Higher Dimensional Discrete Cheeger Inequalities

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Abstract

For graphs there exists a strong connection between spectral and combinatorial expansion properties. This is expressed, e.g., by the discrete Cheeger inequality, the lower bound of which states that $\lambda(G) \leq h(G)$, where $\lambda(G)$ is the second smallest eigenvalue of the Laplacian of a graph G and $h(G)$ is the Cheeger constant measuring the edge expansion of G. We are interested in generalizations of expansion properties to finite simplicial complexes of higher dimension (or uniform hypergraphs).

Whereas higher dimensional Laplacians were introduced already in 1945 by Eckmann, the generalization of edge expansion to simplicial complexes is not straightforward. Recently, a topologically motivated notion analogous to edge expansion that is based on $\mathbb{Z}_2$-cohomology was introduced by Gromov and independently by Linial, Meshulam and Wallach and by Newman and Rabinovich. It is known that for this generalization there is no higher dimensional analogue of the lower bound of the Cheeger inequality. A different, combinatorially motivated generalization of the Cheeger constant, denoted by $h(X)$, was studied by Parzanchevski, Rosenthal and Tessler. They showed that indeed $\lambda(X) \leq h(X)$, where $\lambda(X)$ is the smallest non-trivial eigenvalue of the ($(k-1)$-dimensional upper) Laplacian, for the case of k-dimensional simplicial complexes X with complete $(k-1)$-skeleton.

Whether this inequality also holds for k-dimensional complexes with non-complete $(k-1)$-skeleton has been an open question. We give two proofs of the inequality for arbitrary complexes. The proofs differ strongly in the methods and structures employed, and each allows for a different kind of additional strengthening of the original result.

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Introduction

Roughly speaking, a graph is an expander if it is sparse and at the same time well-connected. Such graphs have found various applications, in theoretical computer science as well as in pure mathematics. Expander graphs have, e.g., been used to construct certain classes of error correcting codes, in a proof of the PCP Theorem, a deep result in computational complexity theory, and in the theory of metric embeddings. See, e.g., the surveys [10] and [14] for these and other applications.

In recent years, the combinatorial study of simplicial complexes - considering them as a higher-dimensional generalization of graphs - has attracted increasing attention and the profitability of the concept of expansion for graphs has inspired the search for a corresponding higher-dimensional notion, see, e.g., [9, 15, 22, 24]

The expansion of a graph G can be measured by the *Cheeger constant* [1]

$$h(G) := \min_{\substack{A \subseteq V \\ 0<|A|<|V|}} \frac{|V||E(A, V \setminus A)|}{|A||V \setminus A|}.$$

Here $E(A, V \setminus A)$ is the set of edges with one endpoint in A and the other in $V \setminus A$. A straightforward higher-dimensional analogue is the following *Cheeger constant* of a k-dimensional simplicial complex X with *complete* $(k-1)$-skeleton, studied in [22]:

$$h(X) := \min_{\substack{V = \amalg_{i=0}^{k} A_i \\ A_i \neq \emptyset}} \frac{|V||F(A_0, A_1, \ldots, A_k)|}{|A_0| \cdot |A_1| \cdot \ldots \cdot |A_k|}.$$

Here $F(A_0, A_1, \ldots, A_k)$ is the set of k-dimensional faces of X with exactly one vertex in each set A_i.

For graphs, this combinatorial notion of expansion is closely related to the spectra of certain matrices associated with the graph: the adjacency matrix and the Laplacian. This connection between combinatorial and spectral expansion properties of a graph is established, e.g., by the discrete Cheeger inequality [1, 2, 5, 25]. For a d-regular graph G with second smallest eigenvalue $\lambda(G)$ of the Laplacian $L(G)$ (see Section 1), it states that

$$\lambda(G) \leq h(G) \leq \sqrt{8d\lambda(G)}.$$

A different approach to generalizing expansion is hence to consider higher-dimensional analogues of graph Laplacians. Higher-dimensional Laplacians were first introduced by Eckmann [6] in the 1940s and have since then been used in various contexts, see [12] for an example. We denote by $\lambda(X)$ the smallest non-trivial eigenvalue of this Laplacian. More precisely, $\lambda(X)$ is the smallest eigenvalue of the upper Laplacian $L_{k-1}^{\mathrm{up}}(X)$ on $(B^{k-1}(X; \mathbb{R}))^{\perp}$. (See Section 1 for further details.)

The Cheeger inequality for graphs has proven to be a useful tool. Computing the Cheeger constant is difficult, from the standpoint of complexity theory [19, 3] but often

[1] Often the Cheeger constant is defined by $\phi(G) = \min_{\substack{A \subseteq V \\ 0<|A|\leq|V|/2}} \frac{|E(A,V \setminus A)|}{|A|}$. Since $\phi(G) \leq h(G) \leq 2\phi(G)$, the two concepts are closely related. $h(G)$ and $\phi(G)$ are also called *(edge) expansion ratio.*

also for explicit examples. The lower bound – even though easy to prove – hence gives a helpful, polynomially computable, lower bound on the Cheeger constant. Essentially all constructions of families of expander graphs (graphs $(G_n)_{n\in\mathbb{N}}$ on n vertices with constant edge degree, where the Cheeger constant is bounded from below by a constant) use eigenvalues to establish a lower bound on the combinatorial expansion [7, 16, 17, 18, 23].

Parzanchevski, Rosenthal and Tessler [22] recently showed the following analogue of this lower bound of the Cheeger inequality for k-dimensional simplicial complexes with complete $(k-1)$-skeleton.

Theorem 1 (Parzanchevski, Rosenthal, Tessler [22])**.** *Let X be a k-dimensional simplicial complex with complete $(k-1)$-skeleton. Then $\lambda(X) \le h(X)$.*

Here, we present two ways to extend this result to k-dimensional complexes with non-complete $(k-1)$-skeleton, addressing an open question that was posed in [22]. Both proofs allow for an additional strengthening of the original result.

To make an extension to arbitrary complexes possible, it is necessary to adapt the definition of $h(X)$ as it is easily seen that $h(X)$ as defined above is non-zero only for k-dimensional X with complete $(k-1)$-skeleton. For any k-dimensional complex X, define its *k-dimensional completion* as $K(X) := X \cup \{\tau^\partial \in \binom{V}{k+1} : \tau^\partial \setminus \{v\} \in X \text{ for all } v \in \tau^\partial\}$. If X has a complete $(k-1)$-skeleton, we get $K(X) = K_n^k$, the complete k-dimensional complex on n vertices. We then define, as suggested in [22],

$$h(X) := \min_{\substack{V=\amalg_{i=0}^k A_i \\ A_i \neq \emptyset}} \frac{|V||F(A_0, A_1, \ldots, A_k)|}{|F^\partial(A_0, A_1, \ldots, A_k)|},$$

where $F^\partial(A_0, A_1, \ldots, A_k) = \{\tau^\partial \in \binom{V}{k+1} : \tau^\partial \in K(X), |\tau^\partial \cap A_i| = 1 \text{ for } i = 0, 1, \ldots, k\}$ is the set corresponding to $F(A_0, A_1, \ldots, A_k)$ in the completion $K(X)$ – and hence the largest possible set of k-simplices with one vertex in each A_i in a simplicial complex with the $(k-1)$-skeleton of X. For $F^\partial(A_0, A_1, \ldots, A_k) = 0$ define $\frac{|V||F(A_0,A_1,\ldots,A_k)|}{|F^\partial(A_0,A_1,\ldots,A_k)|} := \infty$.

Our first result is as follows:

Theorem 2. *Let X be a k-dimensional simplicial complex. For a $(k-1)$-face $\sigma \in X_{k-1}$ let $d(\sigma) := |\{\tau^\partial \in F^\partial(A_0, A_1, \ldots, A_k) : \sigma \subseteq \tau^\partial\}|$ and let*

$$C(X) := \max_{\tau^\partial \in F^\partial(A_0,A_1,\ldots,A_k)} \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} d(\sigma).$$

Then $\lambda(X) \le \frac{C(X)}{|V|} h(X)$.

Note that if A_l is the unique block not containing a vertex of σ, then $d(\sigma) \le |A_l|$ and that this bound is tight for k-complexes X with complete $(k-1)$-skeleton. So by definition $C(X) \le |V|$ and Theorem 2 implies the statement of Theorem 1 for arbitrary k-dimensional simplicial complexes. Whereas $C(X) = |V|$ for X with complete $(k-1)$-skeleton, in extreme cases $C(X)$ can be arbitrary small compared to $|V|$.

Our second result gives a different kind of strengthening. It is possible to rephrase $h(X)$ in terms of $\mathbb{Z}_2$-coboundaries as follows (see Section 1 for the necessary definitions).

For a partition $V = \amalg_{i=0}^{k} A_i$ let $F(A_0, A_1, \ldots, A_{k-1})$ be the set of $(k-1)$-dimensional faces of X with exactly one vertex in each set A_i, $i = 0, 1, \ldots, k-1$, and let $\mathbf{1}_{F(A_0,A_1,\ldots,A_{k-1})}$ be its characteristic function, interpreted as a $\mathbb{Z}_2$-cochain. Then the support of the $\mathbb{Z}_2$-coboundary $\delta_X \mathbf{1}_{F(A_0,A_1,\ldots,A_{k-1})}$ in X is exactly the set $F(A_0, A_1, \ldots, A_k)$ and the coboundary $\delta_{K(X)} \mathbf{1}_{F(A_0,A_1,\ldots,A_{k-1})}$ in $K(X)$ has support $F^{\partial}(A_0, A_1, \ldots, A_k)$. Thus,

$$h(X) = \min_{\substack{V=\amalg_{i=0}^{k} A_i \\ A_i \neq \emptyset}} \frac{|V| \cdot |\delta_X \mathbf{1}_{F(A_0,A_1,\ldots,A_{k-1})}|}{|\delta_{K(X)} \mathbf{1}_{F(A_0,A_1,\ldots,A_{k-1})}|},$$

where $|\cdot|$ denotes the Hamming norm. In order to strengthen the bound on $\lambda(X)$ given by Theorem 1, we define

$$h'(X) := \min_{\substack{V=\amalg_{i=0}^{k-1} A_i,\, f \in C^{k-1}(X,\mathbb{Z}_2), \\ \mathrm{supp}(f) \subset F(A_0,A_1,\ldots,A_{k-1})}} \frac{|V| \cdot |\delta_X f|}{|\delta_{K(X)} f|}.$$

If $|\delta_{K(X)} f| = 0$, we again define $\frac{|V| \cdot |\delta_X f|}{|\delta_{K(X)} f|} = \infty$. Note that here we consider partitions of V into $k-1$ parts. For a partition $V = \amalg_{i=0}^{k} A_i$ into k parts, clearly $\mathrm{supp}(\mathbf{1}_{F(A_0,A_1,\ldots,A_{k-1})}) \subset F(A_0, A_1, \ldots, A_{k-2}, A_{k-1} \cup A_k)$. Hence, as we minimize over a larger set of cochains, we have $h'(X) \leq h(X)$. See appendix for an example where $h'(X) < h(X)$. We show:

Theorem 3. *Let X be a k-dimensional simplicial complex. Then $\lambda(X) \leq h'(X)$.*

Discussion of Results. The inspiration for the definition of $h'(X)$ is a different analogue of the Cheeger constant for graphs, introduced by Gromov and independently by Linial, Meshulam and Wallach and by Newman and Rabinovich. It is based on $\mathbb{Z}_2$-cohomology and emerged in various contexts as a useful notion, see, e.g., [13, 20, 8, 21]. For a k-complex with complete $(k-1)$-skeleton, this notion can be described[2] by

$$\phi(X) := \min_{f \in C^{k-1}(X,\mathbb{Z}_2)} \frac{|V| \cdot |\delta_X f|}{|\delta_{K_n^k} f|},$$

similar to the definitions of $h(X)$ and $h'(X)$, but without any restriction on the cochains considered. As this seems to be an important and useful concept, one might wish for an inequality as in Theorems 1, 2 and 3 also for this notion of expansion. It was, however, shown that such an inequality can not exist, see [9, 24].

Theorem 2 and Theorem 3 can indeed give a stronger bound than Theorem 1, see appendix for examples that also show that it depends on the complex X whether $h'(X)$ or $\frac{C(X)}{|V|} h(X)$ presents the stronger upper bound on $\lambda(X)$.

Recall that the Cheeger inequality for graphs also gives an upper bound of $h(G)$ in terms of $\lambda(G)$. As $\lambda(X) = 0$ does not imply $h(X) = 0$, see [22], a higher-dimensional analogue of this upper bound of the form $C \cdot h(X)^m \leq \lambda(X)$ is hence not possible.

[2] Usually, one considers $\min_{f \in C^{k-1}(X,\mathbb{Z}_2)} \frac{|V| \cdot |\delta_X f|}{|[f]|}$, where $|[f]| = \min\{|f + \delta g| : g \in C^{k-2}(X; \mathbb{Z}_2)\}$. The two notions are closely related, because of expansion properties of the complete complex.

1 Preliminaries

Graph Laplacian. Let $G = (V, E)$ be a finite simple undirected graph. The *Laplacian* of G is the $|V| \times |V|$-matrix $L(G) = D(G) - A(G)$, where $A(G)$ is the *adjacency matrix* given by $A_{u,v} = 1$ if and only if $\{u, v\} \in E$ and $D(G)$ is the diagonal matrix with entries $D_{v,v} = \deg_G(v)$, the *degrees* of the vertices. The Laplacian is a symmetric positive semi-definite matrix and hence has n real non-negative eigenvalues. As $L\mathbf{1} = 0$, the smallest eigenvalue is always 0, and we denote by $\lambda(G)$ the second smallest eigenvalue of $L(G)$. A graph G is connected if and only if $\lambda(G)$ is non-zero (see, e.g., [10]).

Simplicial Complexes. Let V be a finite set. A *(finite abstract) simplicial complex* (or *complex*) X with vertex set V is a collection of subsets of V that is closed under taking subsets, i.e., $\sigma \subseteq \tau \in X$ implies $\sigma \in X$. An element $\tau \in X$ is called a *simplex* or *face* of X, the *dimension* of τ is $\dim \tau = |\tau| - 1$. A simplex τ with $\dim \tau = i$ is also called an *i-simplex*. The *dimension* of the complex X is $\dim X = \max_{\tau \in X} \dim \tau$. A simplicial complex of dimension k is called a *k-dimensional simplicial complex* or a *k-complex*. The one-element sets $\{v\}$, $v \in V$, are the *vertices* of X. We identify the singleton $\{v\}$ with its unique element v. For an $(i-1)$-simplex σ the *degree* of σ is defined as $\deg \sigma = |\{\tau \supseteq \sigma | \dim \tau = i\}|$. The set of all i-simplices of X is denoted by X_i, the collection of all simplices of dimension at most i, the *i-skeleton* of X, by $X^{(i)}$. The *complete k-complex* K_n^k has vertex set $V = [n] = \{1, \ldots, n\}$ and $X_i = \binom{[n]}{i+1}$ for all $i \leq k$.

Cohomology. Let X be a k-dimensional simplicial complex with vertex set V and assume we have a fixed linear ordering on V. We consider the faces of X with the *orientation* given by the order of their vertices. Formally, consider an i-simplex $\tau = \{v_0, v_1, \ldots, v_i\}$ where $v_0 < v_1 < \cdots < v_i$. For an $(i-1)$-simplex $\sigma \in X_{i-1}$ the *oriented incidence number* $[\tau : \sigma]$ is defined as $(-1)^j$ if $\sigma = \tau \setminus \{v_j\}$, for some $j = 0, 1, \ldots, i$ and zero otherwise, i.e., if $\sigma \nsubseteq \tau$. In particular for $v \in X_0$ and the unique face $\emptyset \in X_{-1}$ we have $[v : \emptyset] = 1$.

Let $\mathbb{G}$ be an Abelian group (we will be concerned with the cases $\mathbb{G} = \mathbb{Z}_2$ and $\mathbb{G} = \mathbb{R}$). The *group of i-dimensional cochains* on X (with coefficients in $\mathbb{G}$) is $C^i(X, \mathbb{G}) := \{f : X_i \to \mathbb{G}\}$, i.e., the group of maps from the set of i-simplices to $\mathbb{G}$. For $i < -1$ or $i > \dim X$ we conveniently define $C^i(X, \mathbb{G}) = 0$. Note that since the empty set is the unique element of X_{-1} we have $C^{-1}(X, \mathbb{G}) \cong \mathbb{G}$. The characteristic functions e_τ of faces $\tau \in X_i$ form a basis of $C^i(X, \mathbb{G})$, they are called the *elementary cochains*.

The *coboundary operator* $\delta_i \colon C^i(X, \mathbb{G}) \to C^{i+1}(X, \mathbb{G})$ is the linear function given by

$$\delta_i f(\tau) := \sum_{\sigma \in X_i} [\tau : \sigma] f(\sigma),$$

for τ an $(i+1)$-simplex, $f \in C^i(X, \mathbb{G})$ and $-1 \leq i < \dim X$. We let $\delta_i = 0$ otherwise. Define $Z^i(X; \mathbb{G}) = \ker \delta_i$ the group of i-dimensional *cocycles* and $B^i(X; \mathbb{G}) = \operatorname{im} \delta_{i-1}$ the group of i-dimensional *coboundaries*. A straightforward calculation shows that $\delta_i \delta_{i-1} = 0$, i.e., $B^i(X; \mathbb{G}) \subseteq Z^i(X; \mathbb{G})$. Hence, we can define the *i-th cohomology group with coefficients in* $\mathbb{G}$ as $H^i(X; \mathbb{G}) := Z^i(X; \mathbb{G}) / B^i(X; \mathbb{G})$.

Real Cohomology and Higher-Dimensional Laplacians. We endow $C^i(X;\mathbb{R})$ with the *inner product*

$$\langle f, g\rangle = \sum_{\tau\in X_i} f(\tau)g(\tau)$$

for $f, g \in C^i(X;\mathbb{R})$ and denote by $\partial_i\colon C^i(X;\mathbb{R}) \to C^{i-1}(X;\mathbb{R})$ the dual operator of δ_{i-1}, i.e., for $f \in C^i(X;\mathbb{R})$ and $g \in C^{i-1}(X;\mathbb{R})$ we have $\langle \partial_i f, g\rangle = \langle f, \delta_{i-1} g\rangle$. The map ∂_i is also called the *boundary operator* and $Z_i(X;\mathbb{R}) = \ker \partial_i$ and $B_i(X;\mathbb{R}) = \operatorname{im} \partial_{i+1}$ are called the group of i-dimensional *cycles* and the group of i-dimensional *boundaries*, respectively. Setting $\mathcal{H}_i = \mathcal{H}_i(X;\mathbb{R}) := Z_i(X;\mathbb{R}) \cap Z^i(X;\mathbb{R})$, one gets a *Hodge decomposition* of the vector space $C^i(X;\mathbb{R})$ into pairwise orthogonal subspaces

$$C^i(X;\mathbb{R}) = \mathcal{H}_i \oplus B^i(X;\mathbb{R}) \oplus B_i(X;\mathbb{R}), \tag{1}$$

in particular, $\mathcal{H}_i \cong H^i(X;\mathbb{R})$ (see [6, 11]).

The higher-dimensional analogue of the graph Laplacian is based on these notions. From now on, write C^{k-1} for $C^{k-1}(X;\mathbb{R})$, B^{k-1} for $B^{k-1}(X;\mathbb{R})$ and Z_{k-1} for $Z_{k-1}(X;\mathbb{R})$. The *upper, lower* and *full Laplacian* $L^{\mathrm{up}}_{k-1}(X), L^{\mathrm{down}}_{k-1}(X), L_{k-1}(X)\colon C^{k-1} \to C^{k-1}$ are defined as

$$L^{\mathrm{up}}_{k-1}(X) = \partial_k\delta_{k-1}, \quad L^{\mathrm{down}}_{k-1}(X) = \delta_{k-2}\partial_{k-1} \text{ and } L_{k-1}(X) = L^{\mathrm{up}}_{k-1}(X) + L^{\mathrm{down}}_{k-1}(X),$$

respectively. More generally the upper Laplacian in dimension i is defined as $L^{\mathrm{up}}_i(X) = \partial_{i+1}\delta_i$ and the lower and full Laplacian similarly. We solely focus on the case $i = k-1$.

Analogously to the case of graphs ($k = 1$) we can express $L^{\mathrm{up}}_{k-1}(X)$ as a matrix: With respect to the orthogonal basis of elementary cochains it corresponds to the matrix $L^{\mathrm{up}}_{k-1}(X) = D_{k-1}(X) - A_{k-1}(X)$. Here we let $D_{k-1}(X)$ denote the diagonal matrix with entry $(D_{k-1}(X))_{\tau,\tau} = \deg(\tau)$ for $\tau \in X_{k-1}$ and define the *signed adjacency matrix* $A_{k-1}(X)$ by

$$(A_{k-1}(X))_{\tau,\tau'} = \begin{cases} -[\tau\cup\tau' : \tau][\tau\cup\tau' : \tau'] & \text{if } \tau \sim \tau', \\ 0 & \text{otherwise}, \end{cases}$$

where $\tau, \tau' \in X_{k-1}$ and we write $\tau \sim \tau'$ if τ and τ' share a common $(k-2)$-face and $\tau \cup \tau' \in X_k$. This shows that $L^{\mathrm{up}}_0(G)$ for a graph G agrees with the Laplacian $L(G)$.

Note that $L^{\mathrm{up}}_{k-1}(X)$ (as well as $L^{\mathrm{down}}_{k-1}(X)$ and $L_{k-1}(X)$) is a *self-adjoint* and *positive semidefinite* linear operator on C^{k-1}. It is furthermore not hard to see that $\ker L^{\mathrm{up}}_{k-1}(X) = Z^{k-1}$. Since $B^{k-1} \subseteq Z^{k-1}$, this implies that $L^{\mathrm{up}}_{k-1}(X)$ is zero on B^{k-1}. Hence, non-zero eigenvalues can only occur in the space $(B^{k-1})^\perp$ and we define the *spectral gap* of X as

$$\lambda(X) := \operatorname{minSpec}(L^{\mathrm{up}}_{k-1}(X)|_{(B^{k-1})^\perp}) = \operatorname{minSpec}(L^{\mathrm{up}}_{k-1}(X)|_{Z_{k-1}}),$$

where the equality holds because we have $Z_{k-1} = (B^{k-1})^\perp$ by the Hodge decomposition (1). We remark that even though the spaces B^{k-1} and Z_{k-1} depend on the choice of orientations for the faces of X, the spectrum of L^{up}_{k-1} and the value of $\lambda(X)$ do not.

Note that $\lambda(X)$ is also the minimal eigenvalue of the full Laplacian $L_{k-1}(X)$ on Z_{k-1}, since $Z_{k-1} = \ker L_{k-1}^{\mathrm{down}}(X)$. We have $\lambda(X) = 0$, i.e., there exist more zero eigenvalues than the ones corresponding to functions in B^{k-1}, if and only if $H^{k-1}(X;\mathbb{R}) \neq 0$.

For a graph G the space B^0 is the space of constant functions, spanned by the all-ones vector $\mathbf{1}$, so this definition of the spectral gap coincides with $\lambda(G)$ as defined previously.

2 The Cheeger Inequality for k-Complexes with complete $(k-1)$-Skeleton.

In the following part we describe the basic ideas of the proof of Theorem 1 from [22]. By the variational characterization of eigenvalues we know that

$$\lambda(X) = \min_{f \in Z_{k-1}} \frac{\langle L_{k-1}^{\mathrm{up}}(X) f, f\rangle}{\langle f, f\rangle}. \tag{2}$$

The key idea is to find a function $f \in Z_{k-1}$ that satisfies

$$\frac{\langle L_{k-1}^{\mathrm{up}}(X) f, f\rangle}{\langle f, f\rangle} = h(X).$$

In order to define a function satisfying this equation, we fix a partition $A_0, A_1, \dots, A_k$ of V which realizes the minimum in $h(X)$. We call the A_i's *blocks of the partition* or shortly just *blocks*. Since the value of $\lambda(X)$ does not depend on the chosen orientation, we are free to choose an orientation depending on this partition. For reasons of simplicity we choose a linear ordering on V such that for all $i < j$, $v \in A_i$, $w \in A_j$ we have $v < w$.

Let $\sigma = \{v_0, v_1, \dots v_{k-1}\} \in X_{k-1}$, with $v_0 < v_1 < \dots < v_{k-1}$. Then $f \in C^{k-1}$ is defined as

$$f(\sigma) = \begin{cases} (-1)^l |A_l| & \text{if } A_l \text{ is the unique block not containing any of the } v_i, \\ 0 & \text{otherwise, i.e., if } \exists l, i \neq j \text{ with } v_i, v_j \in A_l. \end{cases} \tag{3}$$

The following two statements describing essential properties of f give the proof of Theorem 1.

Lemma 4. *[22] Let X be a k-dimensional simplicial complex with* complete $(k-1)$-skeleton *and let f be defined as above.*

Then $f \in Z_{k-1}$ and $\langle f, f\rangle = |V||F^{\partial}(A_0, A_1, \dots, A_k)| = |V||A_0||A_1| \cdots |A_k|$.

Lemma 5. *[22] Let X be* any *k-dimensional simplicial complex and let f be defined as above. Then*

$$\langle L_{k-1}^{up}(X) f, f\rangle = \langle \delta_{k-1} f, \delta_{k-1} f\rangle = |V|^2 |F(A_0, A_1, \dots, A_k)|.$$

For the first lemma, which can be proven by a straightforward calculation, there is no trivial generalization for arbitrary simplicial complexes. The latter lemma does not require any assumptions on the $(k-1)$-skeleton and we will be able to use it for our purposes. See the appendix for a proof of Lemma 5.

3 Proof of Theorem 2

In this section we give the proof of Theorem 2. As in Section 2 we fix a partition $A_0, A_1, \ldots, A_k$ of V realizing the minimum in $h(X)$ and choose an orientation accordingly. We define f as in (3). A key ingredient of the proof of Theorem 1 presented in Section 2 is that $f \in Z^{k-1}$. This does not hold in general. To extend the proof to arbitrary complexes, we instead study the projection of f onto the space Z^{k-1}:

Lemma 6. *Let $f \in C^{k-1}$ be as previously defined. Then there exist unique $z \in Z_{k-1}$, $b \in B^{k-1}$ such that $f = z + b$ and*

$$\lambda(X) \leq \frac{|V|^2 |F(A_0, A_1, \ldots, A_k)|}{\langle z, z \rangle}.$$

Proof. Since $Z_{k-1} = (B^{k-1})^{\perp}$ there exist unique $z \in Z_{k-1}$ and $b \in B^{k-1}$ such that $f = z + b$. Hence the claim follows by combining (2) with Lemma 5 and the fact that $\langle L_{k-1}^{\mathrm{up}}(X)z, z\rangle = \langle L_{k-1}^{\mathrm{up}}(X)f, f\rangle$ because $b \in \ker L_{k-1}^{\mathrm{up}}(X)$. □

From now on we always use z and b in the context of Lemma 6. To prove Theorem 2 we need to find a lower bound for $\langle z, z\rangle$. To the best of our knowledge, there is no way of explicitly finding z by knowing f. We will instead make use of the fact that $b \in B^{k-1}$, i.e., there exists $g \in C^{k-2}$ such that $b = \delta_{k-2} g$, and estimate the distance of f to any cochain of this form. Recall $d(\sigma)$ and $C(X)$ as defined in the introduction.

Lemma 7. *Let $f \in C^{k-1}$ be as previously defined and let $g \in C^{k-2}$ be arbitrary.*

a) $\|f - \delta_{k-2} g\|^2 \geq \sum_{\tau^\partial \in F^\partial(A_0, A_1, \ldots, A_k)} \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2.$

b) For $\tau^\partial = \{v_0, v_1, \ldots, v_k\} \in F^\partial(A_0, A_1, \ldots, A_k)$ with $v_0 < v_1 < \cdots < v_k$ let $d_j := d(\tau^\partial \setminus \{v_j\})$. Then:

$$q(\tau^\partial, g) := \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2 \geq \frac{|V|^2}{\sum_{j=0}^k d_j}.$$

We first show how to use Lemma 7 to prove Theorem 2 and then prove Lemma 7.

Proof of Theorem 2. Since $b \in B^{k-1}$ there exists $g \in C^{k-2}$ such that $f - z = b = \delta_{k-2} g$. By Lemma 7 we have:

$$\langle z, z \rangle = \|f - \delta_{k-2} g\|^2 \geq \sum_{\tau^\partial \in F^\partial(A_0, A_1, \ldots, A_k)} \frac{|V|^2}{\sum_{j=0}^k d_j} \geq |F^\partial(A_0, A_1, \ldots, A_k)| \cdot \frac{|V|^2}{C(X)},$$

by definition of $C(X)$. Combined with Lemma 6 this proves Theorem 2. □

Proof of Lemma 7. a) Consider the right hand sum. Note that for any $\sigma \in X_{k-1}$ such that $\sigma \subseteq \tau^\partial$ for some $\tau^\partial \in F^\partial(A_0, A_1, \ldots, A_k)$, the corresponding term appears exactly $d(\sigma)$ times by definition. For $\sigma \nsubseteq \tau^\partial$ the term does not appear at all. The statement follows by definition of the inner product.

b) In the first part of the proof assume that $\tau^\partial = \tau \in F(A_0, A_1, \dots, A_k)$. The proof for $\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)$ will work almost analogously.

Let $\tau = \{v_0, v_1, \dots, v_k\} \in X_k$ with $v_i \in A_i$ for $i = 0, 1, \dots, k$. Then

$$\begin{aligned} q(\tau, g) &= \sum_{i=0}^{k} \frac{1}{d_i} \left((-1)^i |A_i| - \delta_{k-2} g(\tau \setminus \{v_i\})\right)^2 \\ &= \sum_{i=0}^{k} \frac{1}{d_i} \left(|A_i| - [\tau : \tau \setminus \{v_i\}]\delta_{k-2} g(\tau \setminus \{v_i\})\right)^2 . \end{aligned}$$

We observe that for $x_i := -[\tau : \tau \setminus \{v_i\}]\delta_{k-2} g(\tau \setminus \{v_i\})$ it holds that $-\sum_{i=0}^{k} x_i = \delta_{k-1}(\delta_{k-2} g)(\tau) = 0$. Instead of $q(\tau, g)$ we now study the function $q(\tau, x) = \sum_{i=0}^{k} \frac{1}{d_i}(|A_i| + x_i)^2$, which we try to minimize. By the previous equation we can rewrite $q(\tau, x)$ as

$$q(\tau, x) = \sum_{i=0}^{k-1} \frac{1}{d_i}(|A_i| + x_i)^2 + \frac{1}{d_k}\left(|A_k| - \sum_{i=0}^{k-1} x_i\right)^2 .$$

By checking the partial derivatives we know that the extremal points must satisfy $\frac{\partial q(\tau,x)}{\partial x_i} = \frac{2}{d_i}(|A_i| + x_i) - \frac{2}{d_k}\left(|A_k| - \sum_{i=0}^{k-1} x_i\right) = 0$ for all $i \in \{0, 1, \dots, k-1\}$. One can check that this equality system is satisfied by $y_i = \frac{d_i |V|}{\sum_{j=0}^{k} d_j} - |A_i|$.

We will show that $q(\tau, x)$ attains its unique global minimum in $y = (y_0, y_1, \dots, y_{k-1})$. It is a well known fact from basic calculus that an extremal point is a local minimum if the Hessian matrix at this point is positive-definite. A straightforward calculation shows that the Hessian matrix of $q(\tau, x)$ is strictly positive-definite everywhere. Therefore $q(\tau, x)$ is strictly convex (see, e.g., [4]) and we can conclude that it attains its unique minimum in y. Hence,

$$q(\tau, x) \geq q(\tau, y) = \sum_{i=0}^{k-1} \frac{1}{d_i}\left(\frac{d_i |V|}{\sum_{j=0}^{k} d_j}\right)^2 + \frac{1}{d_k}\left(\frac{d_k |V|}{\sum_{j=0}^{k} d_j}\right)^2 = \frac{|V|^2}{\sum_{i=0}^{k} d_j}.$$

It remains to prove the statement for $\tau^\partial = \{v_0, v_1, \dots, v_k\} \in F^\partial(A_0, A_1, \dots, A_k)$ with $v_0 < v_1 < \dots < v_k$. Observe that the whole proof works analogously except for the part that we have not defined the "incidence number" $[\tau^\partial : \sigma]$ for $\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k) \setminus F(A_0, A_1, \dots, A_k)$. By defining it the obvious way as $(-1)^i$ if $\sigma = \tau^\partial \setminus \{v_i\}$, $i = 0, 1, \dots, k$ and zero otherwise, i.e., if $\sigma \not\subseteq \tau^\partial$, we observe that $\delta_{k-1}\delta_{k-2}(\tau^\partial) = 0$ and the proof works analogously.

□

4 Proof of Theorem 3

In this section we give the proof of Theorem 3. Since we consider real as well as $\mathbb{Z}_2$-cohomology, we denote the real coboundary operator by $\delta^{\mathbb{R}}$, the $\mathbb{Z}_2$-coboundary by $\delta^{\mathbb{Z}_2}$. The space of $\mathbb{Z}_2$-cochains is denoted by $C^{k-1}(X, \mathbb{Z}_2)$, the space of real cochains by $C^{k-1}(X)$ instead of $C^{k-1}(X, \mathbb{R})$. Also, $B^{k-1}(X)$ stands for $B^{k-1}(X; \mathbb{R})$ (We now add the space X to the notation, because we will consider cochains in different spaces.)

The following lemma points out a special behaviour of the $\mathbb{Z}_2$-cochains appearing in the definition of $h'(X)$ that will be central to our argument: The size of the $\mathbb{Z}_2$-boundary of such a cochain agrees with the size of its real coboundary.

Lemma 8. *Let X be a k-complex with n vertices. Let $A_0, A_1, \dots, A_{k-1} \subset V = V(X)$ be pairwise disjoint and let $f \in C^{k-1}(X, \mathbb{Z}_2)$ such that* $\text{supp}(f) \subset F(A_0, A_1, \dots, A_{k-1})$. *Choose an orientation of the simplices of X by fixing a linear ordering on V such that for all $i < j \in \{0, 1, \dots, k-1\}$, $v \in A_i$, $w \in A_j$ we have $v < w$. Then, interpreting f also as an $\mathbb{R}$-cochain with values in $\{0, 1\}$, we have*

$$\|\delta^{\mathbb{R}} f\|^2 = \langle L^{\text{up}}_{k-1} f, f \rangle = |\delta^{\mathbb{Z}_2} f|.$$

Here, $\|\cdot\|$ denotes the ℓ_2-norm and $|\cdot|$ denotes the Hamming norm.

Proof. Note that any k-face $\tau \in X_k$ can have at most two $(k-1)$-faces that are contained in $F(A_0, A_1, \dots, A_{k-1})$, and the same holds for $\text{supp}(f) \subset F(A_0, A_1, \dots, A_{k-1})$.

For $\tau \in X_k$ consider $\delta^{\mathbb{R}} f(\tau) = \sum_{\sigma \subset \tau, \sigma \in X_{k-1}} [\tau : \sigma] f(\sigma)$. If τ has no faces in $\text{supp}(f)$ this sum is empty. It is ± 1 if t has exactly one face in $\text{supp}(f)$. Otherwise τ has exactly two faces σ and σ' with $f(\sigma) = f(\sigma') = 1$. By our choice of orientations, we have $[\tau : \sigma] = -[\tau : \sigma']$ and hence $\delta^{\mathbb{R}} f(\tau) = 0$.

This shows that $\langle L^{\text{up}}_{k-1} f, f \rangle = \|\delta^{\mathbb{R}} f\|^2$ equals the number of k-faces with exactly one face in $\text{supp}(f)$. As $\text{supp}(f) \subset F(A_0, A_1, \dots, A_{k-1})$, this is $|\delta^{\mathbb{Z}_2} f|$. □

Before we come to the proof of Theorem 3, we give an upper bound for the eigenvalue $\lambda(X)$. By the variational characterization of eigenvalues, $\lambda(X)$ is the minimum over all $f \in C^{k-1}(X, \mathbb{R})$ of unit norm that are orthogonal to $B^{k-1}(X)$. The key observation here is that we can get rid of this orthogonality constraint.

Lemma 9. *Let X be a k-complex with n vertices and let $\lambda(X)$ be the smallest eigenvalue of the upper Laplacian $L^{\text{up}}_{k-1}(X)$ on $(B^{k-1})^{\perp}$. Then*

$$\lambda(X) \leq \min_{\substack{f \in C^{k-1}(X), \\ f \notin B^{k-1}(X)}} \frac{n \cdot \langle L^{\text{up}}_{k-1}(X) f, f \rangle}{\langle L^{\text{up}}_{k-1}(K(X)) f, f \rangle}. \tag{4}$$

If $\langle L^{\text{up}}_{k-1}(K(X)) f, f \rangle = 0$, we define $\frac{n \cdot \langle L^{\text{up}}_{k-1}(X) f, f \rangle}{\langle L^{\text{up}}_{k-1}(K(X)) f, f \rangle} = \infty$. For X with complete $(k-1)$-skeleton (4) holds with equality.

Proof. First assume that X has a complete $(k-1)$-skeleton. The following equality is contained implicitly in [12] and follows from a straightforward calculation using the matrix representations of the Laplacians: $L^{\text{up}}_{k-1}(K^k_n) + L^{\text{down}}_{k-1}(K^k_n) = nI$. Hence, we have $n\langle f, f\rangle = \langle L^{\text{up}}_{k-1}(K^k_n)f, f\rangle + \langle L^{\text{down}}_{k-1}(K^k_n)f, f\rangle$ for any $f \in C^{k-1}(X) = C^{k-1}(K^k_n)$. Combining this with the variational characterization of eigenvalues and the fact that $L^{\text{down}}_{k-1}(K^k_n)f = 0$ for $f \perp B^{k-1}(X) = B^{k-1}(K^k_n)$, we get:

$$\lambda(X) = \min_{\substack{f\in C^{k-1}(X),\\ f\perp B^{k-1}(X)}} \frac{\langle L^{\text{up}}_{k-1}(X)f, f\rangle}{\langle f, f\rangle} = \min_{\substack{f\in C^{k-1}(X),\\ f\perp B^{k-1}(X)}} \frac{n\cdot\langle L^{\text{up}}_{k-1}(X)f, f\rangle}{\langle L^{\text{up}}_{k-1}(K^k_n)f, f\rangle}.$$

For $f \notin B^{k-1}(X)$ that is not orthogonal to $B^{k-1}(X)$, let b be the projection of f onto $B^{k-1}(X)$ and let $z = f - b$. Then $z \perp B^{k-1}(X)$ and it holds that $\langle L^{\text{up}}_{k-1}(X)z, z\rangle = \langle L^{\text{up}}_{k-1}(X)f, f\rangle$ as well as $\langle L^{\text{up}}_{k-1}(K^k_n)z, z\rangle = \langle L^{\text{up}}_{k-1}(K^k_n)f, f\rangle$. This shows that we can omit the orthogonality constraint.

Now, consider the general case of a k-complex X with an arbitrary $(k-1)$-skeleton. Let $f \in C^{k-1}(X)$. We extend f to $\tilde{f} \in C^{k-1}(K^k_n)$ defined by $\tilde{f}(\sigma) = f(\sigma)$ if $\sigma \in X$ and $\tilde{f}(\sigma) = 0$ otherwise. A straightforward calculation shows that $\tilde{f} \perp B^{k-1}(K^k_n)$ if $f \perp B^{k-1}(X)$. Hence, we can argue as above to see that for $f \perp B^{k-1}(X)$ we get $n\langle f, f\rangle = n\langle \tilde{f}, \tilde{f}\rangle = \langle L^{\text{up}}_{k-1}(K^k_n)\tilde{f}, \tilde{f}\rangle \geq \langle L^{\text{up}}_{k-1}(K(X))f, f\rangle$. Thus,

$$\lambda(X) = \min_{\substack{f\in C^{k-1}(X),\\ f\perp B^{k-1}(X)}} \frac{n\cdot\langle L^{\text{up}}_{k-1}(X)f, f\rangle}{\langle L^{\text{up}}_{k-1}(K^k_n)\tilde{f}, \tilde{f}\rangle} \leq \min_{\substack{f\in C^{k-1}(X),\\ f\perp B^{k-1}(X)}} \frac{n\cdot\langle L^{\text{up}}_{k-1}(X)f, f\rangle}{\langle L^{\text{up}}_{k-1}(K(X))f, f\rangle}.$$

For $f \notin B^{k-1}(X)$ that is not orthogonal to $B^{k-1}(X)$, we again consider the projection b of f onto $B^{k-1}(X)$. For $z = f - b$ we have $z \perp B^{k-1}(X) = B^{k-1}(K(X))$ and $\langle L^{\text{up}}_{k-1}(X)z, z\rangle = \langle L^{\text{up}}_{k-1}(X)f, f\rangle$ as well as $\langle L^{\text{up}}_{k-1}(K(X))z, z\rangle = \langle L^{\text{up}}_{k-1}(K(X))f, f\rangle$, which shows that also in this case we can omit the orthogonality constraint □

Now we can prove Theorem 3:

Proof of Theorem 3. Fix sets $A_0, A_1, \ldots, A_{k-1} \subset V = V(X)$ and $f \in C^{k-1}(X, \mathbb{Z}_2)$ with $\text{supp}(f) \subset F(A_0, A_1, \ldots, A_{k-1})$ such that $h'(X) = n \cdot |\delta^{\mathbb{Z}_2}_X f|/|\delta^{\mathbb{Z}_2}_{K(X)} f|$. If $|\delta^{\mathbb{Z}_2}_{K(X)} f| = 0$, we have $h'(X) = \infty$ and there is nothing to show. Otherwise, we apply Lemmas 8 and 9 as follows: Since the value of $\lambda(X)$ does not depend on the chosen orientations of the simplices of X, we are free to choose the orientations as in Lemma 8, i.e., we fix a linear ordering on V such that for all $i < j$, $v \in A_i$, $w \in A_j$ we have $v < w$. Then by Lemma 8 we get $\langle L^{\text{up}}_{k-1}(X)f, f\rangle = |\delta^{\mathbb{Z}_2}_X f|$ and $\langle L^{\text{up}}_{k-1}(K(X))f, f\rangle = |\delta^{\mathbb{Z}_2}_{K(X)} f|$. As $|\delta^{\mathbb{Z}_2}_{K(X)} f| \neq 0$, we have $f \notin B^{k-1}(X)$ and can apply Lemma 9 to obtain

$$\lambda(X) \leq \frac{n\cdot\langle L^{\text{up}}_{k-1}(X)f, f\rangle}{\langle L^{\text{up}}_{k-1}(K(X))f, f\rangle} = h'(X).$$

□

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Appendix

Examples

The following examples show that Theorem 2 and Theorem 3 can indeed give a stronger bound than Theorem 1 and that it depends on X whether $h'(X)$ or $\frac{C(X)}{|V|}h(X)$ presents the stronger upper bound on $\lambda(X)$.

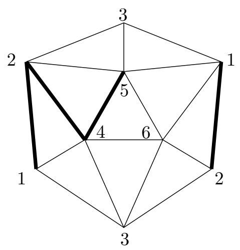

Figure 1: Real Projective Plane

Consider the complex X given in Figure 1, which shows a triangulation of the real projective plane. It has 6 vertices, a complete 1-skeleton and all triangles that are visible in the figure. Let $A = \{\{1,2\},\{2,4\},\{4,5\}\}$ be the edge set depicted by bold lines and let $\mathbf{1}_A$ be its characteristic function, interpreted as a $\mathbb{Z}_2$-cochain. Then $|\delta_X \mathbf{1}_A| = 2$ and $|\delta_{K_6^2} \mathbf{1}_A| = 8$ and hence we see that $h'(X) \leq \frac{1}{4}$. We will show that $h(X) \geq \frac{1}{3}$. Note that, since X has a complete 1-skeleton, we furthermore have $\frac{C(X)}{|V|}h(X) = h(X)$.

Consider a 3-coloring of the vertices of X. In the case where there exists a color class $\{v\}$ of size one, the five neighbors of v (which belong to the other two colors classes) span at least two 3-colored triangles with v. In the case where all color classes have size two, one can show in a similar fashion that every such 3-coloring (one has to distinguish between two cases) has exactly four 3-colored triangles. Therefore $h(X) \geq \frac{2}{6} = \frac{1}{3}$.

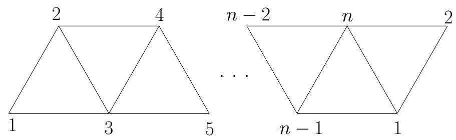

Figure 2: Band

On the other hand consider the 2-dimensional complex Y depicted in Figure 2. It has vertex set $V = [n]$ and the edges that are present in the figure. The set of triangles is $\{\{i, i+1 \bmod n, i+2 \bmod n\}, i \in [n]\}$. Then $F(A_0, A_1, A_2) = F^{\partial}(A_0, A_1, A_2)$ and $\delta_X A = \delta_{K(X)} A$ for any partition A_0, A_1, A_2 and any A. Therefore $\frac{C}{|V|}h(X) \leq \frac{5}{n}n = 5$ gives a constant bound whereas $h(X) = h'(X) = n$ yields a linear bound.

Proof of Lemma 5

Proof of Lemma 5. Let $\tau = \{v_0, v_1, \ldots, v_k\} \in X_k$ with $v_0 < v_1 < \cdots < v_k$. By definition of the coboundary operator it is enough to prove that

$$(\delta_{k-1} f)(\tau) = \begin{cases} |V| & \text{if } \tau \in F(A_0, A_1, \ldots, A_k), \\ 0 & \text{otherwise.} \end{cases}$$

First suppose that $\tau \notin F(A_0, A_1, \ldots, A_k)$. If τ has three vertices in the same block A_i or four vertices in two blocks, then every term $\tau \setminus \{v_i\}$ in

$$(\delta_{k-1} f)(\tau) = \sum_{i=0}^{k} [\tau : \tau \setminus \{v_i\}] f(\tau \setminus \{v_i\})$$

has two vertices in the same block and hence the sum vanishes. If we assume that v_j, v_l with $v_j < v_l$ is the only pair of vertices in the same block, then by our linear ordering $j + 1 = l$ and since $f(v \setminus \{v_j\}) = f(v \setminus \{v_l\})$, the two non-vanishing terms $[\tau : \tau \setminus \{v_j\}] f(\tau \setminus \{v_j\})$ and $[\tau : \tau \setminus \{v_{j+1}\}] f(\tau \setminus \{v_{j+1}\})$ in $(\delta_{k-1} f)(\tau)$ cancel out.

In the case where $\tau \in F(A_0, A_1, \ldots, A_k)$, i.e., where $v_i \in A_i$ for all $i = 0, 1, \ldots, k$, we have

$$(\delta_{k-1} f)(\tau) = \sum_{i=0}^{k} (-1)^i f(\tau \setminus \{v_i\}) = \sum_{i=0}^{k} (-1)^i (-1)^i |A_i| = |V|.$$

□