Exact solution of higher order recurrences

Method of generating functions

sequence $f_0, f_1, f_2, \dots$ $\longleftrightarrow$ generating function $\sum_{n=0}^{\infty} f_n x^n = F($

second order example:

$$f_0 = 0,\ f_1 = 1,\ f_k = f_{k-1} + f_{k-2},\ k \geq 2 \text{ (Fibonacci numbers)}$$

1.) Multiply by x^k and sum

$$\sum f_k x^k = \sum f_{k-1} x^k + \sum f_{k-2} x^k$$

2.) Determine summation boundaries (here: $k \geq 2$)

$$\sum_{k=2}^{\infty} f_k x^k = x \sum_{k=1}^{\infty} x^k + x^2 \sum_{k=0}^{\infty} x^k$$

3.) Use generating function

$$F(x) - x = x(F(x)) + x^2 F(x)$$

4.) Solve

$$F(x) = \frac{x}{1 - x - x^2} \longleftarrow \text{characteristic polynomial of recurrence}$$

normalize to 1

5.) Partial fraction expansion

$$F(x) = \frac{x}{(1-\alpha x)(1-\alpha' x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\alpha' x} \quad (\alpha \neq \alpha$$

$1/\alpha, 1/\alpha'$ are zeros of $1 - x - x^2$

$\Longleftrightarrow \alpha, \alpha'$ " $x^2 - x - 1$ (mirrored polynomial)

$$\alpha, \alpha' = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$$

$$A = F(x)(1-\alpha x)\Big|_{x=\frac{1}{\alpha}} = \frac{x}{1-\alpha' x}\Big|_{x=\frac{1}{\alpha}} = \frac{1}{\sqrt{5}}$$

$$B = F(x)(1-\alpha' x)\Big|_{x=\frac{1}{\alpha'}} = \frac{x}{1-\alpha x}\Big|_{x=\frac{1}{\alpha'}} = -\frac{1}{\sqrt{5}}$$

6.) Evolve into series

$$F(x) = A \sum_{k=0}^{\infty} \alpha^k x^k + B \sum_{k=0}^{\infty} (\alpha')^k x^k$$

7.) Read off f_k: $f_k = A\alpha^k + B(\alpha')^k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$

Enables solving recurrences of the form

$$g_1 = c,\ g_2 = d,\ g_n = a\, g_{n/2} + b\, g_{n/4},\quad n = 2^k,\ k \geq 2$$

by substituting $n = 2^k$, $g_n = f_k$