$ cannot be simplified

Cost analysis: cost measure

Example MMM, $k \times m$ matrix times $m \times n$ matrix

asymptotic: $O(kmn)$

cost measure $C(k,m,n)$ = number of adds/mults: $2kmn - kn$

cost measure $C(k,m,n)$ = (# adds, # mults): $(kmn, kmn - kn)$

Cost analysis: solving recurrences

Easy case: $f_0 = c$, $f_k = a f_{k-1} + s_k$, $k \ge 1$ $\qquad$ a, c constants

$$\Rightarrow f_k = a^k c + \sum_{i=0}^{k-1} a^i s_{k-i}$$

Example: $f_0 = 0$, $f_k = 2 f_{k-1} + 3 \cdot 2^{k-1} - 1$

$$\Rightarrow f_k = \frac{3}{2} k 2^k - 2^k + 1$$

Exponential version: $g_1 = c,\ g_n = a\, g_{n/2} + t_n$

$$\left[\text{substitute } n = 2^k,\ g_{2^k} = f_k,\ t_{2^k} = s_k\right]$$

$$\Longrightarrow\ f_0 = c,\ f_k = a f_{k-1} + s_k$$

solve as before, translate back

Example: $g_1 = 0,\ g_n = 2 g_{n/2} + \frac{3}{2} n - 1 \qquad (\longrightarrow g_n = \Theta(n \log(n)))$

$$\xrightarrow{n=2^k} f_0 = 0,\ f_k = 2 f_{k-1} + \frac{3}{2} 2^k - 1$$

$$\xrightarrow{\text{solve}} f_k = \frac{3}{2} k 2^k - 2^k + 1$$

$$\xrightarrow{\text{translate}} g_n = \frac{3}{2} \log_2(n) n - n + 1 \quad ✓$$

Solving recurrences using generating functions

Method — Example: $f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$ (Fibonacci numbers)

1.) Multiply by x^n and sum: $\sum f_n x^n = \sum f_{n-1} x^n + \sum f_{n-2} x^n$

2.) Determine bounds of sum (avoid f_n, $n < 0$): $\sum_{n\geq 2} f_n x^n = \sum_{n\geq 2} f_{n-1} x^n + \sum_{n \geq 2} f_{n-2} x^n$

3.) Translate to equation for $F(x) = \sum_{n\geq 0} f_n x^n$: $F(x) - x = x F(x) + x^2 F(x)$ (needs initial conditions)

4.) Solve for $F(x)$: $F(x) = \frac{x}{1 - x - x^2}$

5.) Partial fraction expansion: $F(x) = \frac{x}{(1-\alpha x)(1-\alpha' x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\alpha' x}$

$1/\alpha,\ 1/\alpha'$ zeros of $1 - x - x^2$

$\Longleftrightarrow \alpha, \alpha'$ zeros of $x^2 - x - 1$

[Note: α zero of $a_n x^n + \ldots + a_0 \Longleftrightarrow 1/\alpha$ zero of $a_n + \ldots + a_0 x^n$]

$$\alpha, \alpha' = \frac{1 \pm \sqrt{5}}{2}$$

$$A = F(x) \cdot (1 - \alpha x)\Big|_{x = 1/\alpha} = \frac{x}{1-\alpha' x}\Big|_{x=1/\alpha} = \frac{1}{\sqrt{5}}$$

$$B = \ldots\ldots\ldots = -\frac{1}{\sqrt{5}}$$

6.) Evolve into series: $F(x) = \frac{1}{\sqrt{5}} \sum_{n\geq 0} \alpha^n x^n - \frac{1}{\sqrt{5}} \sum_{n\geq 0} (\alpha')^n x^n$

7.) Read off f_n: $f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$