How to Write Fast Numerical Code

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Lecture: Spiral (Computer generation of FFT code)

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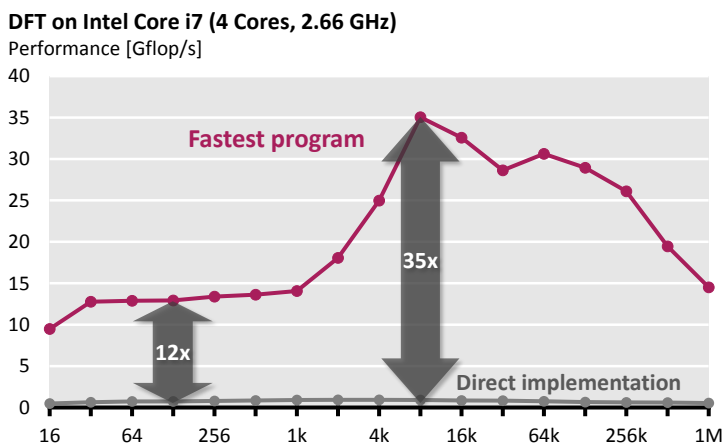
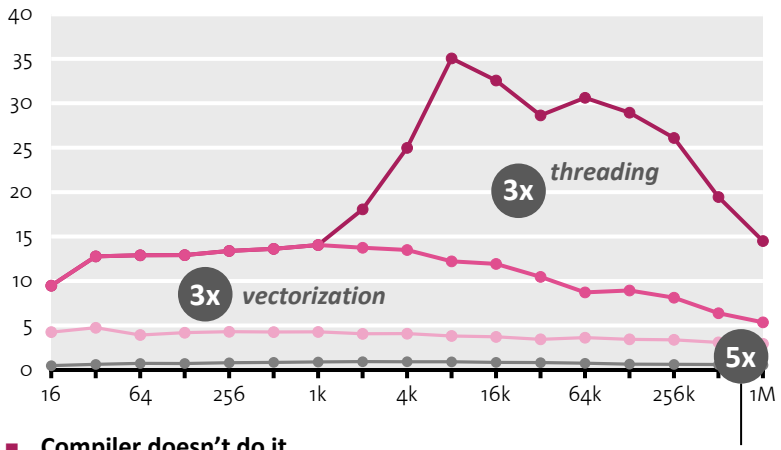
The Problem: Example DFT



- **Same number of operations**
- **Best compiler**

DFT: Analysis

DFT (single precision) on Intel Core i7 (4 cores, 2.66 GHz)
Performance [Gflop/s]



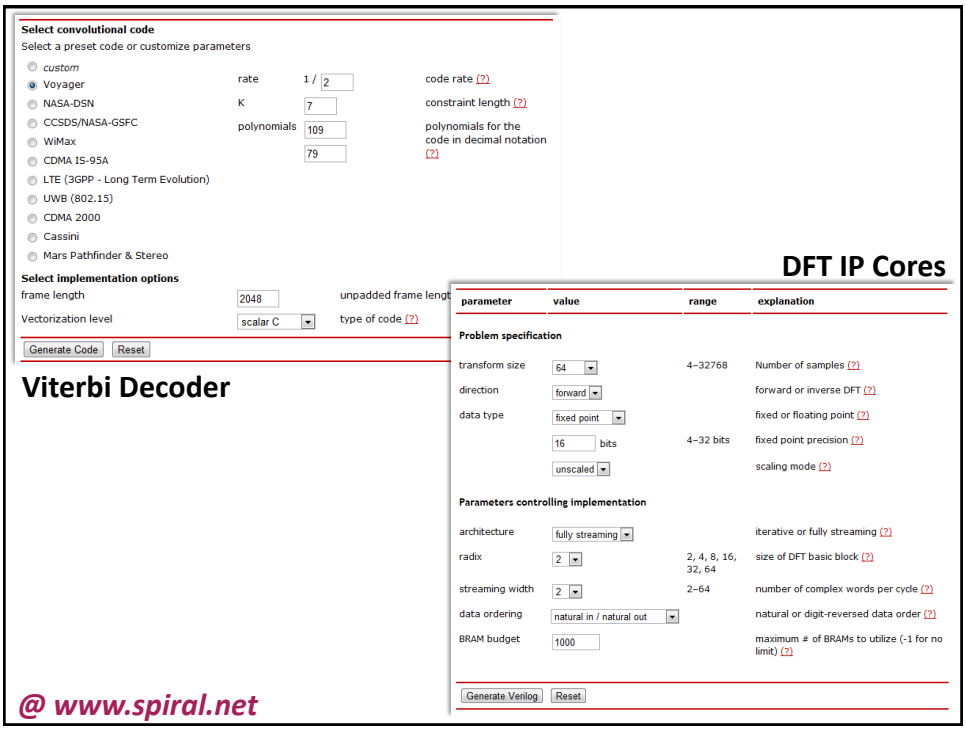
- **Compiler doesn't do it**
- **Doing by hand: Very tough**

Our Goal:

Computer writes high performance library code

Generate Code

"click"



Possible Approach:

Capturing algorithm knowledge:
Domain-specific languages (DSLs)

Structural optimization:
Rewriting systems

High performance code style:
Compiler

Decision making for choices:
Machine learning

Organization

- *Spiral: Basic system*
- Vectorization
- General input size
- Results
- Final remarks

Algorithms: Example FFT, n = 4

Fast Fourier transform (FFT)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} x = \begin{bmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & i \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} x$$

Representation using matrix algebra

$$\mathrm{DFT}_4 = (\mathrm{DFT}_2 \otimes \mathrm{I}_2)\, \mathrm{T}^4_2 (\mathrm{I}_2 \otimes \mathrm{DFT}_2)\, \mathrm{L}^4_2$$

- *SPL (Signal processing language):* **Mathematical, declarative, point-free**
- **Divide-and-conquer algorithms = breakdown rules in SPL**

Decomposition Rules (>200 for >40 Transforms)

$$\mathbf{DFT}_n \to P_{k/2,2m}^{\top}\left(\mathbf{DFT}_{2m} \oplus \left(I_{k/2-1} \quad {}_i C_{2m}\, \mathbf{rDFT}_{2m}(i/k)\right)\right)\left(\mathbf{RDFT}'_k \quad I_m\right), \quad k \text{ even},$$

$$\begin{vmatrix}\mathbf{RDFT}_n\\ \mathbf{RDFT}'_n\\ \mathbf{DHT}_n\\ \mathbf{DHT}'_n\end{vmatrix} \to (P_{k/2,m}^{\top} \quad I_2)\left(\begin{vmatrix}\mathbf{RDFT}_{2m}\\ \mathbf{RDFT}'_{2m}\\ \mathbf{DHT}_{2m}\\ \mathbf{DHT}'_{2m}\end{vmatrix} \oplus \left(I_{k/2-1} \quad {}_i D_{2m} \begin{vmatrix}\mathbf{rDFT}_{2m}(i/k)\\ \mathbf{rDFT}_{2m}(i/k)\\ \mathbf{rDHT}_{2m}(i/k)\\ \mathbf{rDHT}_{2m}(i/k)\end{vmatrix}\right)\right)\begin{pmatrix}\mathbf{RDFT}'_k\\ \mathbf{RDFT}'_k\\ \mathbf{DHT}'_k\\ \mathbf{DHT}'_k\end{pmatrix} \quad I_m\Big), \quad k \text{ even},$$

$$\begin{vmatrix}\mathbf{rDFT}_{2n}(u)\\ \mathbf{rDHT}_{2n}(u)\end{vmatrix} \to L_m^{2n}\left(I_k \quad {}_i \begin{vmatrix}\mathbf{rDFT}_{2m}((i+u)/k)\\ \mathbf{rDHT}_{2m}((i+u)/k)\end{vmatrix}\right)\left(\begin{vmatrix}\mathbf{rDFT}_{2k}(u)\\ \mathbf{rDHT}_{2k}(u)\end{vmatrix} \quad I_m\right),$$

$$\mathbf{RDFT\text{-}3}_n \to (Q_{k/2,m}^{\top} \quad I_2)\,(I_k \quad {}_i\mathbf{rDFT}_{2m})(i+1/2)/k))\,(\mathbf{RDFT\text{-}3}_k \quad I_m), \quad k \text{ even},$$

$$\mathbf{DCT\text{-}2}_n \to P_{k/2,2m}^{\top}\left(\mathbf{DCT\text{-}2}_{2m}\, K_2^{2m} \oplus \left(I_{k/2-1} \quad N_{2m}\, \mathbf{RDFT\text{-}3}_{2m}^{\top}\right)\right) B_n(L_{k/2}^{n/2} \quad I_2)(I_m \quad \mathbf{RDFT}'_k)Q_{m/2,k},$$

$$\mathbf{DCT\text{-}3}_n \to \mathbf{DCT\text{-}2}_n^{\top},$$

$$\mathbf{DFT}_n \to (\mathbf{DFT}_k \otimes I_m)\, T_m^{km}(\mathbf{DFT}_m \otimes I_k)\, L_k^{km}, \quad n = km$$

$$\mathbf{DFT}_n \to P_n(\mathbf{DFT}_m \otimes \mathbf{DFT}_{m-1})P_n^{\top}, \quad n = pm-1, \ \gcd(k,m) = 1$$

$$\mathbf{DCT\text{-}3}_n \to (I_m \oplus J_m)\, L_m^n(\mathbf{DCT\text{-}3}_m(1/4) \oplus \mathbf{DCT\text{-}3}_m(3/4))$$

$$\cdot(\mathsf{F}_2 \otimes I_m)\begin{bmatrix} I_m & 0 \\ 0 & J_{m-1} \end{bmatrix}\begin{bmatrix} I_m & 0 \\ 0 & \frac{1}{\sqrt{2}}(I_1 \oplus 2I_m) \end{bmatrix}, \quad n = 2m$$

$$\mathbf{DCT\text{-}4}_n \to S_n \mathbf{DCT\text{-}2}_n \,\mathrm{diag}_{0 \le k < n}(1/(2\cos((2k+1)\pi/4n)))$$

$$\mathbf{IMDCT}_{2m} \to (J_m \oplus I_m \oplus I_m \oplus J_m)\left(\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes I_m\right) \oplus \left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} \otimes I_m\right)\right) J_{2m}\, \mathbf{DCT\text{-}4}_{2m}$$

$$\mathbf{WHT}_{2^k} \to \prod_{i=1}^{t} (I_{2^{k_1+\cdots+k_{i-1}}} \otimes \mathbf{WHT}_{2^{k_i}} \otimes I_{2^{k_{i+1}+\cdots+k_t}}), \quad k = k_1 + \cdots + k_t$$

$$\mathbf{DFT}_2 \to \mathsf{F}_2$$

$$\mathbf{DCT\text{-}2}_2 \to \mathrm{diag}(1, 1/\sqrt{2})\,\mathsf{F}_2$$

$$\mathbf{DCT\text{-}4}_2 \to J_2 R_{13\pi/8}$$

Decomposition rules = Algorithm knowledge in Spiral (from ≈100 publications)

Combining these rules yields many algorithms for every given transform

SPL to Code

SPL S	Pseudo code for $y = Sx$
$A_n B_n$	`<code for: t = Bx>` `<code for: y = At>`

$I_m \otimes A_n$	`for (i=0; i<m; i++)`
	`  <code for:`
	`  y[i*n:1:i*n+n-1] = A(x[i*n:1:i*n+n-1])>`

$A_m \otimes I_n$	`for (i=0; i<n; i++)`
	`  <code for:`
	`  y[i:n:i+m*n-n] = A(x[i:n:i+m*n-n])>`

D_n	`for (i=0; i<n; i++)` `  y[i] = D[i]*x[i];`

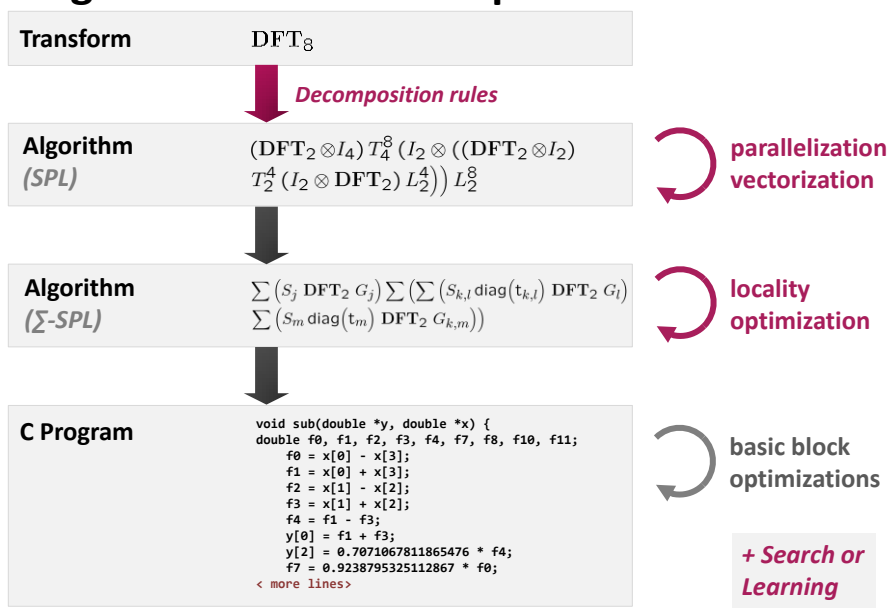
L_k^{km}	`for (i=0; i<k; i++)`
	`  for (j=0; j<m; j++)`
	`  y[i*m+j] = x[j*k+i];`

F_2	`y[0] = x[0] + x[1];` `y[1] = x[0] - x[1];`

$$\mathrm{I}_m \otimes A_n = \begin{bmatrix} A_n & & \\ & \ddots & \\ & & A_n \end{bmatrix}$$

Correct code: easy ***fast code: very difficult***

Program Generation in Spiral




```
void sub(double *y, double *x) {
double f0, f1, f2, f3, f4, f7, f8, f10, f11;
    f0 = x[0] - x[3];
    f1 = x[0] + x[3];
    f2 = x[1] - x[2];
    f3 = x[1] + x[2];
    f4 = f1 - f3;
    y[0] = f1 + f3;
    y[2] = 0.7071067811865476 * f4;
    f7 = 0.9238795325112867 * f0;
< more lines>
```

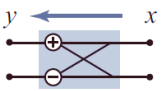
Organization

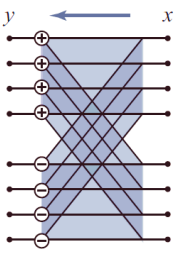
- Spiral: Basic system
- *Vectorization*
- General input size
- Results
- Final remarks

Example: Vectorization in Spiral

- **Relationship SPL expressions ↔ vectorization?**

$$y = \mathrm{DFT}_2\, x \qquad y = \begin{pmatrix} \mathrm{DFT}_2 & \mathrm{I}_4 \end{pmatrix} x$$





one addition
one subtraction

one (4-way) vector addition
one (4-way) vector subtraction

Step 1: Identify "Good" Vector Constructs

- **Vector length:** ν
- **Good (= easily vectorizable) SPL constructs:**

  $A \quad \mathrm{I}_\nu$

  $\mathrm{L}^{\nu^2}_{\nu}, \ \mathrm{L}^{2\nu}_{2}, \mathrm{L}^{2\nu}_{\nu}$ *base cases*

  SPL expressions recursively built from those

- ***Idea:*** **Convert a given SPL expression into a "good" SPL expression through rewriting (structural manipulation)**

Step 2: Find Manipulation Rules

$$
\begin{aligned}
L^{n\nu}_{n} &\rightarrow \left(\mathrm{I}_{n/\nu} \quad L^{\nu^2}_{\nu}\right)\left(L^{n}_{n/\nu} \quad \mathrm{I}_{\nu}\right)\\
L^{n\nu}_{\nu} &\rightarrow \left(L^{n}_{\nu} \quad \mathrm{I}_{\nu}\right)\left(\mathrm{I}_{n/\nu} \quad L^{\nu^2}_{\nu}\right)\\
L^{mn}_{m} &\rightarrow \left(L^{mn/\nu}_{m} \quad \mathrm{I}_{\nu}\right)\left(\mathrm{I}_{mn/\nu^2} \quad L^{\nu^2}_{\nu}\right)\left(\mathrm{I}_{n/\nu} \quad L^{m}_{m/\nu} \quad \mathrm{I}_{\nu}\right)\\
\mathrm{I}_l \quad L^{kmn}_{n} \quad \mathrm{I}_r &\rightarrow \left(\mathrm{I}_l \quad L^{kn}_{n} \quad \mathrm{I}_{mr}\right)\left(\mathrm{I}_{kl} \quad L^{mn}_{n} \quad \mathrm{I}_r\right)\\
\mathrm{I}_l \quad L^{kmn}_{n} \quad \mathrm{I}_r &\rightarrow \left(\mathrm{I}_l \quad L^{kmn}_{kn} \quad \mathrm{I}_r\right)\left(\mathrm{I}_l \quad L^{kmn}_{mn} \quad \mathrm{I}_r\right)\\
\mathrm{I}_l \quad L^{kmn}_{km} \quad \mathrm{I}_r &\rightarrow \left(\mathrm{I}_{kl} \quad L^{mn}_{m} \quad \mathrm{I}_r\right)\left(\mathrm{I}_l \quad L^{kn}_{k} \quad \mathrm{I}_{mr}\right)\\
\mathrm{I}_l \quad L^{kmn}_{km} \quad \mathrm{I}_r &\rightarrow \left(\mathrm{I}_l \quad L^{kmn}_{k} \quad \mathrm{I}_r\right)\left(\mathrm{I}_{kl} \quad L^{mn}_{m} \quad \mathrm{I}_r\right)\\
\left(\mathrm{I}_m \quad A^{n\times n}\right)L^{mn}_{m} &\rightarrow \left(\mathrm{I}_{m/\nu} \quad L^{n\nu}_{\nu}\left(A^{n\times n} \quad \mathrm{I}_{\nu}\right)\right)\left(L^{mn/\nu}_{m/\nu} \quad \mathrm{I}_{\nu}\right)\\
L^{mn}_{n}\left(\mathrm{I}_m \quad A^{n\times n}\right) &\rightarrow \left(L^{mn/\nu}_{n} \quad \mathrm{I}_{\nu}\right)\left(\mathrm{I}_{m/\nu} \quad \left(A^{n\times n} \quad \mathrm{I}_{\nu}\right)L^{n\nu}_{n}\right)\\
\left(\mathrm{I}_k \quad \left(\mathrm{I}_m \quad A^{n\times n}\right)L^{mn}_{m}\right)L^{kmn}_{k} &\rightarrow \left(L^{km}_{k} \quad \mathrm{I}_n\right)\left(\mathrm{I}_m \quad \left(\mathrm{I}_k \quad A^{n\times n}\right)L^{kn}_{k}\right)\left(L^{mn}_{m} \quad \mathrm{I}_k\right)\\
L^{kmn}_{mn}\left(\mathrm{I}_k \quad L^{mn}_{n}\left(\mathrm{I}_m \quad A^{n\times n}\right)\right) &\rightarrow \left(L^{mn}_{n} \quad \mathrm{I}_k\right)\left(\mathrm{I}_m \quad L^{kn}_{n}\left(\mathrm{I}_k \quad A^{n\times n}\right)\right)\left(L^{km}_{m} \quad \mathrm{I}_n\right)\\
\overline{AB} &\rightarrow \overline{A}\,\overline{B}\\
\overline{A^{m\times m} \quad \mathrm{I}_\nu} &\rightarrow \left(\mathrm{I}_m \quad L^{2\nu}_{\nu}\right)\left(\overline{A^{m\times m}} \quad \mathrm{I}_\nu\right)\left(\mathrm{I}_m \quad L^{2\nu}_{2}\right)\\
\overline{\mathrm{I}_m \quad A^{n\times n}} &\rightarrow \mathrm{I}_m \quad \overline{A^{n\times n}}\\
\overline{D} &\rightarrow \left(\mathrm{I}_{n/\nu} \quad L^{2\nu}_{\nu}\right)\vec{D}\left(\mathrm{I}_{n/\nu} \quad L^{2\nu}_{2}\right)\\
\overline{P} &\rightarrow P \quad \mathrm{I}_2
\end{aligned}
$$

Manipulation rules = Processor knowledge in Spiral

Example

$$
\underbrace{\mathbf{DFT}_{mn}}_{\mathrm{vec}(\nu)} \rightarrow \underbrace{(\mathbf{DFT}_m \quad \mathrm{I}_n)\,T^{mn}_{n}\,(\mathrm{I}_m \quad \mathbf{DFT}_n)\,L^{mn}_{m}}_{\mathrm{vec}(\nu)}
$$

...

...

...

$$
\rightarrow \underline{\left(\mathrm{I}_{\frac{mn}{\nu}} \quad L^{2\nu}_{\nu}\right)}\,\underline{\left(\overline{\mathbf{DFT}_m} \quad \mathrm{I}_{\frac{n}{\nu}} \quad \mathrm{I}_\nu\right)}\,\overline{T}^{mn}_{n}
$$

$$
\underline{\left(\mathrm{I}_{\frac{m}{\nu}} \quad \left(L^{2n}_{\nu} \quad \mathrm{I}_\nu\right)\left(\mathrm{I}_{\frac{2n}{\nu}} \quad L^{\nu^2}_{\nu}\right)\left(\mathrm{I}_{\frac{n}{\nu}} \quad L^{2\nu}_{2} \quad \mathrm{I}_\nu\right)\right)}\,\underline{\left(\overline{\mathbf{DFT}_n} \quad \mathrm{I}_\nu\right)}\,\underline{\left(L^{\frac{mn}{\nu}}_{\frac{m}{\nu}} \quad L^{2\nu}_{2}\right)}
$$

vectorized arithmetic

vectorized data accesses

Automatically Generate Base Case Library

- ***Goal:*** **Given instruction set, generate base cases**

  $\nu = 4:\quad \left\{ \mathsf{L}^4_2, \mathrm{I}_2 \quad \mathsf{L}^4_2, \mathsf{L}^4_2 \quad \mathrm{I}_2, \mathsf{L}^8_2, \mathsf{L}^8_4 \right\}$

- ***Idea:*** **Instructions as matrices + search**

```
y = _mm_unpacklo_ps(x0, x1);
y = _mm_shuffle_ps(x0, x1, _MM_SHUFFLE(1,2,1,2));
y = _mm_shuffle_ps(x0, x1, _MM_SHUFFLE(3,4,3,4));
```

$$y = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0 \end{bmatrix} \begin{bmatrix} \vec{x}_0 \\ \vec{x}_1 \end{bmatrix}$$

$$y = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{bmatrix} \begin{bmatrix} \vec{x}_0 \\ \vec{x}_1 \end{bmatrix}$$

$$y = \begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} \vec{x}_0 \\ \vec{x}_1 \end{bmatrix}$$

$$\left[\begin{array}{cccccccc} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&1&0&0\\ 0&1&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0\\ \hline 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{array}\right]$$

```
y0 = _mm_shuffle_ps(x0, x1,
                    _MM_SHUFFLE(1,2,1,2));
y1 = _mm_shuffle_ps(x0, x1,
                    _MM_SHUFFLE(3,4,3,4));
```

$\mathsf{L}^4_2 \otimes \mathrm{I}_2$

no base case

Base case

Same Approach for Different Paradigms

Threading:

$$\underbrace{\mathrm{DFT}_{mn}}_{\mathrm{smp}(p,\mu)} \rightarrow \underbrace{(\mathrm{DFT}_m \otimes \mathrm{I}_n)\,\mathrm{T}^{mn}_n\,(\mathrm{I}_m \otimes \mathrm{DFT}_n)\,\mathsf{L}^{mn}_m}_{\mathrm{smp}(p,\mu)}$$

$$\rightarrow \underbrace{(\mathrm{DFT}_m \otimes \mathrm{I}_n)}_{\mathrm{smp}(p,\mu)}\ \underbrace{\mathrm{T}^{mn}_n}_{\mathrm{smp}(p,\mu)}\ \underbrace{(\mathrm{I}_m \otimes \mathrm{DFT}_n)}_{\mathrm{smp}(p,\mu)}\ \underbrace{\mathsf{L}^{nn}_m}_{\mathrm{smp}(p,\mu)}$$

$$\rightarrow (\mathsf{L}^{mp}_m \otimes \mathrm{I}_{n/p\mu}) \otimes_\mu \mathrm{I}_\mu \Big)\Big(\mathrm{I}_p \otimes_\parallel (\mathrm{DFT}_m \otimes \mathrm{I}_{n/p})\Big)\Big((\mathsf{L}^{mp}_p \otimes \mathrm{I}_{n/p\mu}) \otimes_\mu \mathrm{I}_\mu\Big)$$
$$\Big(\bigoplus_{i=0}^{p-1}{}_\parallel\, \mathrm{T}^{mn,i}_n\Big)\Big(\mathrm{I}_p \otimes_\parallel (\mathrm{I}_{m/p} \otimes \mathrm{DFT}_n)\Big)\Big(\mathrm{I}_p \otimes_\parallel \mathsf{L}^{mn/p}_{m/p}\Big)\Big((\mathsf{L}^{m}_p \otimes \mathrm{I}_{mn/p\mu}) \otimes_\mu \mathrm{I}_\mu\Big)$$

Vectorization:

$$\underbrace{\overline{(\mathrm{DFT}_{mn})}}_{\mathrm{vec}(\nu)} \rightarrow \underbrace{\overline{(\mathrm{DFT}_m \otimes \mathrm{I}_n)\,\mathrm{T}^{mn}_n\,(\mathrm{I}_m \otimes \mathrm{DFT}_n)\,\mathsf{L}^{mn}_m}}_{\mathrm{vec}(\nu)}$$

$$\rightarrow \underbrace{\overline{(\mathrm{DFT}_m \otimes \mathrm{I}_n)}}_{\mathrm{vec}(\nu)}\ \underbrace{\overline{(\mathrm{T}^{mn}_n)}}_{\mathrm{vec}(\nu)}\ \underbrace{\overline{(\mathrm{I}_m \otimes \mathrm{DFT}_n)\,\mathsf{L}^{mn}_m}}_{\mathrm{vec}(\nu)}$$

$$\rightarrow (\mathrm{I}_{mn/\nu} \otimes \underbrace{\mathsf{L}^{2\nu}_\nu}_{\mathrm{sse}})\,\overline{(\mathrm{DFT}_m \otimes \mathrm{I}_{n/\nu})}\,\underbrace{\overline{(\mathrm{T}^{mn}_n)}}_{\mathrm{sse}}$$
$$\Big(\mathrm{I}_{m/\nu} \otimes (\overline{\mathsf{L}^n_\nu} \bar{\otimes} \mathrm{I}_\nu)(\mathrm{I}_{n/\nu} \otimes (\mathsf{L}^{2\nu}_\nu \bar{\otimes} \mathrm{I}_\nu)(\mathrm{I}_2 \otimes \underbrace{\mathsf{L}^{\nu^2}_\nu}_{\mathrm{sse}})(\mathsf{L}^{2\nu}_2 \bar{\otimes} \mathrm{I}_\nu))\,\overline{(\mathrm{DFT}_n \otimes \mathrm{I}_\nu)}\Big)$$
$$\big((\mathsf{L}^{mn}_m \otimes \mathrm{I}_2) \bar{\otimes} \mathrm{I}_\nu\big)(\mathrm{I}_{mn/\nu} \otimes \underbrace{\mathsf{L}^{2\nu}_2}_{\mathrm{sse}})$$

GPUs:

$$\underbrace{(\mathrm{DFT}_{p^k})}_{\mathrm{gpu}(t,c)} \rightarrow \underbrace{\left(\prod_{i=0}^{k-1} \mathsf{L}^{p^k}_r \left(\mathrm{I}_{p^{k-1}} \otimes \mathrm{DFT}_r\right)\left(\mathsf{L}^{p^k}_{p^{k-1}} (\mathrm{I}_{p^i} \otimes \mathrm{T}^{p^{k-i}}_{p^{k-1}})\,\underbrace{\mathsf{L}^{p^k}_{p^{i+1}}}_{\mathrm{vec}(c)}\right)\right) \mathsf{R}^{p^k}_r}_{\mathrm{gpu}(t,c)}$$

$$\rightarrow \left(\prod_{i=0}^{k-1} (\mathsf{L}^{r^n/2} \bar{\otimes} \mathrm{I}_2)\left(\mathrm{I}_{r^{n-1/2}} \otimes_\times \underbrace{(\mathrm{DFT}_r \bar{\otimes} \mathrm{I}_2)\,\mathsf{L}^{2r}_r}_{\mathrm{shd}(t,c)}\right) \mathrm{T}_i\right)$$
$$(\mathsf{L}^{r^n/2}_r \bar{\otimes} \mathrm{I}_2)(\mathrm{I}_{r^{n-1/2}} \otimes_\times \underbrace{\mathsf{L}^{2r}_2}_{\mathrm{shd}(t,c)})(\mathsf{R}^{r^{n-1}}_r \bar{\otimes} \mathrm{I}_r)$$

Verilog for FPGAs:

$$\underbrace{(\mathrm{DFT}_{r^k})}_{\mathrm{stream}(r^s)} \rightarrow \underbrace{\left[\prod_{i=0}^{k-1} \mathsf{L}^{r^k}_{r^{k-i}} \left(\mathrm{I}_{r^{k-1}} \otimes \mathrm{DFT}_r\right)\left(\mathsf{L}^{r^k}_{r^{k-i-1}} (\mathrm{I}_{r^i} \otimes \mathrm{T}^{r^{k-i}}_{r^{k-i-1}})\,\mathsf{L}^{r^k}_{r^{i+1}}\right)\right] \mathsf{R}^{r^k}_r}_{\mathrm{stream}(r^s)}$$

$$\rightarrow \left[\prod_{i=0}^{k-1} \underbrace{\mathsf{L}^{r^k}_r}_{\mathrm{stream}(r^s)} \left(\mathrm{I}_{r^{k-1}} \otimes \mathrm{DFT}_r\right) \underbrace{\left(\mathsf{L}^{r^k}_{r^{k-i-1}} (\mathrm{I}_{r^i} \otimes \mathrm{T}^{r^{k-i}}_{r^{k-i-1}})\,\mathsf{L}^{r^k}_{r^{i+1}}\right)}_{\mathrm{stream}(r^s)}\right] \underbrace{\mathsf{R}^{r^k}_r}_{\mathrm{stream}(r^s)}$$

$$\rightarrow \left[\prod_{i=0}^{k-1} \underbrace{\mathsf{L}^{r^k}_r}_{\mathrm{stream}(r^s)} \left(\mathrm{I}_{r^{k-i-1}} \otimes_s (\mathrm{I}_{r^{i-1}} \otimes \mathrm{DFT}_r)\right) \underbrace{\mathsf{T}^{r^k}_r}_{\mathrm{stream}(r^s)}\right] \underbrace{\mathsf{R}^{r^k}_r}_{\mathrm{stream}(r^s)}$$

- Rigorous, correct by construction
- Overcomes compiler limitations

Organization

- Spiral: Basic system
- Vectorization
- *General input size*
- Results
- Final remarks

Challenge: General Size Libraries

So far:
Code specialized to fixed input size

```
DFT_384(x, y) {
  …
  for(i = …) {
    t[2i]   = x[2i] + x[2i+1]
    t[2i+1] = x[2i] - x[2i+1]
  }
  …
}
```

- **Algorithm fixed**
- **Nonrecursive code**

Challenge:
Library for general input size

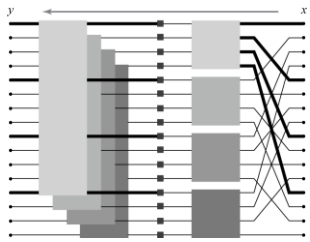
```
DFT(n, x, y) {
  …
  for(i = …) {
    DFT_strided(m, x+mi, y+i, 1, k)
  }
  …
}
```

- **Algorithm cannot be fixed**
- **Recursive code**
- **Creates many challenges**

Challenge: Recursion Steps

- **Cooley-Tukey FFT**

$$y = (\mathbf{DFT}_k \otimes I_m) T_m^{km} (I_k \otimes \mathbf{DFT}_m) L_k^{km} x$$



- **Implementation that increases locality (e.g., FFTW 2.x)**

```
void DFT(int n, cpx *y, cpx *x) {
  int k = choose_dft_radix(n);

  for (int i=0; i < k; ++i)
    DFTrec(m, y + m*i, x + i, k, 1);
  for (int j=0; j < m; ++j)
    DFTscaled(k, y + j, t[j], m);
}
```

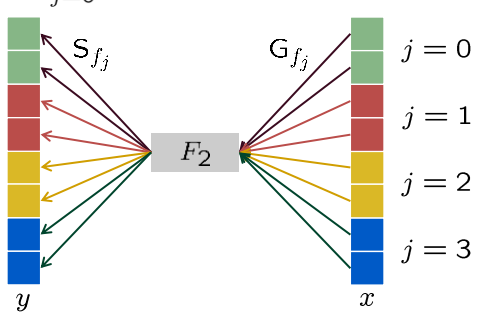
Σ–SPL : Basic Idea

- **Four additional matrix constructs: Σ, G, S, Perm**
  - **Σ (sum)** — explicit loop
  - **G_f (gather)** — load data with index mapping f
  - **S_f (scatter)** — store data with index mapping f
  - **Perm_f** — permute data with the index mapping f

- **Σ-SPL formulas = matrix factorizations**

Example: $y = (I_4 \otimes F_2)x \;\rightarrow\; y = \sum_{j=0}^{3} \mathsf{S}_{f_j} F_2 \mathsf{G}_{f_j} x$

$$y = \begin{bmatrix} F_2 & & & \\ & F_2 & & \\ & & F_2 & \\ & & & F_2 \end{bmatrix} x$$



Find Recursion Step Closure

Voronenko, 2008

$$\{\mathbf{DFT}_n\}$$

$$(\{\mathbf{DFT}_{n/k}\} \otimes I_k) T^n_k (I_{n/k} \otimes \{\mathbf{DFT}_k\}) L^n_{n/k}$$

$$\left(\sum_{i=0}^{k-1} \mathsf{S}_{h_{i,k}} \{\mathbf{DFT}_{n/k}\} \mathsf{G}_{h_{i,k}}\right) \mathsf{diag}(f) \left(\sum_{j=0}^{n/k-1} \mathsf{S}_{h_{jk,1}} \{\mathbf{DFT}_k\} \mathsf{G}_{h_{jk,1}}\right) \mathsf{perm}(\ell^n_{n/k})$$

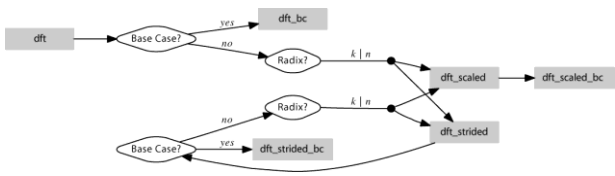
$$\sum_{i=0}^{k-1} \mathsf{S}_{h_{i,k}} \{\mathbf{DFT}_{n/k}\} \mathsf{diag}(f \circ h_{i,k}) \mathsf{G}_{h_{i,k}} \sum_{j=0}^{n/k-1} \mathsf{S}_{h_{jk,1}} \{\mathbf{DFT}_k\} \mathsf{G}_{h_{j,n/k}}$$

$$\sum_{i=0}^{k-1} \left\{ \mathsf{S}_{h_{i,k}} \mathbf{DFT}_{n/k} \mathsf{diag}(f \circ h_{i,k}) \mathsf{G}_{h_{i,k}} \right\} \sum_{j=0}^{n/k-1} \left\{ \mathsf{S}_{h_{jk,1}} \mathbf{DFT}_k \mathsf{G}_{h_{j,n/k}} \right\}$$

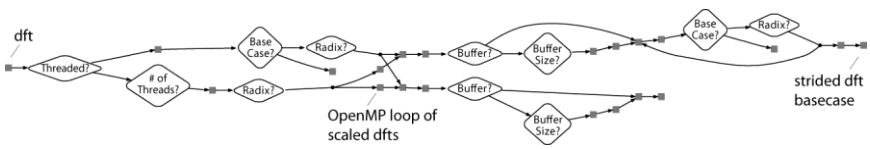
Repeat until closure

Recursion Step Closure: Examples

DFT: scalar code



DFT: full-fledged (vectorized and parallel code)



Summary: Complete Automation for Transforms

- **Memory hierarchy optimization**
  Rewriting and search for algorithm selection
  Rewriting for loop optimizations
- **Vectorization**
  Rewriting
- **Parallelization**
  Rewriting

fixed input size code

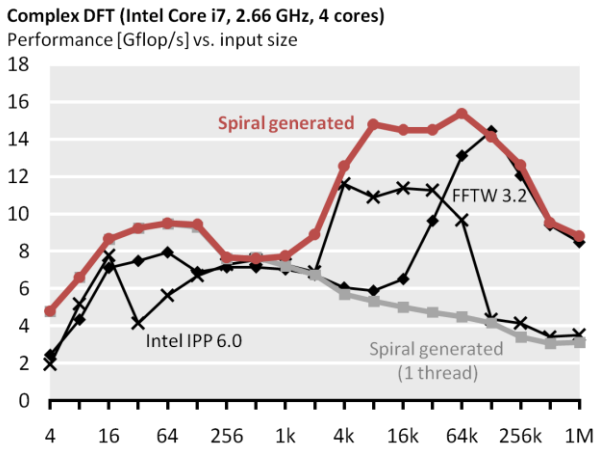
- **Derivation of library structure**
  Rewriting
  Other methods

general input size library

Organization

- Spiral: Basic system
- Vectorization
- General input size
- ***Results***
- Final remarks

DFT on Intel Multicore



$$\mathbf{DFT}_n \to (\mathbf{DFT}_k \otimes I_m) T^{m}_{m} (I_k \otimes \mathbf{DFT}_m) L^{n}_{k}$$
$$\mathbf{DFT}_n \to P^{\top}_{k/2,2m} \left(\mathbf{DFT}_{2m} \oplus \left(I_{k/2-1} \otimes_i C_{2m}\, \mathbf{rDFT}_{2m}(i/k)\right)\right) (\mathbf{RDFT}_k \otimes I_m)$$
$$\mathbf{RDFT}_n \to (P^{\top}_{k/2,m} \otimes I_2) \left(\mathbf{RDFT}_{2m} \oplus \left(I_{k/2-1} \otimes_i D_{2m}\, \mathbf{rDFT}_{2m}(i/k)\right)\right) (\mathbf{RDFT}_k \otimes I_m)$$
$$\mathbf{rDFT}_{2n}(u) \to L^{2n}_{m} (I_k \otimes_i \mathbf{rDFT}_{2m}((i+u)/k)) (\mathbf{rDFT}_{2k}(u) \otimes I_m)$$

Spiral

5MB vectorized, threaded, general-size, adaptive library

Generating 100s of FFTWs

PhD thesis Voronenko, 2009

$$\mathbf{DFT}_n \to P^{\top}_{k/2,2m} \left(\mathbf{DFT}_{2m} \oplus \left(I_{k/2-1} \;\; {}_i C_{2m}\, \mathbf{rDFT}_{2m}(i/k)\right)\right) \left(\mathbf{RDFT}'_k \;\; I_m\right), \quad k \text{ even},$$

$$\begin{vmatrix} \mathbf{RDFT}_n \\ \mathbf{RDFT}'_n \\ \mathbf{DHT}_n \\ \mathbf{DHT}'_n \end{vmatrix} \to (P^{\top}_{k/2,m} \;\; I_2) \left(\begin{vmatrix} \mathbf{RDFT}_{2m} \\ \mathbf{RDFT}'_{2m} \\ \mathbf{DHT}_{2m} \\ \mathbf{DHT}'_{2m} \end{vmatrix} \oplus \left(I_{k/2-1} \;\; {}_i D_{2m} \begin{vmatrix} \mathbf{rDFT}_{2m}(i/k) \\ \mathbf{rDFT}_{2m}(i/k) \\ \mathbf{rDHT}_{2m}(i/k) \\ \mathbf{rDHT}_{2m}(i/k) \end{vmatrix} \right) \right) \begin{pmatrix} \mathbf{RDFT}'_k \\ \mathbf{RDFT}'_k \\ \mathbf{DHT}'_k \\ \mathbf{DHT}'_k \end{vmatrix} \;\; I_m \end{pmatrix}, \quad k \text{ even},$$

$$\begin{vmatrix} \mathbf{rDFT}_{2n}(u) \\ \mathbf{rDHT}_{2n}(u) \end{vmatrix} \to L^{2n}_{m} \left(I_k \;\; {}_i \begin{vmatrix} \mathbf{rDFT}_{2m}((i+u)/k) \\ \mathbf{rDHT}_{2m}((i+u)/k) \end{vmatrix} \right) \left(\begin{vmatrix} \mathbf{rDFT}_{2k}(u) \\ \mathbf{rDHT}_{2k}(u) \end{vmatrix} \;\; I_m \right),$$

$$\mathbf{RDFT\text{-}3}_n \to (Q^{\top}_{k/2,m} \;\; I_2) \left(I_k \;\; {}_i \mathbf{rDFT}_{2m})(i+1/2)/k)\right) (\mathbf{RDFT\text{-}3}_k \;\; I_m), \quad k \text{ even},$$

$$\mathbf{DCT\text{-}2}_n \to P^{\top}_{k/2,2m} \left(\mathbf{DCT\text{-}2}_{2m}\, K^{2m}_{2} \oplus \left(I_{k/2-1} \;\; N_{2m}\, \mathbf{RDFT\text{-}3}^{\top}_{2m}\right)\right) B_n (L^{n/2}_{k/2} \;\; I_2)(I_m \;\; \mathbf{RDFT}'_k) Q_{m/2,k},$$

$$\mathbf{DCT\text{-}3}_n \to \mathbf{DCT\text{-}2}^{\top}_{n},$$

$$\mathbf{DCT\text{-}4}_n \to Q^{\top}_{k/2,2m} \left(I_{k/2} \;\; N_{2m}\, \mathbf{RDFT\text{-}3}^{\top}_{2m}\right) B'_n (L^{n/2}_{k/2} \;\; I_2)(I_m \;\; \mathbf{RDFT\text{-}3}_k) Q_{m/2,k}.$$

$$\begin{aligned}
\mathbf{DFT}_n &\to (\mathbf{DFT}_k \;\; \mathsf{I}_m)\, \mathsf{T}^n_m (\mathsf{I}_k \;\; \mathbf{DFT}_m)\, \mathsf{L}^n_k, \quad n = km \\
\mathbf{DFT}_n &\to P_n (\mathbf{DFT}_k \;\; \mathbf{DFT}_m) Q_n, \quad n = km,\ \gcd(k,m) = 1 \\
\mathbf{DFT}_p &\to R^T_p (\mathsf{I}_1 \oplus \mathbf{DFT}_{p-1}) D_p (\mathsf{I}_1 \oplus \mathbf{DFT}_{p-1}) R_p, \quad p \text{ prime} \\
\mathbf{DCT\text{-}3}_n &\to (\mathsf{I}_m \oplus \mathsf{J}_m)\, \mathsf{L}^n_m (\mathbf{DCT\text{-}3}_m(1/4) \oplus \mathbf{DCT\text{-}3}_m(3/4)) \\
&\quad \cdot (\mathsf{F}_2 \;\; \mathsf{I}_m) \begin{bmatrix} \mathsf{I}_m & 0 \oplus -\mathsf{J}_{m-1} \\ & \frac{1}{\sqrt{2}}(\mathsf{I}_1 \oplus 2\,\mathsf{I}_m) \end{bmatrix}, \quad n = 2m \\
\mathbf{DCT\text{-}4}_n &\to S_n \mathbf{DCT\text{-}2}_n\, \mathrm{diag}_{0 \le k < n}(1/(2\cos((2k+1)\pi/4n))) \\
\mathbf{IMDCT}_{2m} &\to (\mathsf{J}_m \oplus \mathsf{I}_m \oplus \mathsf{I}_m \oplus \mathsf{J}_m) \left(\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \;\; \mathsf{I}_m \right) \oplus \left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} \;\; \mathsf{I}_m \right) \right) \mathsf{J}_{2m}\, \mathbf{DCT\text{-}4}_{2m} \\
\mathbf{WHT}_{2^k} &\to \prod_{i=1}^{t} (\mathsf{I}_{2^{k_1+\cdots+k_{i-1}}} \;\; \mathbf{WHT}_{2^{k_i}} \;\; \mathsf{I}_{2^{k_{i+1}+\cdots+k_t}}), \quad k = k_1 + \cdots + k_t \\
\mathbf{DFT}_2 &\to \mathsf{F}_2 \\
\mathbf{DCT\text{-}2}_2 &\to \mathrm{diag}(1, 1/\sqrt{2})\, \mathsf{F}_2 \\
\mathbf{DCT\text{-}4}_2 &\to \mathsf{J}_2 \mathsf{R}_{13\pi/8}
\end{aligned}$$

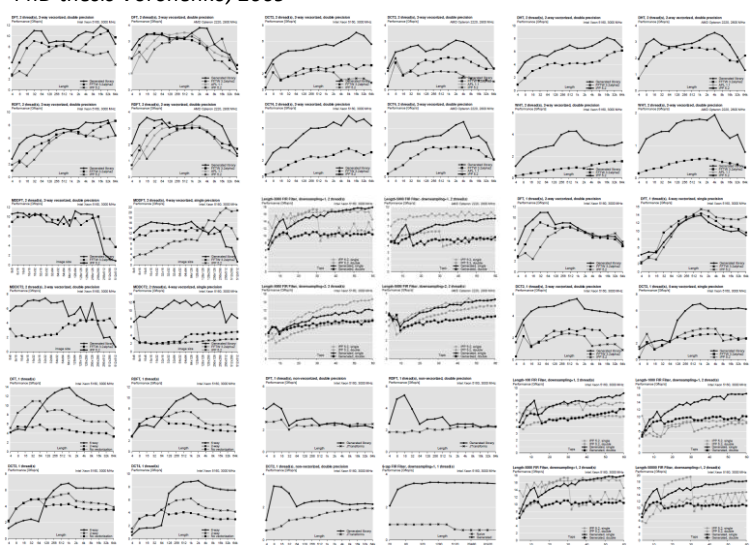
Generating 100s of FFTWs

PhD thesis Voronenko, 2009

Transform	Code size non-parallelized	Code size parallelized
no vectorization		
DFT	13.1 KLOC / 0.59 MB	10.3 KLOC / 0.45 MB
RDFT	8.5 KLOC / 0.36 MB	8.8 KLOC / 0.39 MB
DHT	9.1 KLOC / 0.40 MB	9.4 KLOC / 0.39 MB
DCT-2	12.0 KLOC / 0.55 MB	12.4 KLOC / 0.57 MB
DCT-3	12.0 KLOC / 0.56 MB	12.3 KLOC / 0.59 MB
DCT-4	6.8 KLOC / 0.33 MB	7.1 KLOC / 0.35 MB
WHT	5.6 KLOC / 0.21 MB	—
2-way vectorization		
DFT	14.8 KLOC / 0.73 MB	15.0 KLOC / 0.74 MB
RDFT	15.6 KLOC / 0.76 MB	16.0 KLOC / 0.81 MB
scaled RDFT	16.0 KLOC / 0.78 MB	—
DHT	16.9 KLOC / 0.83 MB	17.2 KLOC / 0.87 MB
DCT-2	20.7 KLOC / 1.10 MB	21.0 KLOC / 1.09 MB
DCT-3	27.9 KLOC / 1.56 MB	28.2 KLOC / 1.59 MB
DCT-4	7.8 KLOC / 0.47 MB	8.1 KLOC / 0.50 MB
WHT	6.9 KLOC / 0.32 MB	5.8 KLOC / 0.26 MB
FIR Filter	167 KLOC / 7.75 MB	120 KLOC / 5.12 MB
Downsampled FIR Filter	100 KLOC / 4.2 MB	68 KLOC / 2.76 MB
4-way vectorization		
DFT	17.9 KLOC / 1.09 MB	18.2 KLOC / 1.11 MB
RDFT	16.2 KLOC / 0.86 MB	16.5 KLOC / 0.91 MB
scaled RDFT	16.5 KLOC / 0.88 MB	—
DHT	17.9 KLOC / 1.02 MB	18.3 KLOC / 1.04 MB
DCT-2	23.3 KLOC / 1.50 MB	23.6 KLOC / 1.53 MB
DCT-3	32.0 KLOC / 2.17 MB	32.3 KLOC / 2.20 MB
DCT-4	8.3 KLOC / 0.63 MB	8.6 KLOC / 0.66 MB
WHT	8.5 KLOC / 0.53 MB	6.9 KLOC / 0.4 MB
2D DFT	20.6 KLOC / 1.32 MB	20.8 KLOC / 1.33 MB
2D DCT-2	27.0 KLOC / 2.1 MB	27.2 KLOC / 2.11 MB
FIR Filter	109 KLOC / 5.69 MB	74 KLOC / 3.44 MB
Downsampled FIR Filter	151 KLOC / 7.7 MB	92 KLOC / 4.61 MB

Generating 100s of FFTWs

PhD thesis Voronenko, 2009



Computer generated Functions for Intel IPP 6.0

3984 C functions

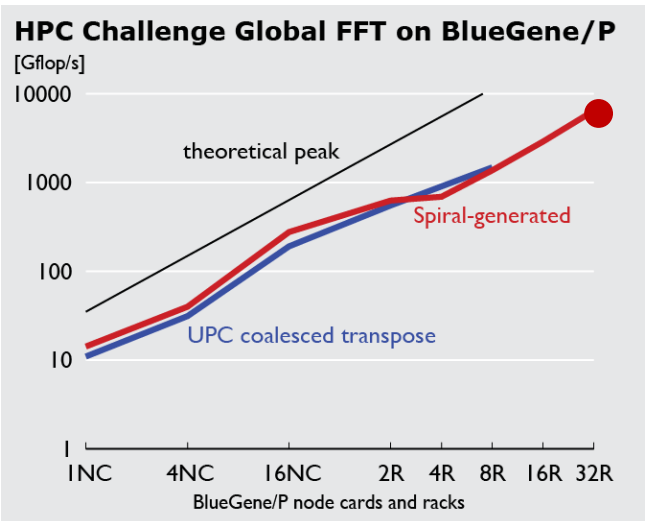
1M lines of code

Transforms: DFT (fwd+inv), RDFT (fwd+inv), DCT2, DCT3, DCT4, DHT, WHT
Sizes: 2–64 (DFT, RDFT, DHT); 2-powers (DCTs, WHT)
Precision: single, double
Data type: scalar, SSE, AVX (DFT, DCT), LRB (DFT)

Computer generated

Results: SpiralGen Inc.

Very Large Scale: BG/P



6.4 Tflop/s

32 racks
= 32K node cards
= 128K cores

2010 HPC Challenge Class I Award, Almasi et al.

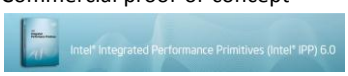
Organization

- Spiral: Basic system
- Vectorization
- General input size
- Results
- ***Final remarks***

Spiral: Summary

- **Spiral:**
  Successful approach to automating
  the development of computing software

  Commercial proof-of-concept



$$\mathbf{DFT}_{64}$$

```
void dft64(float  *Y, float  *X) {
  __m512 U912, U913, U914, U915,...
  __m512  *a2153, *a2155;
  a2153 = ((__m512  *) X);  a1107 = *(a2153);
  a1108 = *((a2153 + 4));   t1323 = _mm512_add_ps(a1107,a1108);
  t1324 = _mm512_sub_ps(a1107,a1108);
    <many more lines>
  U926 = _mm512_swizupconv_r32(,);
  a1121 = _mm512_madd231_ps(_mm512_mul_ps(_mm512_mask_or_pi(
    _mm512_set_1to16_ps(0.707106781186547571),0xAAAA,a2154,U926),t1341),
    _mm512_mask_sub_ps(_mm512_set_1to16_ps(0.707106781186547571),…),
    _mm512_swizupconv_r32(t1341,_MM_SWIZ_REG_CDAB));
  U927 = _mm512_swizupconv_r32
    <many more lines>
}
```

- **Key ideas:**
  Algorithm knowledge:
  Domain specific symbolic representation

  Platform knowledge:
  Tagged rewrite rules, SIMD specification

$$\mathbf{DFT}_4 \rightarrow (\mathbf{DFT}_2 \otimes \mathrm{I}_2) \mathrm{T}^4_2 (\mathrm{I}_2 \otimes \mathbf{DFT}_2) \mathrm{L}^4_2$$

$$\underbrace{\mathrm{I}_m \otimes A_n}_{\mathrm{smp}(p,\mu)} \rightarrow \mathrm{I}_p \otimes_{\parallel} \left(\mathrm{I}_{m/p} \otimes A_n \right)$$