Eidgenössische Technische Hochschule Zürich
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On the Cheeger Inequality for Simplicial Complexes

Master's Thesis

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August, 2013

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Abstract

The lower bound of the Cheeger inequality for a graph G estabilishes a connection between the spectral and expansion properties of the graph, namely $\lambda(G) \leq h(G)$, where $\lambda(G)$ is the second smallest eigenvalue of the Laplacian of G and $h(G)$ the Cheeger constant. We show a generalization of the above inequality for simplicial complexes. Parzanchevski, Rosenthal and Tessler [27] showed, that for suitable generalizations of $\lambda(G)$ and $h(G)$ the inequality $\lambda(X) \leq h(X)$ holds, if X is a k-dimensional simplicial complex with complete $(k-1)$-skeleton. In this thesis we prove that $\lambda(X) \leq h(X)$ holds for arbitrary k-dimensional simplicial complexes.

Acknowledgments

First and foremost I thank my advisor Anna Gundert for her constant support, patience and useful inputs in the many discussions we had throughout the whole process of writing this thesis. I am also grateful to my advisor Prof. Dr. Uli Wagner for his useful ideas and great advice. Last but not least I thank my supervisor Prof. Dr. Emo Welzl, without his support this thesis would not have been possible.

Contents

Chapter 1

Introduction

A simplicial complex of dimension k, also called a k-complex, is a higher dimensional analogue of graphs for $k \geq 2$. They are a family of hypergraphs which are mainly used in algebraic topology and every graph corresponds to a 1-complex.

In recent years there has been much interest in generalizing known results on graphs for simplicial complexes. For instance Linial and Meshulam [19] introduced a higher dimensional analogue $X^k(n,p)$ of the binomial random graph $G(n,p)$: On a simplicial complex with n vertices and complete $(k-1)$-skeleton, every k-simplex is inserted independently with the same probability p. Since then, this model has been studied extensively and thresholds for several topological properties have been established. For example the sharp threshold for the vanishing of the $(k-1)$-st cohomology group $H^{k-1}(X^k(n,p);\mathbb{F})$ is $p = \frac{k \log n}{n}$ for any abelian group $\mathbb{F}$ [19, 24]. For more threshold results see e.g. [4, 3, 7, 8, 17, 31].

In this thesis we are interested in a higher dimensional analogue of the Cheeger inequality. For a graph $G = (V,E)$ the Cheeger inequality connects the spectral gap $\lambda(G)$ and the Cheeger constant $h(G)$, where the lower bound of the Cheeger inequality is given by $\lambda(G) \leq h(G)$. The spectral gap $\lambda(G)$ is defined as the second smallest eigenvalue of the Laplacian $L(G) = D - A$, where D is the diagonal matrix containing the degrees of the vertices V in the diagonal entries and A is the adjacency matrix of G. The Cheeger constant $h(G)$, which measures the edge expansion, is given by

$$h(G) = \min_{\substack{A \subseteq V \\ 0<|A|<|V|}} \frac{|V||e(A, V \setminus A)|}{|A||V \setminus A|}$$

where $e(A, V \setminus A)$ denotes the set of edges with one vertex in A and the other one in $V \setminus A$.

The lower bound of the Cheeger inequality turns out to be useful in various contexts. For instance although $\lambda(G)$ is well studied and easy to compute, we know that it is NP-hard to find $h(G)$ for a given graph [23, 5]. Another example are expanders, i.e. families of graphs $(G_n)_{n\in\mathbb{N}}$ on n vertices with constant vertex degree, where the Cheeger constants of the graphs are bounded from below by a constant. In several proofs the Cheeger inequality was used to show that a families of graphs $(G_n)_{n\in\mathbb{N}}$ are expanding [21, 11, 20, 22, 3].

A generalization of the Laplace operator on simplicial complexes was introduced by Eckmann in 1945 [10]. For the Cheeger constant the generalization is not straight forward. A topologically motivated one was recently introduced by Gromov [12] and independently by Linial, Meshulam and Wallach [19, 24] and by Newman and Rabinovich [26]. Gundert and Wagner [13] showed that for the above generalization there is no higher dimensional analogue of the lower bound of the Cheeger inequality, i.e., there exists no $c > 0$ such that $c\lambda(X) \leq h(X)$. A similar result can be found in [29].

In this thesis we consider a combinatorially motivated higher dimensional analogue $h(X)$, introduced by Parzanchevski, Rosenthal and Tessler [27]. In their paper they prove that for a k-dimensional simplicial complex X with complete $(k-1)$-skeleton, we have $\lambda(X) \leq h(X)$. In this thesis we will show that $\lambda(X) \leq h(X)$ holds for arbitrary k-complexes, which is also the main result of this work.

This can be found in Chapter 4, where for sparse k-complexes an even stronger bound than $\lambda(X) \leq h(X)$ is established. Chapter 2 gives an introduction to simplicial complexes and Chapter 3 gives a formal introduction into the Cheeger inequality for graphs and complexes, where in the last part we repeat the result of Parzanchevski et al. [27].

In a first attempt we tried to generalize the Cheeger inequality for 2-simplicial complexes with strong assumptions on the underlying graph, to get a statement of the form $(1-\varepsilon)\lambda(X) \leq h(X)$. This has strong disadvantages compared to the result in Chapter 4. Firstly we only get a bound $(1-\varepsilon)\lambda(X) \leq h(X)$, which is strictly worse than the bound obtained in Chapter 4 and secondly, this bound only holds with strong preconditions which are hard to verify. Since the two approaches differ very much, the reader might still be interested in the proof of this weaker statement which can be found in Chapter 5.

Chapter 2

Simplicial Complexes and Basics of (Co-)Homology

This chapter is based the books ”Elements of Algebraic Topology” by V.V. Prasolov [28] and ”Elements of Homology Theory” by J.R. Munkres [25].

Definition 2.1 *Let* $V = \{w_0, w_1, \dots, w_n\}$*,* $n \in \mathbb{N}$*, be a finite set. A* finite abstract simplicial complex X *is a collection of sets such that*

1. $X \subseteq 2^V$*, where* $2^{|V|}$ *is the set of all subsets of* V*,*
2. $(\sigma \subseteq \tau \wedge \tau \in X) \Rightarrow \sigma \in X$ *i.e. if* τ *is an element of* X*, then so is every subset of* τ*.*

In this thesis we always consider *finite abstract* simplicial complexes, usually the terms finite and abstract will be omitted.

An element $\tau \in X$ is called a *simplex* or *face* of X, the *dimension* of τ is $|\tau| - 1$ and is denoted $\dim \tau$. A simplex τ with $\dim \tau = i$ is also called an *i-simplex*. The *dimension* of the simplicial complex X is defined as $\dim X = \max_{\tau \in X} \dim \tau$ and a simplicial complex of dimension k is called a *k-dimensional simplicial complex* or short a *k-complex*. For an $(i-1)$-simplex σ the *degree* of σ is defined as $\deg \sigma = |\{\tau \supseteq \sigma | \dim \tau = i\}|$. The one element sets $\{w_0\}, \{w_1\}, \dots, \{w_n\}$ are called the *vertices* of X. We will identify the singletons $\{w_i\}$ with its unique element w_i. The collection of all simplices of X that have dimension at most p is called the *p-skeleton* of X, denoted by $X^{(p)}$. $X^{(p)}$ is called *complete* if for every set $\tau \in 2^V$ with $|\tau| \leq p + 1$, it holds that $\tau \in X$. We denote the set of all i-simplices of X by X_i, it follows that X_0 is the set of the vertices of X.

In the rest of this chapter we introduce basics of (co-)homology, which are important tools in algebraic topology. Throughout this chapter let $X \subseteq 2^V$ be a k-dimensional simplicial complex for $k \geq 1$, $V = \{w_0, w_1, \dots, w_n\}$, $n \in \mathbb{N}$.

Throughout the thesis assume we have a fixed linear ordering $w_0 < w_1 < \cdots < w_n$ on the vertex set $V = X_0$.

Definition 2.2 *Consider an i-simplex* $\tau = \{v_0, v_1, \ldots, v_i\}$ *where* $v_0 < v_1 < \cdots < v_i$. *For an* $(i-1)$*-simplex* $\sigma \in X_{i-1}$ *the* oriented incidence number $[\tau : \sigma]$ *is defined as*

$$[\tau : \sigma] = \begin{cases} (-1)^j & \text{if } \sigma \subseteq \tau \text{ and } \sigma = \tau \setminus \{v_j\}, 0 \leq j \leq i, \\ 0 & \text{otherwise i.e. if } \sigma \nsubseteq \tau. \end{cases}$$

In particular for $v \in X_0$ and the unique face $\emptyset \in X_{-1}$ we have $[v : \emptyset] = 1$.

Definition 2.3 *Let X be a finite simplicial complex and $\mathbb{R}$ the real space. The* group of i-chains *on X is defined as*

$$C^i(X, \mathbb{R}) = \{f : X_i \to \mathbb{R}\},$$

i.e., the group of functions from the i-simplices to $\mathbb{R}$.

For $i < -1$ or $i >$ dim X we define $C^i(X, \mathbb{R}) = 0$, the the trivial space.

Note that since the empty set is the unique element of X_{-1} we have $C^{-1}(X, \mathbb{R}) \cong \mathbb{R}$ and $C^i(X, \mathbb{R})$ is a vector space for all i. Throughout the thesis we shortened write C^i instead of $C^i(X, \mathbb{R})$.

Definition 2.4 *The* boundary operator $\partial_i \colon C^i \to C^{i-1}$ *is defined as*

$$\begin{aligned} \partial_i \colon f \mapsto & \partial_i f \\ & \partial_i f(\sigma) := \sum_{\tau \in X_i} [\tau : \sigma] f(\tau), \end{aligned}$$

for σ an $(i-1)$-simplex.

Similarly the coboundary operator $\delta_{i-1} \colon C^{i-1} \to C^i$ *is defined as*

$$\begin{aligned} \delta_{i-1} \colon f \mapsto & \delta_{i-1} f \\ & \delta_{i-1} f(\tau) := \sum_{\sigma \in X_{i-1}} [\tau : \sigma] f(\sigma), \end{aligned}$$

for τ an i-simplex.

The boundary and coboundary operator have some nice and very useful properties as the following lemmas will show.

Lemma 2.5 *[28]* $\partial_i \partial_{i+1} = 0$ *and* $\delta_i \delta_{i-1} = 0$.

Proof Let σ be an $(i-1)$-simplex and $f \in C^{i+1}$. Then by definition

$$\begin{aligned}\partial_i(\partial_{i+1}f)(\sigma) &= \sum_{\substack{\tau \supseteq \sigma \\ \tau \in X_i}} [\tau : \sigma](\partial_{i+1}f)(\tau) \\ &= \sum_{\substack{\tau \supseteq \sigma \\ \tau \in X_i}} [\tau : \sigma] \sum_{\substack{\rho \supseteq \tau \\ \rho \in X_{i+1}}} [\rho : \tau] f(\rho).\end{aligned}$$

Now consider some $\rho \supseteq \sigma$, $\rho \in X_{i+1}$ where $\rho = \{v_0, v_1, \ldots, v_{i+1}\}, v_0 < v_1 < \cdots < v_{i+1}$ and $\sigma = \rho \setminus \{v_j, v_m\}$ for $v_j < v_m$. Then the term containing $f(\rho)$ appears twice in $\partial_i(\partial_{i+1}f)(\sigma)$ namely for $\tau = \sigma \cup \{v_j\}$ as

$$[\sigma \cup \{v_j\} : \sigma][\rho : \sigma \cup \{v_j\}]f(\rho) = (-1)^j(-1)^m f(\rho)$$

and for $\tau = \sigma \cup \{v_m\}$ as

$$[\sigma \cup \{v_m\} : \sigma][\rho : \sigma \cup \{v_m\}]f(\rho) = (-1)^{m-1}(-1)^j f(\rho)$$

with opposite sign. Hence $\partial_i \partial_{i+1} = 0$. The proof of $\delta_i \delta_{i-1} = 0$ works similarly. Let ρ be an $(i+1)$-simplex and $f \in C^{i-1}$. Then

$$\begin{aligned}\delta_i(\delta_{i-1}f)(\rho) &= \sum_{\substack{\tau \subseteq \rho \\ \tau \in X_i}} [\rho : \tau](\delta_{i-1}f)(\tau) \\ &= \sum_{\substack{\tau \subseteq \rho \\ \tau \in X_i}} [\rho : \tau] \sum_{\substack{\sigma \subseteq \tau \\ \sigma \in X_{i-1}}} [\tau : \sigma] f(\sigma).\end{aligned}$$

Again every term appears twice but with opposite sign and hence $\delta_i \delta_{i-1} = 0$. □

Definition 2.6 *Define*

$$\begin{aligned}Z_i &= ker\partial_i \textit{ the set of } i\text{-cycles}, \\ B_i &= im\partial_{i+1} \textit{ the set of } i\text{- boundaries}, \\ Z^i &= ker\delta_i \textit{ the set of } \text{closed } i\text{-forms}, \\ B^j &= im\delta_{i-1} \textit{ the set of } \text{exact } i\text{-forms}.\end{aligned}$$

Since by Lemma 2.5 we know that $B_i \subseteq Z_i$ and $B^i \subseteq Z^i$, the following are well defined.

$$\begin{aligned}H_i &= Z_i/B_i \textit{ the } i\text{-th homology } \textit{of } X, \\ H^i &= Z^i/B^i \textit{ the } i\text{-th cohomology } \textit{of } X.\end{aligned}$$

Definition 2.7 *We endow C^i with the following* inner product. *For $f, g \in C^i$,*

$$\langle f, g \rangle = \sum_{\tau \in X_i} f(\tau) g(\tau).$$

Lemma 2.8 *Let $f \in C^i$ and $g \in C^{i-1}$. Then*

$$\langle \partial_i f, g \rangle = \langle f, \delta_{i-1} g \rangle,$$

i.e., ∂_i and δ_{i-1} are dual operators.

Note that in some literature δ_{i-1} is defined as the dual operator of ∂_i.

Proof By definition of the scalar product

$$\begin{aligned}\langle \partial_i f, g \rangle &= \sum_{\sigma \in X_{i-1}} (\partial_i f)(\sigma) g(\sigma) \\ &= \sum_{\sigma \in X_{i-1}} \left(\sum_{\tau \in X_i} [\tau : \sigma] f(\tau) \right) g(\sigma)\end{aligned}$$

and

$$\begin{aligned}\langle f, \delta_{i-1} g \rangle &= \sum_{\tau \in X_i} f(\tau)(\delta_{i-1} g)(\tau) \\ &= \sum_{\tau \in X_i} f(\tau) \left(\sum_{\sigma \in X_{i-1}} [\tau : \sigma] g(\sigma) \right).\end{aligned}$$

Note that in both sums for every pair $\sigma \in X_{i-1}, \tau \in X_i$, the term $[\tau : \sigma] f(\tau) g(\sigma)$ appears exactly once and hence both sums are equal. $\square$

Chapter 3

On the Cheeger Inequality

3.1 The Cheeger Inequality for Graphs

Let $G = (V, E)$ be a finite simple undirected graph with vertex set V (where $|V| = n$) and edge set E. Note that every such graph corresponds to a finite abstract 1-dimensional simplicial complex, where V and E correspond to X_0 and X_1 respectively.

Definition 3.1 *The* Cheeger constant $h(G)$ *of* $G = (V, E)$ *is defined as*

$$h(G) = \min_{\substack{A \subseteq V \\ 0<|A|<|V|}} \frac{|V||e(A, V \setminus A)|}{|A||V \setminus A|},$$

where $e(A, V \setminus A)$ *is the set of edges with one endpoint in* A *and the other in* $V \setminus A$.

Observe that $h(G) \neq 0$, if and only if G is connected.

Often the Cheeger constant is defined by

$$\varphi(G) = \min_{\substack{A \subseteq V \\ 0<|A|\leq \frac{|V|}{2}}} \frac{|e(A, V \setminus A)|}{|A|}.$$

Since $\varphi(G) \leq h(G) \leq 2\varphi(G)$ one can freely choose which one to use. $h(G)$ and $\varphi(G)$ are also called *(edge) expansion ratio*.

Definition 3.2 *The* spectral gap $\lambda(G)$ *is defined as the second smallest eigenvalue of the* Laplacian for graphs $L(G) : C^0 \to C^0$, *which is defined by*

$$(L(G)f)(v) = \deg(v)f(v) - \sum_{\substack{w \in V \\ \{v,w\} \in E}} f(w),$$

for $f \in C^0$ *and* $v \in V$.

Equivalently one can define the Laplacian for graphs as a matrix $L(G) \in \mathbb{R}^{n \times n}$. Let D be the diagonal matrix with $\deg(w_i)$ in the i-th entry of the diagonal and A the adjacency matrix of G given by

$$A_{ij} = \begin{cases} 1 & \text{if } (w_i, w_j) \in E, \\ 0 & \text{otherwise,} \end{cases}$$

for $i, j \in \{0, 1, \ldots, n\}$. Then

$$L(G) = D - A.$$

The function $g = 1$ corresponds to the eigenvalue zero for any graph G and since $L(G)$ is positive definite (for a proof see Lemma 3.6) this gives us an incentive to consider the second smallest eigenvalue. An easy calculation also shows that the second smallest eigenvalue is not equal zero if and only if G is connected (see for example [14]).

The discrete Cheeger inequality relates the Cheeger constant to the spectral gap, we here state the version for d-regular graphs.

Theorem 3.3 ***[Discrete Cheeger Inequality for d-Regular Graphs]*** *[30, 9, 2, 1] Let $G = (V, E)$ be a d-regular graph. Then*

$$\lambda(G) \leq h(G) \leq \sqrt{8d\lambda(G)}.$$

Before proving the lower bound for general graphs, we discuss why this inequality is useful.

For instance the spectral gap is known to be computable in polynomial time, whereas the Cheeger constant is hard to analyze. It is known that it is NP-hard to compute the latter for a given graph [23, 5]. Even the easy lower bound is used to prove edge expansion on a graph. There are constructions of families $(G_n)_{n \in \mathbb{N}}$ of graphs on n vertices with constant edge degree, where the expansion ratio is bounded from below by a constant [21, 11, 20, 22, 3]. The proofs that those graphs are expanders are obtained by analyzing their eigenvalues.

For the proof of the lower bound of the discrete Cheeger inequality, we introduce Rayleigh's principle which will also be essential throughout the thesis. For a longer introduction see for example [16].

Rayleigh's principle The main statement of Rayleigh's principle is that for a symmetric matrix $A \in \mathbb{R}^{n \times n}$

$$\mathrm{minSpec}(A) = \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\langle Ax, x \rangle}{\langle x, x \rangle} \text{ and}$$

$$\text{maxSpec}(A) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\langle Ax, x \rangle}{\langle x, x \rangle}.$$

By basic linear algebra we know that if A is symmetric, then there exists a orthonormal basis of eigenvectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$. Then every vector $x \in \mathbb{R}^n$ can be expressed as

$$x = \sum_{i=1}^{n} \alpha_i v_i, \text{ where } \alpha_i = \langle x, v_i \rangle.$$

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A corresponding to $v_1, v_2, \ldots, v_n$. For $x \in \mathbb{R}^n \setminus \{0\}$ it follows that

$$\begin{aligned}
\frac{\langle Ax, x \rangle}{\langle x, x \rangle} &= \frac{\langle A \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \rangle}{\langle \sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i \rangle} \\
&= \frac{\sum_{i=1}^n \alpha_i^2 \lambda_i}{\sum_{i=0}^n \alpha_i^2}.
\end{aligned}$$

Since A is symmetric we know that $\lambda_i \geq 0$ and it follows that $\frac{\langle Ax, x \rangle}{\langle x, x \rangle} \in [\lambda_1, \lambda_n]$, where the bound is tight for $x = v_1$ and $x = v_n$ respectively.

Proof of the lower bound of Theorem 3.3 Let $A, V \setminus A$ be a partition that realizes the minimum $h(G)$. Define $f \in C^1$ as

$$f(v) = \begin{cases} |V \setminus A| & \text{if } v \in A, \\ -|A| & \text{if } v \in V \setminus A. \end{cases}$$

Since we have seen that the eigenvalue is zero for $g = 1$ and $\langle f, g \rangle = 0$, by Rayleigh's principle it follows that

$$\begin{aligned}
\lambda(G) &= \min_{q \perp g} \frac{\langle L(G) q, q \rangle}{\langle q, q \rangle} \\
&\leq \frac{\langle L(G) f, f \rangle}{\langle f, f \rangle}.
\end{aligned}$$

Considering the numerator and denominator separately, by definition

$$\begin{aligned}
\langle L(G) f, f \rangle &= \sum_{v \in V} (\deg(v) f(v) - \sum_{\substack{w \in V \\ \{v,w\} \in E}} f(w)) f(v) \\
&= \sum_{v \in A} |e(v, V \setminus A)| (|V \setminus A| + |A|) |V \setminus A| \\
&\quad + \sum_{v \in A \setminus V} |e(v, A)| (-|A| - |V \setminus A|)(-|A|) \\
&= |V|^2 |e(A, V \setminus A)|
\end{aligned}$$

and

$$\begin{aligned}\langle f, f\rangle &= \sum_{v\in V} f(v)^2 \\ &= |A||V\setminus A|^2 + |V\setminus A||A|^2 \\ &= |V||A||V\setminus A|.\end{aligned}$$

Plugging into the above inequality we obtain

$$\lambda(G) \leq \frac{|V|^2|e(A, V\setminus A)|}{|V||A||V\setminus A|} = h(G). \qquad \square$$

3.2 Expansion Properties of Simplicial Complexes

The question which we study is whether some form of the discrete Cheeger inequality holds for k-simplicial complexes X. Of course one needs suitable definitions for the spectral gap $\lambda(X)$ and the Cheeger constant $h(X)$ of simplicial complexes. Throughout the chapter let X be a k-dimensional simplicial complex where $k \in \mathbb{N}$ and $X_0 = V$.

Parzanchevski, Rosenthal and Tessler showed, that for suitable definitions of $\lambda(X)$ and $h(X)$ and X with complete $(k-1)$-skeleton $X^{(k-1)}$, the lower bound $\lambda(X) \leq h(X)$ holds [27]. (This is in Section 3.3.) The main result of this thesis is the proof of $\lambda(X) \leq h(X)$ for arbitrary simplicial complexes (Theorem 4.1).

In the just mentioned paper of Parzanchevski et al. it is also shown that $\lambda(X) = 0$ does not imply $h(X) = 0$, hence no higher dimensional analogue of the upper bound of the Cheeger inequality of form $C \cdot h(X)^m \leq \lambda(X)$ can be found. Therefore throughout the paper we call the inequality $\lambda(X) \leq h(X)$ "the Cheeger inequality for simplicial complexes", although technically it is only "the lower bound of the Cheeger inequality for simplicial complexes".

Since $h(G) = \min_{\substack{A\subseteq V \\ 0<|A|<|V|}} \frac{|V||e(A,V\setminus A)|}{|A||V\setminus A|}$ compares that ratio between the edges $e(A, V\setminus A)$ and the maximal possible number of edges between A and $V\setminus A$ (i.e., $|A||V\setminus A|$) the following combinatorially motivated definition of $h(X)$ arises.

Definition 3.4 *The* Cheeger constant of a k-complex X *is defined as*

$$h(X) = \min_{\substack{V=\amalg_{i=0}^k A_i \\ A_i\neq\emptyset}} \frac{|V||F(A_0, A_1, \ldots, A_k)|}{|F^\partial(A_0, A_1, \ldots, A_k)|},$$

where $F(A_0, A_1, \dots, A_k)$ is the set of k-dimensional faces of X with one vertex in each A_i and

$$F^\partial(A_0, A_1, \dots, A_k) = \{\tau^\partial \in 2^V : \tau^\partial \setminus \{v\} \in X_{k-1} \text{ for all } v \in \tau^\partial\}.$$

Therefore, for a given $(k-1)$-skeleton $X^{(k-1)}$, $|F^\partial(A_0, A_1, \dots, A_k)|$ is the maximum number of k-faces with one vertex in each A_i, that a simplicial complex with skeleton $X^{(k-1)}$ can have. In particular $F(A_0, A_1, \dots, A_k) \subseteq F^\partial(A_0, A_1, \dots, A_k)$.

For $F^\partial(A_0, A_1, \dots, A_k) = 0$ define $h(X) = \infty$.

We call the A_i's *blocks of the partition* or shortly just *blocks*.

Note that in the paper of Parzanchevski et al. [27] $h(X)$ is defined as

$$h(X) = \min_{\substack{V = \amalg_{i=0}^k A_i \\ A_i \neq \emptyset}} \frac{|V||F(A_0, A_1, \dots, A_k)|}{\prod_{i=0}^k |A_i|}.$$

Since they only consider k-complexes with *complete* $(k-1)$-skeletons this coincides with our definition of the Cheeger constant.

As already stated in the introduction, it is important to know that this is not the only generalization of the Cheeger constant $h(X)$. Another approach, the only possible topologically motivated one, was introduced by Gromov [12], and independently by Linial, Meshulam and Wallach [19, 24] and by Newman and Rabinovich [26]. Gundert and Wagner [13] showed that in this case the Cheeger inequality does not hold, i.e., there exists an infinite family of k-dimensional simplicial complexes which are spectrally expanding (there is a large eigenvalue gap), but not combinatorially expanding. A similar result was shown by Steenbergen, Klivans and Mukerjee [29].

We now move on to the Laplacian of simplicial complexes.

Definition 3.5 *The* upper, lower *and* full Laplacian

$$L_{k-1}^{up}(X), L_{k-1}^{down}(X), L_{k-1}(X) \colon C^{k-1} \to C^{k-1}$$

are defined as

$$L_{k-1}^{up}(X) = \partial_k \delta_{k-1}, L_{k-1}^{down}(X) = \delta_{k-2}\partial_{k-1} \text{ and } L_{k-1}(X) = L_{k-1}^{up}(X) + L_{k-1}^{down}(X)$$

respectively.

More generally one could define the i-th upper Laplacian as $L_i^{\mathrm{up}}(X) = \partial_{i+1}\delta_i$ and the lower and full Laplacian similarly. For our purposes the case where $i = k-1$ suffices.

Explicitly $L_{k-1}^{\mathrm{up}}(X)$ can be expressed as follows. For $\sigma \in X_{k-1}, f \in C^{k-1}$

$$L_{k-1}^{\mathrm{up}}(X)f(\sigma) = (\partial_k(\delta_{k-1}f))(\sigma)$$

$$
\begin{aligned}
&= \sum_{\substack{\tau \supseteq \sigma \\ \tau \in X_k}} [\tau : \sigma] (\delta_{k-1} f)(\tau) \\
&= \sum_{\substack{\tau \supseteq \sigma \\ \tau \in X_k}} [\tau : \sigma] \sum_{\substack{\rho \subseteq \tau \\ \rho \in X_{k-1}}} [\tau : \rho] f(\rho) \\
&= \deg(\sigma) f(\sigma) + \sum_{\substack{\tau \supseteq \sigma \\ \tau \in X_k}} [\tau : \sigma] \sum_{\substack{\rho \subseteq \tau, \rho \neq \sigma \\ \rho \in X_{k-1}}} [\tau : \rho] f(\rho). \qquad (3.1)
\end{aligned}
$$

Note that for $k-1=0$ this coincides with $L(G)$, the Laplacian for graphs given in Definition 3.2.

Before defining the spectral gap $\lambda(X)$ for simplicial complexes we discuss some well known, useful properties of the Laplacian. (For more details see for example [15]).

Lemma 3.6 *The spectrum of $L^{up}_{k-1}(X)$ is contained in the interval $[0,(k+1)d^{\max}_{k-1}]$, where $d^{\max}_{k-1}$ is the maximal degree over all $(k-1)$-simplices.*

Proof Let λ be an eigenvalue of $L^{\text{up}}_{k-1}(X)$ corresponding to $f \in C^{k-1}$. By using Lemma 2.8, i.e., that δ_k and ∂_{k-1} are duals we obtain

$$
\begin{aligned}
\lambda \langle f, f \rangle &= \langle L^{\text{up}}_{k-1}(X) f, f \rangle \\
&= \langle \partial_k \delta_{k-1} f, f \rangle \\
&= \langle \delta_{k-1} f, \delta_{k-1} f \rangle.
\end{aligned}
$$

Therefore

$$
\lambda = \frac{\langle \delta_{k-1} f, \delta_{k-1} f \rangle}{\langle f, f \rangle} \geq 0.
$$

For the upper bound let $\sigma \in X_{k-1}$ such that $|f(\sigma)| = \max_{\sigma' \in X_{k-1}} |f(\sigma')|$. Then by equation (3.1)

$$
\begin{aligned}
|\lambda f(\sigma)| &= |L^{\text{up}}_{k-1}(X) f(\sigma)| \\
&\leq \deg(\sigma)|f(\sigma)| + \sum_{\substack{\tau \supseteq \sigma \\ \tau \in X_k}} \sum_{\substack{\rho \subseteq \tau, \rho \neq \sigma \\ \rho \in X_{k-1}}} |f(\rho)| \\
&\leq \deg(\sigma)|f(\sigma)| + \sum_{\substack{\tau \supseteq \sigma \\ \tau \in X_k}} \sum_{\substack{\rho \subseteq \tau, \rho \neq \sigma \\ \rho \in X_{k-1}}} |f(\sigma)| \\
&\leq d^{\max}_{k-1} |f(\sigma)| + d^{\max}_{k-1} \cdot k |f(\sigma)| \\
&= d^{\max}_{k-1} (k+1) |f(\sigma)|.
\end{aligned}
$$

It follows that $|\lambda| \leq d^{\max}_{k-1}(k+1)$. ☐

We proceed by showing that the kernels of the upper and lower Laplacians coincide with $Z^{k-1} = \ker \delta_{k-1}$ and $Z_{k-1} = \ker \partial_{k-1}$ respectively.

Lemma 3.7 $Z^{k-1} = \ker L^{up}_{k-1}(X)$ *and* $Z_{k-1} = \ker L^{down}_{k-1}(X)$.

Proof Since by definition $L_{k-1}^{\text{up}}(X) = \partial_k \delta_{k-1}$, it follows that $Z^{k-1} \subseteq \ker L_{k-1}^{\text{up}}(X)$.

For the other direction let $f \in \ker L_{k-1}^{\text{up}}(X)$. By Lemma 2.8 we know that ∂_k and δ_{k-1} are dual operators, so

$$0 = \langle L_{k-1}^{\text{up}}(X)f, f\rangle = \langle \partial_k \delta_{k-1} f, f\rangle = \langle \delta_{k-1} f, \delta_{k-1} f\rangle,$$

hence $f \in \ker \delta_{k-1} = Z^{k-1}$. The second part of the proof works similarly. □

Lemma 3.8 $(B^i)^\perp = Z_i$ *and* $(B_i)^\perp = Z^i$.

Proof Let $g \in B^i$ and $f \in Z_i$. Then there exists $\overline{g} \in X_{i-1}$ such that $\delta_{i-1}\overline{g} = g$. Again by Lemma 2.8 we obtain

$$\langle f, g\rangle = \langle f, \delta_{i-1}, \overline{g}\rangle = \langle \partial_i f, \overline{g}\rangle = 0,$$

hence $f \in (B^i)^\perp$.

Now for $\sigma \in X_{i-1}$ define the indicator function $\overline{g}_\sigma \in C^{i-1}$ as

$$\overline{g}(\sigma') = \begin{cases} 1 & \text{if } \sigma' = \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Let $g_\sigma = \delta_{i-1}\overline{g}_\sigma \in B^i$ and $f \in (B^i)^\perp$. Then

$$0 = \langle f, g_\sigma\rangle = \langle f, \delta_{i-1}\overline{g}_\sigma\rangle = \langle \partial_i f, \overline{g}_\sigma\rangle = \partial_i f(\sigma).$$

Hence $\partial_i f(\sigma) = 0$ for all $\sigma \in X_{i-1}$ and therefore $f \in Z_{i-1}$. The second part of the proof works similarly. □

We are finally ready to define the spectral gap for k-dimensional simplicial complexes. Since by Lemma 2.5 and Lemma 3.7 we know that $B^{k-1} \subseteq Z^{k-1} = \ker L_{k-1}^{\text{up}}(X)$, $L_{k-1}^{\text{up}}(X)$ is obviously zero on B^{k-1}. Observe that there exist more zero eigenvalues than the ones corresponding to functions in B^{k-1} if and only if $H^{k-1} \neq 0$ i.e. we have non-trivial $(k-1)$-homology. In the case of graphs B^0 is the set of constant functions. Since $(B^{k-1})^\perp = Z_{k-1}$, the following definition of the spectral gap coincides with $\lambda(G)$ for $k-1=0$.

Definition 3.9 *The* spectral gap *of a k-dimensional simplicial complex X, denoted $\lambda(X)$ is defined as*

$$\lambda(X) = minSpec(L_{k-1}^{up}(X)|_{Z_{k-1}}) = minSpec(L_{k-1}(X)|_{Z_{k-1}}),$$

i.e., the minimal eigenvalue of the upper or full Laplacian that is in $(B^{k-1})^\perp$. Equality holds since $L_{k-1}^{down}(X)|_{Z_{k-1}} = 0$.

It follows that $\lambda(X) \neq 0$ if and only if $H^{k-1} \neq 0$.

3.3 The Cheeger Inequality for k-Complexes with complete $(k-1)$-Skeleton

In this section we repeat the result of Parzanchevski, Rosenthal and Tessler [27] in our own notation.

Theorem 3.10 ***[Cheeger Inequality for k-Complexes with Complete $(k-1)$-Skeleton]*** *Let X be a k-complex with complete $(k-1)$-skeleton. Then*

$$\lambda(X) \leq h(X).$$

By Rayleigh's principle we know that

$$\lambda(X) = \min_{f \in Z_{k-1}} \frac{\langle L^{\mathrm{up}}_{k-1}(X)f, f\rangle}{\langle f, f\rangle}. \tag{3.2}$$

As in the case of graphs the idea of the proof of Theorem 3.10 is to find a function $f \in Z_{k-1}$ that satisfies

$$\frac{\langle L^{\mathrm{up}}_{k-1}(X)f, f\rangle}{\langle f, f\rangle} = h(X).$$

We will see that the following function satisfies this equation. From now on throughout the thesis let $A_0, A_1, \dots, A_k$ be a partition of V which realizes the minimum in $h(X)$.

Definition 3.11 *Let $\sigma = \{v_0, v_1, \dots v_{k-1}\} \in X_{k-1}$, with $v_0 < v_1 < \cdots < v_{k-1}$. Then $f \in C^{k-1}$ is defined as*

$$f(\sigma) = \begin{cases} sign(\pi)|A_{\pi(k)}| & \text{if } \exists \pi \in Sym_{\{0,1,\dots,k\}} \text{ with } v_i \in A_{\pi(i)} \text{ for } 0 \leq i \leq k-1 \\ 0 & \text{otherwise, i.e., if } \exists l, i \neq j \text{ with } v_i, v_j \in A_l. \end{cases}$$

Formulated in words this means, if $\sigma \in X_{k-1}$ has exactly one vertex in each block except in A_i, then the value of $f(\sigma)$ is either $|A_i|$ or $-|A_i|$, depending on the permutation π. In any other case $f(\sigma) = 0$.

$\mathrm{Sym}_{\{0,1,\dots,k\}}$ denotes the group of bijections π of the set $\{0, 1, \dots, k\}$. Recall the *cycle notation* for permutations. Let $\pi \in \mathrm{Sym}_{\{0,1,\dots,k\}}$ be a permutation. Choose some element $i \in \{0, 1, \dots, k\}$. Then the cycle containing i is defined as $(i, \pi(i), \pi^2(i), \dots, \pi^l(i))$ where l is the minimal integer such that $\pi(\pi^l(i)) = i$. Note that one can also start this cycle with any other element which is contained in it. One then consecutively chooses elements that are not contained in any cycles yet and finds their corresponding cycles. π can then be expressed as the concatenation of all such cycles, where cycles containing only one element (i.e., of form $\pi(i) = i$) are omitted.

For example let $\pi \in \mathrm{Sym}_{\{0,1,\ldots,5\}}$ be defined $\pi(0) = 4, \pi(1) = 2, \pi(2) = 3, \pi(3) = 1, \pi(4) = 0$ and $\pi(5) = 5$. In the cycle notation π is expressed as $\pi = (0,4)(1,2,3)$.

Using the function f defined above one can prove the following two statements.

Lemma 3.12 *Let X be a k-dimensional simplicial complex with* complete $(k-1)$-skeleton.

Then $f \in Z_{k-1}$ and $\langle f, f \rangle = |V||F^{\partial}(A_0, A_1, \ldots, A_k)| = |V||A_1||A_2| \cdots |A_k|$.

Lemma 3.13 *Let X be* any *k-dimensional simplicial complex. Then*

$$\langle L^{up}_{k-1}(X) f, f \rangle = \langle \delta_{k-1} f, \delta_{k-1} f \rangle = |V|^2 |F(A_0, A_1, \ldots, A_k)|.$$

The latter lemma does not require any assumptions on the $(k-1)$-skeleton and we will be able to use it for arbitrary k-complexes.

Lemma 3.12 and Lemma 3.13 together yield Theorem 3.10.

Proof of Theorem 3.10 By equation (3.2), Lemma 3.12 and Lemma 3.13 it follows that

$$\lambda(X) \leq \frac{|V|^2 |F(A_0, A_1, \ldots, A_k)|}{|V||F^{\partial}(A_0, A_1, \ldots, A_k)|} = h(X).$$

□

It remains to prove the two lemmas.

Proof of Lemma 3.12 We start by proving that $f \in Z_{k-1} = \ker \partial_{k-1}$. Let $\sigma \in X_{k-2}$ where $\sigma = \{v_0, v_1, \ldots, v_{k-2}\}$, $v_0 < v_1 < \cdots < v_{k-2}$. By definition

$$(\partial_{k-1} f)(\sigma) = \sum_{\tau \in X_{k-1}} [\tau : \sigma] f(\tau).$$

Now if for some $i \neq j, l \in \{0, 1, \ldots, k-2\}$, $v_i, v_j \in A_l$, i.e., v_i and v_j are in the same block, this sum vanishes. Otherwise choose $\pi \in \mathrm{Sym}_{\{0,1,\ldots,k\}}$ with $v_i \in A_{\pi(i)}$ for $i \in \{0, 1, \ldots, k-2\}$. Since $X^{(k-1)}$ is complete it follows that

$$(\partial_{k-1} f)(\sigma) = \sum_{v \in A_{\pi(k-1)}} [\sigma \cup \{v\} : \sigma] f(\sigma \cup \{v\}) + \sum_{v \in A_{\pi(k)}} [\sigma \cup \{v\} : \sigma] f(\sigma \cup \{v\}).$$

Fix some $v \in A_{\pi(k-1)}$. We want to analyze the value of $[\sigma \cup \{v\} : \sigma] f(\sigma \cup \{v\})$ and show that it only depends on the block v is in. By our fixed linear ordering on the vertices, v satisfies exactly one of the following

- $v_j < v < v_{j+1}$ for some $j \in \{0, 1, \ldots, k-3\}$,
- $v < v_0 := v_{j+1}$,

- $v > v_{k-2} := v_j$.

In any case by definition

$$[\sigma \cup \{v\} : \sigma] = (-1)^{j+1}.$$

Since $v \in A_{\pi(k-1)}$ it follows that

$$f(\sigma \cup \{v\}) = \operatorname{sign}(\pi(k-2, k-1, \dots, j))|A_{\pi(k)}| = \operatorname{sign}\pi(-1)^{k-2-j}|A_{\pi(k)}|.$$

Hence we have

$$[\sigma \cup \{v\} : \sigma] f(\sigma \cup \{v\}) = \operatorname{sign}\pi(-1)^{k-1}|A_{\pi(k)}|.$$

Similarly for $v \in A_{\pi(k)}$ one can show

$$[\sigma \cup \{v\} : \sigma] f(\sigma \cup \{v\}) = \operatorname{sign}\pi(-1)^{k}|A_{\pi(k-1)}|.$$

Therefore

$$\begin{aligned}(\partial_{k-1} f)(\sigma) &= \sum_{v \in A_{\pi(k-1)}} [\sigma \cup \{v\} : \sigma] f(\sigma \cup \{v\}) + \sum_{v \in A_{\pi(k)}} [\sigma \cup \{v\} : \sigma] f(\sigma \cup \{v\}) \\ &= \operatorname{sign}\pi(-1)^{k-1}|A_{\pi(k-1)}||A_{\pi(k)}| + \operatorname{sign}\pi(-1)^{k}|A_{\pi(k)}||A_{\pi(k-1)}| \\ &= 0\end{aligned}$$

and hence $f \in Z_{k-1}$.

It remains to prove that

$$\langle f, f \rangle = |V||F^{\partial}(A_0, A_1, \dots, A_k)|.$$

By definition $\langle f, f \rangle = \sum_{\tau \in X_{k-1}} f(\tau)^2$. A $(k-1)$-simplex τ contributes to this sum if and only if all its vertices are in different blocks of the partition. In that case there exists a unique block A_i where τ does not have a vertex and the contribution is $|A_i|^2$. Since the $(k-1)$-skeleton is complete, there are exactly $|A_0| \cdots |A_{i-1}||A_{i+1}| \cdots |A_k|$ simplices of that form. It follows that

$$\langle f, f \rangle = \sum_{i=0}^{k} \Big(\prod_{\substack{j=0 \\ j \neq i}}^{k} |A_j|\Big)|A_i|^2 = \prod_{i=0}^{k} |A_i||V| = |V||F^{\partial}(A_0, A_1, \dots, A_k)|.$$

□

It is important in this proof, that X has complete $(k-1)$-skeleton. We will see in the following chapter that the main obstacle in the proof for non-complete $(k-1)$-skeletons will be to find a way around Lemma 3.12.

Proof of Lemma 3.13 Let $\tau = \{v_0, v_1, \ldots, v_k\} \in X_k$ with $v_0 < v_1 < \cdots < v_k$.
Claim.

$$|(\delta_{k-1}f)(\tau)| = \begin{cases} |V| & \text{if } \tau \in F(A_0, A_1, \ldots, A_k), \\ 0 & \text{otherwise.} \end{cases}$$

This claim proves our lemma since together with Lemma 2.8 it implies

$$\begin{aligned} \langle L^{\text{up}}_{k-1}(X)f, f\rangle &= \langle \delta_{k-1}f, \delta_{k-1}f\rangle \\ &= \sum_{\tau\in X_k} ((\delta_{k-1}f)(\tau))^2 \\ &= \sum_{\tau\in F(A_0,A_1,\ldots,A_k)} ((\delta_{k-1}f)(\tau))^2 \\ &= |V|^2|F(A_0, A_1, \ldots, A_k)|. \end{aligned}$$

It remains to prove the claim. First suppose that $\tau \notin F(A_0, A_1, \ldots, A_k)$. If τ has three vertices in the same block A_i or four vertices in two blocks, then every term $\tau \setminus \{v_i\}$ in

$$(\delta_{k-1}f)(\tau) = \sum_{i=0}^{k} [\tau : \tau \setminus \{v_i\}] f(\tau \setminus \{v_i\})$$

has two vertices in the same block and hence

$$(\delta_{k-1}f)(\tau) = 0.$$

Assume now that v_j, v_l with $v_j < v_l$ is the only pair of vertices in the same block. Then the two non-vanishing terms are

$$(\delta_{k-1}f)(\tau) = [\tau : \tau \setminus \{v_j\}] f(\tau \setminus \{v_j\}) + [\tau : \tau \setminus \{v_l\}] f(\tau \setminus \{v_l\}).$$

Let $\pi \in \mathrm{Sym}_{\{0,1,\ldots,k\}}$ such that

$$\begin{aligned} v_i &\in A_{\pi(i)} \text{ for } i < l, \\ v_i &\in A_{\pi(i-1)} \text{ for } i > l. \end{aligned}$$

In particular $v_l \in A_{\pi(j)}$. It follows that

$$\begin{aligned} [\tau : \tau \setminus \{v_j\}] f(\tau \setminus \{v_j\}) &= (-1)^j \mathrm{sign}(\pi(l-1, l-2, \ldots, j))|A_{\pi(k)}| \\ &= (-1)^{l-1} \mathrm{sign}\pi |A_{\pi(k)}| \text{ and} \\ [\tau : \tau \setminus \{v_l\}] f(\tau \setminus \{v_l\}) &= (-1)^l \mathrm{sign}\pi |A_{\pi(k)}|. \end{aligned}$$

Therefore

$$\begin{aligned} (\delta_{k-1}f)(\tau) =& [\tau : \tau \setminus \{v_j\}] f(\tau \setminus \{v_j\}) + [\tau : \tau \setminus \{v_l\}] f(\tau \setminus \{v_l\}) \\ =& ((-1)^{l-1} + (-1)^l) \mathrm{sign}\pi |A_{\pi(k)}| = 0. \end{aligned}$$

It remains to show the case for $\tau \in F(A_0, A_1, \ldots, A_k)$. In this case there exists $\pi \in \mathrm{Sym}_{\{0,1,\ldots,k\}}$ such that $v_i \in A_{\pi(i)}$ for $0 \leq i \leq k$. Then

$$\begin{aligned} f(\tau \setminus \{v_i\}) = \mathrm{sign}(\pi(k, k-1, \ldots, i))|A_{\pi(i)}| \\ =(-1)^{k-i}\mathrm{sign}\pi|A_{\pi(i)}|. \end{aligned}$$

Therefore

$$\begin{aligned} (\delta_{k-1} f)(\tau) &= \sum_{i=0}^{k} (-1)^i f(\tau \setminus \{v_i\}) \\ &= \sum_{i=0}^{k} (-1)^k \mathrm{sign}\pi |A_{\pi(i)}| \\ &= (-1)^k \mathrm{sign}\pi |V|. \end{aligned}$$

□

Chapter 4

The Cheeger Inequality for Arbitrary k-Complexes

4.1 The Main Result

Throughout this chapter let X be an arbitrary k-dimensional simplicial complex for $k \geq 1$. As shown in the previous chapter, in case where X has a complete $(k-1)$-skeleton, the lower bound of the Cheeger inequality $\lambda(X) \leq h(X)$ holds. One can easily check that for non complete $(k-1)$-skeletons in general $f \notin Z_{k-1}$. Hence Lemma 3.12 cannot be used in the case of arbitrary simplicial complexes.

In this section we will show that the Cheeger inequality holds for arbitrary k-complexes, namely

Theorem 4.1 *[The Cheeger Inequality for Arbitrary Simplicial Complexes]*

$$\lambda(X) \leq h(X).$$

Before proving Theorem 4.1 we show a somewhat weaker statement that is easier to prove, namely

Theorem 4.2

$$\frac{1}{k+1}\lambda(X) \leq h(X).$$

Before moving on to the proof consider the following lemma.

Lemma 4.3 *Let $f \in C^{k-1}$ be as previously defined. Then there exist unique $z \in Z_{k-1}$, $b \in B^{k-1}$ such that $f = z + b$ and*

$$\lambda(X) \leq \frac{\langle L_{k-1}^{up}(X)f, f\rangle}{\langle z, z\rangle} = \frac{\langle L_{k-1}^{up}(X)f, f\rangle}{\langle f, f\rangle - \langle b, b\rangle}.$$

Proof of Lemma 4.3 Since by Lemma 3.8 $Z_{k-1} = (B^{k-1})^{\perp}$ there exist unique $z \in Z_{k-1}$ and $b \in B^{k-1}$ such that $f = z + b$. By Rayleigh's principle it follows that

$$\lambda(X) = \min \operatorname{Spec}(L^{\mathrm{up}}_{k-1}|_{Z_{k-1}}) \leq \frac{\langle L^{\mathrm{up}}_{k-1}(X)z, z\rangle}{\langle z, z\rangle}.$$

The claim follows since

$$\begin{aligned} \langle f, f\rangle &= \langle z, z\rangle + 2\langle z, b\rangle + \langle b, b\rangle \\ &= \langle z, z\rangle + \langle b, b\rangle \text{ and} \\ \langle L^{\mathrm{up}}_{k-1}(X)z, z\rangle &= \langle \delta_{k-1}(f-b), \delta_{k-1}(f-b)\rangle \\ &= \langle \delta_{k-1}f, \delta_{k-1}f\rangle \\ &= \langle L^{\mathrm{up}}_{k-1}(X)f, f\rangle. \end{aligned}$$

□

From now on we always use z and b in the context of the lemma above.

Corollary 4.4

$$\lambda(X) \leq \frac{|V|^2 |F(A_0, A_1, \ldots, A_k)|}{\langle z, z\rangle}.$$

Proof The result follows by Lemma 4.3 and Lemma 3.13. □

In the case of complete $(k-1)$-skeleton we have seen that $\langle z, z\rangle = \langle f, f\rangle$. This is usually not the case for arbitrary simplicial complexes, so we need another way to find a lower bound of $\langle z, z\rangle$.

4.2 Proof of the Weak Version

The following proposition gives us a lower bound on $\langle z, z\rangle$, the unknown term in Corollary 4.4.

Proposition 4.5

$$\langle z, z\rangle \geq \frac{1}{k+1} |V| |F^{\partial}(A_0, A_1, \ldots, A_k)|.$$

To the best of our knowledge there is no straightforward way to compute z (by knowing f) for arbitrary simplicial complexes. Since $\langle z, z\rangle = \|f - b\|^2$, the idea is to show that for any $g \in C^{k-2}$

$$\|f - \delta_{k-2} g\|^2 \geq \frac{1}{k+1} |V| |F^{\partial}(A_0, A_1, \ldots, A_k)|.$$

Since by definition $b \in B^{k-1} = \mathrm{im}\delta_{k-2}$ in particular this implies

$$\|f - b\|^2 \geq \frac{1}{k+1}|V||F^{\partial}(A_0, A_1, \ldots, A_k)|$$

and therefore Proposition 4.5.

Proposition 4.5 suffices to prove Theorem 4.2.

Proof of Theorem 4.2 By Corollary 4.4 we know

$$\lambda(X) \leq \frac{|V|^2|F(A_0, A_1, \ldots, A_k)|}{\langle z, z\rangle}.$$

Plugging in the result of Proposition 4.5 we obtain

$$\begin{aligned}\lambda(X) &\leq \frac{|V|^2|F(A_0, A_1, \ldots, A_k)|}{\frac{1}{k+1}|V||F^{\partial}(A_0, A_1, \ldots, A_k)|}\\ &= (k+1)h(X),\end{aligned}$$

since by definition

$$h(X) = \frac{|V||F(A_0, A_1, \ldots, A_k)|}{|F^{\partial}(A_0, A_1, \ldots, A_k)|}. \qquad \square$$

For the proof of Proposition 4.5, we assume the following lemma which we will prove at the end of this section.

Throughout the chapter let $g \in C^{k-2}$ be an arbitrary function.

Lemma 4.6 *Let $\tau^{\partial} \in F^{\partial}(A_0, A_1, \ldots, A_k)$. Then*

$$\sum_{\substack{\sigma \subseteq \tau^{\partial} \\ \sigma \in X_{k-1}}} (f(\sigma) - \delta_{k-2}g(\sigma))^2 \geq \frac{1}{k+1}|V|^2.$$

Proof of Proposition 4.5 Consider the following inequality.

$$\sum_{\tau^{\partial} \in F^{\partial}(A_0, A_1, \ldots, A_k)} \sum_{\substack{\sigma \subseteq \tau^{\delta} \\ \sigma \in X_{k-1}}} (f(\sigma) - \delta_{k-2}g(\sigma))^2 \leq |V| \sum_{\sigma \in X_{k-1}} (f(\sigma) - \delta_{k-2}g(\sigma))^2. \tag{4.1}$$

Note that in the left hand sum for any $\sigma \in X_{k-1}$ the term $(f(\sigma) - \delta_{k-2}g(\sigma))^2$ appears at most $\max_i |A_i| \leq |V|$ times. Therefore every term on the left hand side is counted at most $|V|$ times and the inequality follows.

By inequality (4.1) and Lemma 4.6 it follows that

$$\|f - \delta_{k-2}g\|^2 = \sum_{\sigma \in X_{k-1}} (f(\sigma) - \delta_{k-2}g(\sigma))^2$$

$$\geq \frac{1}{|V|} \sum_{\tau^\partial \in F^\partial(A_0,A_1,\ldots,A_k)} \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} (f(\sigma) - \delta_{k-2}g(\sigma))^2$$
$$\geq \frac{1}{|V|} \sum_{\tau^\partial \in F^\partial(A_0,A_1,\ldots,A_k)} \frac{1}{k+1}|V|^2$$
$$= \frac{1}{k+1}|V||F^\partial(A_0, A_1, \ldots, A_k)|.$$

Since g was arbitrary, in particular

$$\langle z, z\rangle = \|f - b\|^2 \geq \frac{1}{k+1}|V||F^\partial(A_0, A_1, \ldots, A_k)|. \qquad \square$$

The only thing left to complete the proof of Theorem 4.2 is to prove Lemma 4.6.

Proof of Lemma 4.6 In the first part of the proof assume that $\tau^\partial = \tau \in F(A_0, A_1, \ldots, A_k)$. We will see later that the proof for $\tau^\partial \in F^\partial(A_0, A_1, \ldots, A_k)$ works almost analougusly.

Let $\tau = \{v_0, v_1, \ldots, v_k\} \in F(A_0, A_1, \ldots, A_k)$ with $v_0 < v_1 < \cdots < v_k$, and $\pi \in \mathrm{Sym}_{\{0,1,\ldots,k\}}$ such that $v_i \in A_{\pi(i)}$, for all $i \in \{0, 1, \ldots, k\}$. Then

$$\sum_{\substack{\sigma \subseteq \tau \\ \sigma \in X_{k-1}}} (f(\sigma) - \delta_{k-2}g(\sigma))^2 = \sum_{i=0}^{k}(f(\tau \setminus \{v_i\}) - \delta_{k-2}g(\tau \setminus \{v_i\}))^2.$$

As in the proof of Lemma 3.13 we observe that

$$f(\tau \setminus \{v_i\}) = (-1)^{k-i}\mathrm{sign}\pi|A_{\pi(i)}|.$$

Therefore it follows that

$$\begin{aligned}
&\sum_{i=0}^{k} (f(\tau \setminus \{v_i\}) - \delta_{k-2}g(\tau \setminus \{v_i\}))^2 \\
&\quad = \sum_{i=0}^{k} \Big((-1)^{k-i}\mathrm{sign}\pi|A_{\pi(i)}| - \delta_{k-2}g(\tau \setminus \{v_i\})\Big)^2 \\
&\quad = \sum_{i=0}^{k} \Big((-1)^{k}\mathrm{sign}\pi|A_{\pi(i)}| - (-1)^i\delta_{k-2}g(\tau \setminus \{v_i\})\Big)^2 \\
&\quad = \sum_{i=0}^{k} \Big((-1)^{k}\mathrm{sign}\pi|A_{\pi(i)}| - [\tau : \tau \setminus \{v_i\}]\delta_{k-2}g(\tau \setminus \{v_i\})\Big)^2 \\
&\quad \geq \frac{1}{k+1} \left(\sum_{i=0}^{k}(-1)^{k}\mathrm{sign}\pi|A_{\pi(i)}| - \sum_{i=0}^{k}[\tau : \tau \setminus \{v_i\}]\delta_{k-2}g(\tau \setminus \{v_i\})\right)^2,
\end{aligned} \tag{4.2}$$

where the last step follows from the Cauchy-Schwarz inequality. Note that by definition of the coboundary operator

$$\delta_{k-1}(\delta_{k-2}g)(\tau) = \sum_{i=0}^{k}[\tau : \tau \setminus \{v_i\}]\delta_{k-2}g(\tau \setminus \{v_i\}).$$

Hence by Lemma 2.5 the second term of (4.2) vanishes, i.e.,

$$\sum_{i=0}^{k}[\tau : \tau \setminus \{v_i\}]\delta_{k-2}g(\tau \setminus \{v_i\}) = 0.$$

Therefore

$$\begin{aligned}\sum_{i=0}^{k}(f(t \setminus \{v_i\}) - \delta_{k-2}g(\tau \setminus \{v_i\}))^2 &\geq \frac{1}{k+1}\left(\sum_{i=0}^{k}(-1)^k \mathrm{sign}\pi |A_{\pi(i)}|\right)^2 \\ &= \frac{1}{k+1}|V|^2\end{aligned}$$

as desired.

It remains to prove the proposition for $\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)$ where $\tau^\partial = \{v_0, v_1, \dots, v_k\}$ with $v_0 < v_1 < \cdots < v_k$. Observe that the whole proof works analogously except for the part that we have not defined the "incidence number" $[\tau^\partial : \sigma]$ for $\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k) \setminus F(A_0, A_1, \dots, A_k)$. Define it the obvious way as in Definition 2.2, namely as

$$[\tau^\partial : \sigma] = \begin{cases}(-1)^i & \text{if } \sigma \subseteq \tau^\partial \text{ and } \sigma = \tau^\partial \setminus \{v_i\}, 0 \leq i \leq k \\ 0 & \text{otherwise, i.e., if } \sigma \nsubseteq \tau^\partial.\end{cases}$$

We observe that $\delta_{k-1}\delta_{k-2}(\tau^\partial) = 0$. (The proof works identically as the proof of $\delta_{k-1}\delta_{k-2}(\tau) = 0$ for $\tau \in F(A_0, A_1, \dots, A_k)$ in Lemma 2.5.) Consider the complex X_{complete} defined as the k-complex with maximum possible size of $|X_k|$ for the given $(k-1)$-skeleton $X^{(k-1)}$. Then the "incidence number" and "coboundary operator" for X as defined above are nothing but the usual incidence number and coboundary operator on X_{complete}.

With this definition the proof of the Lemma works identically for $\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)$ as for $\tau \in F(A_0, A_1, \dots, A_k)$, which concludes the proof of Theorem 4.2. □

4.3 Proof of the Strong Version

We now proceed to the proof of Theorem 4.1, i.e., that $\lambda(X) \leq h(X)$. The main weakness of the proof of Theorem 4.1 is inequality (4.1), which is obviously not tight, since it is not possible that every $\sigma \in X_{k-1}$ is contained in $|V|$ different τ^∂'s.

Definition 4.7 *For $\sigma \in X_{k-1}$ define*

$$d(\sigma) = \left|\{\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k) \colon \sigma \subseteq \tau^\partial\}\right|.$$

Note that if σ has vertices in $k-1$ different blocks A_j, by definition $d(\sigma) \leq |A_i|$, where A_i is the unique block such that $\sigma \cap A_i = \emptyset$. This bound is tight for the case where X has complete $(k-1)$-skeleton.

Consider the following sum

$$\sum_{\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)} \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2.$$

Note that for any $\sigma \in X_{k-1}$ such that $\sigma \subseteq \tau^\partial$ for some $\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)$ the term

$$\frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2$$

appears exactly $d(\sigma)$ times by definition.

For $\sigma \nsubseteq \tau^\partial$ the term does not appear at all. Hence

$$\begin{aligned}
&\sum_{\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)} \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2 \\
&= \sum_{\substack{\sigma \in X_{k-1} \\ \sigma \subseteq \tau^\partial, \tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)}} d(\sigma) \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2 \\
&\leq \sum_{\sigma \in X_{k-1}} (f(\sigma) - \delta_{k-2} g(\sigma))^2 \\
&= \|f - \delta_{k-2} g\|^2
\end{aligned} \tag{4.3}$$

Remember that for the proof of the weak version we used the weaker inequality

$$\|f - \delta_{k-2} g\|^2 \geq \frac{1}{|V|} \sum_{\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)} \sum_{\substack{\sigma \in \tau^\partial \\ \sigma \in X_{k-1}}} (f(\sigma) - \delta_{k-2} g(\sigma))^2.$$

We use the better new bound to our advantage to prove the following proposition, which gives a stronger statement than Proposition 4.5.

Proposition 4.8

$$\langle z, z \rangle \geq |V| |F^\partial(A_0, A_1, \dots, A_k)|.$$

This is obviously enough to prove Theorem 4.1.

Proof of Theorem 4.1 Plugging in the result of Proposition 4.8 into Corollary 4.4 we get

$$\lambda(X) \leq \frac{|V|^2 |F(A_0, A_1, \dots, A_k)|}{|V||F^\partial(A_0, A_1, \dots, A_k)|} = h(X). \qquad \square$$

The main focus of this section is on proving the following Lemma, which implies Proposition 4.8.

Lemma 4.9 *For any $\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)$ and $g \in C^{k-2}$*

$$q(\tau^\partial, g) := \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2 \geq |V|.$$

Proof of Proposition 4.8. We have observed in inequality (4.3) that

$$\|f - \delta_{k-2} g\|^2 \geq \sum_{\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)} \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2.$$

Hence by Lemma 4.9

$$\begin{aligned}
\langle z, z \rangle \geq & \sum_{\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)} \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2 \\
\geq & \sum_{\tau^\partial \in F^\partial(A_0, A_1, \dots, A_k)} |V| \\
= & |F^\partial(A_0, A_1, \dots, A_k)||V|. \qquad \square
\end{aligned}$$

It remains to prove Lemma 4.9.

Proof of Lemma 4.9 As in the proof of Lemma 4.6 assume that $\tau^\partial = \tau \in F(A_0, A_1, \dots, A_k)$. Let $\tau = \{v_0, v_1, \dots, v_k\}$ with $v_0 < v_1 < \cdots < v_k$ and $\pi \in \mathrm{Sym}_{\{0,1,\dots,k\}}$ such that $v_i \in A_{\pi(i)}$ for all $i \in \{0, 1, \dots, k\}$. For simplicity of notation let $d_i := d(\tau \setminus \{v_i\})$. Therefore

$$\begin{aligned}
q(\tau, g) = & \sum_{\substack{\sigma \subseteq \tau \\ \sigma \in X_{k-1}}} \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2 \\
= & \sum_{i=0}^{k} \frac{1}{d_i} \left(f(\tau \setminus \{v_i\}) - \delta_{k-2} g(\tau \setminus \{v_i\}) \right)^2.
\end{aligned}$$

The same way as in the proof of Proposition 4.5 we find that

$$q(\tau, g) = \sum_{i=0}^{k} \frac{1}{d_i} \left((-1)^k \mathrm{sign}\pi |A_{\pi(i)}| - [\tau : \tau \setminus \{v_i\}] \delta_{k-2} g(\tau \setminus \{v_i\}) \right)^2,$$

where for

$$x_i := -[\tau : \tau \setminus \{v_i\}]\delta_{k-2}g(\tau \setminus \{v_i\})$$

again it holds that

$$-\sum_{i=0}^{k} x_i = \delta_{k-1}(\delta_{k-2}g)(\tau) = 0,$$

and hence

$$\sum_{i=0}^{k-1} x_i = -x_k.$$

Unfortunately, here one can not proceed by just using the Cauchy-Schwarz inequality since the unknown terms containing $\delta_{k-2}g(\tau \setminus \{v_i\})$ will not cancel out as beautifully as in the proof of Lemma 4.6. To make life easier we assume without loss of generality that $(-1)^k \mathrm{sign}\pi = 1$. Otherwise just replace g by $-g$.

Instead of $q(\tau, g)$ we now study the function $q(\tau, x) = \sum_{i=0}^{k} \frac{1}{d_i}(|A_{\pi(i)}| + x_i)^2$, which we try to minimize.

By definition of $d(\sigma)$ we know that

$$d(\tau \setminus \{v_i\}) \leq |A_{\pi(i)}|.$$

Therefore

$$\begin{aligned} q(\tau, x) &= \sum_{i=0}^{k} \frac{1}{d_i}(|A_{\pi(i)}| + x_i)^2 \\ &= \sum_{i=0}^{k-1} \frac{1}{d_i}(|A_{\pi(i)}| + x_i)^2 + \frac{1}{d_k}\left(|A_{\pi(k)}| - \sum_{i=0}^{k-1} x_i\right)^2 \qquad (4.4) \\ &\geq \sum_{i=0}^{k-1} \frac{1}{|A_{\pi(i)}|}\Big(|A_{\pi(i)}| + x_i\Big)^2 + \frac{1}{|A_{\pi(k)}|}\left(|A_{\pi(k)}| - \sum_{i=0}^{k-1} x_i\right)^2 =: \tilde{q}(\tau, x) \end{aligned}$$

The idea is to check the partial derivatives of $\tilde{q}(\tau, x)$ to find a lower bound on $q(\tau, x)$ by determining the minimum of $\tilde{q}(\tau, x)$. By elementary calculus we know that if the gradient is zero in x, then x is a local minimum, local maximum or saddle point. Hence the extremal points must satisfy for all $i \in \{0, 1, \ldots, k-1\}$

$$\frac{\partial \tilde{q}(\tau, x)}{\partial x_i} = \frac{2}{|A_{\pi(i)}|}\Big(|A_{\pi(i)}| + x_i\Big) - \frac{2}{|A_{\pi(k)}|}\left(|A_{\pi(k)}| - \sum_{j=0}^{k-1} x_j\right) = 0,$$

and therefore for all $i \in \{0, 1, \ldots, k-1\}$

$$x_i + \sum_{j=0}^{k-1} x_j = 0. \qquad (4.5)$$

By subtracting the m-th equation from the l-th equation in (4.5) we obtain

$$x_l - x_m = 0.$$

Hence it must hold that

$$x_0 = x_1 = \cdots = x_{k-1},$$

and so

$$x_0 = x_1 = \cdots = x_{k-1} = 0$$

is the only solution to the equality system (4.5). Since

$$\lim_{|x_0|+|x_1|+\cdots+|x_{k-1}|\to\infty} \tilde{q}(\tau, x) = +\infty$$

it follows that $\tilde{q}(\tau, x)$ attains its unique global minimum for $x = (x_0, x_1, \ldots, x_{k-1}) = 0$. Therefore

$$\begin{aligned} q(\tau, x) \geq \tilde{q}(\tau, x) \geq \tilde{q}(\tau, 0) \\ &= \sum_{i=0}^{k-1} \frac{1}{|A_{\pi(i)}|} \left(|A_{\pi(i)}|\right)^2 + \frac{1}{|A_{\pi(k)}|} \left(|A_{\pi(k)}|\right)^2 \\ &= \sum_{i=0}^{k} |A_{\pi(i)}| = |V|. \end{aligned}$$

As in the proof of Proposition 4.5 the case for $\tau^\partial \in F^\partial(A_0, A_1, \ldots, A_k)$ follows analogously. $\square$

4.4 Case of Sparse $(k-1)$-Skeletons

In the proof of Theorem 4.1 we used the inequality $d_i \leq |A_{\pi(i)}|$. This is obviously tight in the case where X has complete $(k-1)$-skeleton, but may become arbitrarily bad for "sparse" $(k-1)$-skeletons. How tight is the bound of the Cheeger inequality in those cases? To get some idea of what we mean by sparsity in this context consider the following definition.

Definition 4.10

$$X_k^\partial = \{\tau^\partial \subseteq 2^V : |\tau^\partial| = k+1 \wedge (\sigma \subseteq \tau^\partial, |\sigma| = k \Rightarrow \sigma \in X)\}.$$

As introduced earlier in the chapter one can think of $X_{\text{complete}} = X \cup X_k^\partial$ as the k-complex with the maximum possible size of $|X_k|$ for the given $(k-1)$-skeleton $X^{(k-1)}$ of X. Intuitively X is sparse if every $(k-1)$-simplex is contained in only a few elements of X_k^∂.

Lemma 4.11 *For any $\tau^\partial \in F^\partial(A_0, A_1, \ldots, A_k)$ with $\tau^\partial = \{v_0, v_1, \ldots, v_k\}$, $v_0 < v_1 < \cdots < v_k$ it holds that*

$$\begin{aligned} q(\tau^\partial, g) &= \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2 \\ &\geq \frac{1}{\sum_{j=0}^k d_j} |V|^2. \end{aligned}$$

Note that for $d_j = |A_{\pi(j)}|$ this is the same bound as given in Proposition 4.9. Before continuing to the proof observe the following consequence.

Proposition 4.12 *Let $C = \max_{\tau^\partial \in F^\partial(A_0, A_1, \ldots, A_k)} \sum_{\substack{\sigma \in \tau^\partial \\ \sigma \in X_{k-1}}} d(\sigma)$. Then*

$$\lambda(X) \leq \frac{C}{|V|} h(X).$$

Since $C \leq n$ by definition, this is a stronger statement than the Cheeger inequality for simplicial complexes.

It follows that if every $(k-1)$-simplex is contained in at most c elements of X_k^∂ then $C \leq c \cdot (k+1)$ and hence

$$|V| \lambda(X) \leq c \cdot (k+1) \cdot h(X).$$

One can see that this is much stronger than the Cheeger inequality for simplicial complexes if c is small.

Proof Using Lemma 4.11 in inequality (4.3) we get

$$\begin{aligned} \langle z, z \rangle &\geq \| f - \delta_{k-2} g \|^2 \\ &\geq \sum_{\tau^\partial \in F^\partial(A_0, A_1, \ldots, A_k)} \sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} \frac{1}{d(\sigma)} (f(\sigma) - \delta_{k-2} g(\sigma))^2 \\ &\geq \sum_{\tau^\partial \in F^\partial(A_0, A_1, \ldots, A_k)} \frac{1}{\sum_{\substack{\sigma \subseteq \tau^\partial \\ \sigma \in X_{k-1}}} d(\sigma)} |V|^2 \\ &\geq |F^\partial(A_0, A_1, \ldots, A_k)| \cdot \frac{1}{C} \cdot |V|^2. \end{aligned}$$

Hence by Corollary 4.4

$$\begin{aligned} \lambda(X) &\leq \frac{\langle L_{k-1}^{\mathrm{up}}(X) f, f \rangle}{\langle z, z \rangle} \\ &\leq \frac{|V|^2 |F(A_0, A_1, \ldots, A_k)|}{\frac{1}{C} |V|^2 |F^\partial(A_0, A_1, \ldots, A_k)|} = \frac{1}{|V|} C \cdot h(X). \end{aligned}$$ □

Proof of Lemma 4.11 The first part of the proof works analogously as the proof of Proposition 4.9. Up to equation (4.4) we have not used the estimate that $d_i \leq |A_{\pi(i)}|$, i.e., our problem is to find a better lower bound for

$$q(\tau, x) = \sum_{i=1}^{k-1} \frac{1}{d_i}(|A_{\pi(i)}| + x_i)^2 + \frac{1}{d_k}\left(|A_{\pi(k)}| - \sum_{i=0}^{k-1} x_i\right)^2.$$

By checking the partial derivatives we know that the extremal points must satisfy for all $i \in \{0, 1, \ldots, k-1\}$

$$\frac{\partial q(\tau, x)}{\partial x_i} = \frac{2}{d_i}(|A_{\pi(i)}| + x_i) - \frac{2}{d_k}\left(|A_{\pi(k)}| - \sum_{i=0}^{k-1} x_i\right) = 0. \tag{4.6}$$

Observe that for $i \in \{0, 1, \ldots, k-1\}$

$$y_i = \frac{d_i |V|}{\sum_{j=0}^{k-1} d_j} - |A_{\pi(i)}|$$

satisfies (4.6). Note that

$$\begin{aligned} -\sum_{i=0}^{k-1} y_i &= \sum_{i=0}^{k-1} |A_{\pi(i)}| - \frac{\sum_{i=0}^{k-1} d_i |V|}{\sum_{i=0}^{k} d_j} \\ &= |V| - |A_{\pi(k)}| - \frac{\sum_{i=0}^{k-1} d_i |V|}{\sum_{i=0}^{k} d_j} \\ &= \frac{d_k |V|}{\sum_{j=0}^{k} d_j} - |A_{\pi(k)}|. \end{aligned}$$

We will show that $q(\tau, x)$ attains its unique global minimum in $y = (y_0, y_1, \ldots, y_{k-1})$. To do so, we use the following lemma of calculus that we state without proof.

Lemma 4.13 *[6] Let $f : \mathbb{R}^k \to \mathbb{R}^k$, $k \in \mathbb{N}$ be such that the Hessian matrix exists. If the Hessian matrix is strictly positive-definite, then f is strictly convex and there exists at most one unique global minimum.*

It is a well known fact from basic calculus that in every local minimum the Hessian matrix is positive-definite. Hence showing that the Hessian matrix H_q of $q(\tau, x)$ is positive-definite everywhere suffices to prove our claim that $q(\tau, x)$ attains its unique minimum in y.

By definition of the Hessian matrix we observe

$$H_q(i,i) = \frac{2}{d_i} + \frac{2}{d_k} \text{ for } i \in \{0, 1, \ldots, k-1\} \text{ and}$$

$$H_q(i,j) = \frac{2}{d_k} \text{ for } i \neq j \in \{0, 1, \ldots, k-1\}.$$

Hence for $0 \neq x \in \mathbb{R}^k$ we get

$$\begin{aligned} x^T H x &= \sum_{i=0}^{k-1} \frac{2}{d_i} x_i^2 + \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \frac{2}{d_k} x_i x_j \\ &= \sum_{i=0}^{k-1} \frac{2}{d_i} x_i^2 + \frac{2}{d_k} \left(\sum_{i=0}^{k-1} x_i \right)^2 > 0. \end{aligned}$$

It follows that H_q is positive-definite everywhere and $q(\tau, x)$ attains its unique minimum in y. Then

$$\begin{aligned} q(\tau, x) &\geq q(\tau, y) \\ &= \sum_{i=0}^{k-1} \frac{1}{d_i} \left(\frac{d_i |V|}{\sum_{j=0}^{k} d_k} \right)^2 + \frac{1}{d_k} \left(\frac{d_k |V|}{\sum_{j=0}^{k} d_k} \right)^2 \\ &= \frac{\sum_{i=0}^{k} d_i |V|^2}{\left(\sum_{j=0}^{k} d_j \right)^2} \\ &= \frac{1}{\sum_{i=0}^{k} d_j} |V|^2. \end{aligned}$$

□

Chapter 5

A Weaker Bound on 2-Complexes using Jumbledness Condition

In the course of writing this thesis we first used a different approach for 2-complexes using (p, α)-jumbledness of the underlying graph. Unfortunately, this only gives a bound $(1-\varepsilon)\lambda(X) \leq h(X)$, for $\varepsilon > 0$ in special cases (see Theorem 5.9), which is a weaker statement than the result $\lambda(X) \leq h(X)$ obtained in Theorem 4.1. Since the approach is very different from the one in the previous chapter we still introduce the proof of the weaker statement in this chapter.

5.1 A Different Approach for 2-Complexes

For the remaining part let $X = (V, E, T)$ be a 2-dimensional simplicial complex where $V := X_0$, $E := X_1$ and $T := X_2$. Let $f \in C^1$, $b \in B^1$ and $z \in Z_1$ be defined as before. We have seen in Lemma 4.3 that

$$\lambda(X) \leq \frac{\langle L_1^{\mathrm{up}}(X) f, f\rangle}{\langle f, f\rangle - \langle b, b\rangle}.$$

In Chapter 4 we have found a direct way to get a lower bound of $\langle z, z\rangle = \langle f, f\rangle - \langle b, b\rangle$. The approach here, is to find a bound for $\langle f, f\rangle$ and $\langle b, b\rangle$ individually. Since we do not have a given explicit form of b we will bound $\langle b, b\rangle$ through

$$\lambda_{\mathrm{down}}(X) := \min \operatorname{Spec}(L_1^{\mathrm{down}}(X)\big|_{B_1}) \leq \frac{\langle L_1^{\mathrm{down}}(X) b, b\rangle}{\langle b, b\rangle}.$$

Proposition 5.1

$$\left(1 - \frac{\langle L_1^{down}(X) f, f\rangle}{\lambda_{down}(X)\langle f, f\rangle}\right)\lambda(X) = \left(1 - \frac{\langle \partial_1 f, \partial_1 f\rangle}{\lambda_{down}(X)\langle f, f\rangle}\right)\lambda(X) \leq h(X),$$

where $\lambda_{\text{down}}(X)$ is defined as the smallest non-zero eigenvalue of $L_1^{\text{down}}(X)$. Since $\ker L_1^{\text{down}}(X) = Z_1 = (B^1)^{\perp}$ this is equivalent to saying that $\lambda_{\text{down}}(X)$ is the smallest eigenvalue of $L_1^{\text{down}}(X)$ that corresponds to a function in B^1. Also note the following lemma.

Lemma 5.2 *$\lambda_{down}(X)$ is the smallest non-zero eigenvalue of $L_0^{up}(X)$.*

Since by definition $\lambda_{\text{down}}(X) > 0$ and by Lemma 3.6 we know that $\text{Spec} L_0^{\text{up}}(X) \in [0, 2\max_{v\in V} \deg(v)]$, this implies that $\lambda_{\text{down}}(X) \in (0, 2\max_{v\in V} \deg(v)]$.

Proof We show that $\nu \neq 0$ is an eigenvalue of $L_1^{\text{down}}(X)$ if and only if it is also an eigenvalue of $L_0^{\text{up}}(X)$. Let x be an eigenvector of $L_1^{\text{down}}(X)$ with eigenvalue $\nu \neq 0$, i.e., $\delta_0\partial_1 x = \nu x$. Then

$$L_0^{\text{up}}(X)(\partial_1 x) = \partial_1(\delta_0\partial_1)x = \partial_1 \nu x = \nu(\partial_1 x).$$

Since $\delta_0\partial_1 x = \nu x \neq 0$, it follows that $\partial_1 x \neq 0$ and hence ν is an eigenvalue of $L_0^{\text{up}}(X)$. The other direction follows similarly. $\square$

In this section we present two different proofs of Proposition 5.1. In the next two sections we will find an upper bound for $\frac{\langle \partial_1 f, \partial_1 f\rangle}{\lambda_{\text{down}}(X)\langle f,f\rangle}$.

Throughout let A_0, A_1, A_2 be a partition that realizes the minimum $h(X)$.

5.1.1 Analysis Using Rayleigh's Principle

Note that from now on indices $i+1$ and $i+2$ are always understood as $i+1$ modulo 2 and $i+2$ modulo 2. One can easily check that the following definition of $f \in C^1$ coincides with the previous definition in Definition 3.11.

Definition 5.3 *Let $\{v_0, v_1\} \in E$ with $v_0 < v_1$.*

$$f(\{v_0, v_1\}) := \begin{cases} |A_{i+2}| & \text{if } v_0 \in A_i, v_1 \in A_{i+1} \text{ for some } i \in \{0,1,2\}, \\ -|A_{i+2}| & \text{if } v_0 \in A_{i+1}, v_1 \in A_i \text{ for some } i \in \{0,1,2\}, \\ 0 & \text{otherwise, i.e., if } v_0, v_1 \in A_i \text{ for some } i \in \{0,1,2\}. \end{cases}$$

We now bound $\langle L_1^{\text{up}}(X) f, f\rangle$, $\langle f, f\rangle$ and $\langle b, b\rangle$ individually, where we will see that in order to achieve a reasonable upper bound on $\langle b, b\rangle$ we need conditions on the underlying graph. This will be discussed in the next section.

By Lemma 3.13 it follows that

$$\langle L_1^{\text{up}}(X) f, f\rangle = \langle \delta_1 f, \delta_1 f\rangle = |V|^2 |F(A_0, A_1, A_2)|. \tag{5.1}$$

For the first part of the denominator we get a similar bound as given in Lemma 3.12.

Lemma 5.4 $\langle f, f\rangle \geq |V||F^{\partial}(A_0, A_1, A_2)|$.

Proof By definition of the function f we get

$$\begin{aligned}\langle f, f\rangle &= \sum_{e\in E} f(e)^2 \\ &= |e(A_0, A_1)||A_2|^2 + |e(A_1, A_2)||A_0|^2 + |e(A_2, A_0)||A_1|^2. \qquad (5.2)\end{aligned}$$

Note that for $i \in \{0, 1, 2\}$ every edge $e \in e(A_i, A_{i+1})$ can form at most $|A_{i+2}|$ elements of $F^{\partial}(A_0, A_1, A_2)$. Hence

$$|e(A_i, A_{i+1})||A_{i+2}| \geq |F^{\partial}(A_0, A_1, A_2)|.$$

Plugging this into equation (5.2) we obtain

$$\begin{aligned}\langle f, f\rangle &\geq |F^{\partial}(A_0, A_1, A_2)|(|A_0| + |A_1| + |A_2|) \\ &= |V||F^{\partial}(A_0, A_1, A_2)|.\end{aligned}$$

□

Of course this bound may be bad, especially for sparse graphs, but it turns out to be useful.

Proof of Proposition 5.1 Remember by Lemma 4.3 we have

$$\lambda(X) \leq \frac{\langle L_1^{\text{up}}(X)f, f\rangle}{\langle f, f\rangle - \langle b, b\rangle}.$$

Using equation (5.1) and Lemma 5.4 this can be reformulated as follows.

$$\begin{aligned}\lambda(X) &\leq \frac{\langle L_1^{\text{up}}(X)f, f\rangle}{\langle f, f\rangle - \langle b, b\rangle} \\ \Leftrightarrow \lambda(X) &\leq \frac{\frac{\langle L_1^{\text{up}}(X)f, f\rangle}{\langle f, f\rangle}}{1 - \frac{\langle b, b\rangle}{\langle f, f\rangle}} \\ \Rightarrow \lambda(X) &\leq \frac{\frac{|V|^2|F(A_0, A_1, A_2)|}{|V||F^{\partial}(A_0, A_1, A_2)|}}{1 - \frac{\langle b, b\rangle}{\langle f, f\rangle}} \\ \Leftrightarrow \lambda(X) &\leq \frac{h(X)}{1 - \frac{\langle b, b\rangle}{\langle f, f\rangle}} \\ \Leftrightarrow &\left(1 - \frac{\langle b, b\rangle}{\langle f, f\rangle}\right)\lambda(X) \leq h(X). \qquad (5.3)\end{aligned}$$

Since we do not know an explicit formula for b, the idea is to bound $\langle b, b\rangle$ through the eigenvalues of $L_1^{\text{down}}(X)$.

By Rayleigh's principle and definition of $\lambda_{\text{down}}(X)$ we know that

$$\frac{\langle L_1^{\text{down}}(X)b, b\rangle}{\langle b, b\rangle} = \frac{\langle \partial_1 b, \partial_1 b\rangle}{\langle b, b\rangle} \geq \lambda_{\text{down}}(X) > 0.$$

Therefore

$$\begin{aligned}\frac{\langle b, b\rangle}{\langle f, f\rangle} &\leq \frac{\langle \partial_1 b, \partial_1 b\rangle}{\lambda_{\text{down}}(X)\langle f, f\rangle} \\ &= \frac{\langle \partial_1 f, \partial_1 f\rangle}{\lambda_{\text{down}}(X)\langle f, f\rangle},\end{aligned}$$

where the last step follows since $\partial_1 z = 0$.

Plugging this result into inequality (5.3) we obtain

$$\left(1 - \frac{\langle \partial_1 f, \partial_1 f\rangle}{\lambda_{\text{down}}(X)\langle f, f\rangle}\right)\lambda(X) \leq \left(1 - \frac{\langle b, b\rangle}{\langle f, f\rangle}\right) \leq h(X).$$

□

5.1.2 Analysis Using the Lower Laplacian

In this section a new approach is introduced to find an alternate proof for Proposition 5.1. The idea is to find a function $M : C^1 \to C^1$ such that $\lambda(X) = \text{minSpec}(M)$. For finding $\lambda(X)$ one can then use Rayleigh's principle on M and any function $f \in C^1$ instead of restricting f to Z_1. We will find M of form $L_1^{\text{up}}(X) + \beta L_1^{\text{down}}(X)$, where β is a constant depending on the simplicial complex X.

Proposition 5.5 *$\lambda(X)$ is the smallest eigenvalue of*

$$M = L_1^{up}(X) + \beta L_1^{down}(X)$$

for

$$\beta = \frac{\lambda(X)}{\lambda_{down}(X)}.$$

This proposition suffices to prove Theorem 5.1.

Proof of Proposition 5.1 Using Proposition 5.5, by Rayleigh's principle we know that

$$\lambda(X) = \min_{g \in C^1} \frac{\langle Mg, g\rangle}{\langle g, g\rangle}.$$

Hence

$$\lambda(X) \leq \frac{\langle L_1^{\text{up}}(X)f, f\rangle}{\langle f, f\rangle} + \frac{\langle \beta L_1^{\text{down}}(X)f, f\rangle}{\langle f, f\rangle}$$
$$= \frac{\langle L_1^{\text{up}}(X)f, f\rangle}{\langle f, f\rangle} + \frac{\lambda(X)}{\lambda_{\text{down}}(X)} \cdot \frac{\langle L_1^{\text{down}}(X)f, f\rangle}{\langle f, f\rangle}.$$

Since by equation (5.1) and Lemma 5.4 we know that

$$\langle L_1^{\text{up}}(X)f, f\rangle = |V|^2 |F(A_0, A_1, A_2)| \text{ and}$$
$$\langle f, f\rangle \geq |V||F^\partial(A_0, A_1, A_2)|,$$

it follows by definition of $h(X)$ that

$$\lambda(X) \leq h(X) + \frac{\lambda(X)}{\lambda_{\text{down}}(X)} \cdot \frac{\langle L_1^{\text{down}}(X)f, f\rangle}{\langle f, f\rangle}.$$

Hence

$$\left(1 - \frac{\langle L_1^{\text{down}}(X)f, f\rangle}{\lambda_{\text{down}}(X)\langle f, f\rangle}\right)\lambda(X) \leq h(X).$$

□

It remains to prove Proposition 5.5.

Proof of Proposition 5.5 Table 5.1 shows how the eigenvalues of $L_1^{\text{down}}(X)$ and $L_1^{\text{up}}(X)$ behave in the spaces B^1 and Z_1.

	B^1	$(B^1)^\perp = Z_1$
$L_1^{\text{up}}(X)$	0	$\lambda_i \geq 0$
$L_1^{\text{down}}(X)$	$\tilde{\lambda_j} > 0$	0

Table 5.1: Values of the eigenvalues corresponding to $L_1^{\text{up}}(X)$ and $L_1^{\text{down}}(X)$.

We know by Lemma 3.6 that $L_1^{\text{up}}(X)$ is positive-definite, the proof of the positive-definiteness of $L_1^{\text{down}}(X)$ works similarly. Since $B^1 \subseteq Z^1 = \ker L_1^{\text{up}}(X)$ obviously $L_1^{\text{up}}(X)$ is zero on B^1. The eigenvalues of $L_1^{\text{down}}(X)$ are zero on Z_1 and only on Z_1 because $Z_1 = \ker L_1^{\text{down}}(X)$.

Remember that the goal is to construct a matrix M of form

$$M = L_1^{\text{up}}(X) + \beta L_1^{\text{down}}(X),$$

such that $\lambda(X) = \text{minSpec}(M)$, with β a constant depending on X that is yet to be determined.

Consulting table 5.1 one observes that we can find a basis of eigenvectors of M by taking the eigenvectors of $L_1^{\text{up}}(X)$ on Z_1 and the eigenvectors of $L_1^{\text{down}}(X)$ on B^1.

Applying these eigenvectors on our function M we get the following results.

Let $x \in Z_1$ be an eigenvector obtained from $L_1^{\text{up}}(X)$ corresponding to the eigenvalue λ_i in $L_1^{\text{up}}(X)$. Then

$$\begin{aligned} Mx &= (L_1^{\text{up}}(X) + \beta L_1^{\text{down}}(X))x \\ &= \lambda_i x + 0 = \lambda_i x. \end{aligned}$$

Now suppose $y \in B^1$ is an eigenvector obtained from $L_1^{\text{down}}(X)$ corresponding to the eigenvalue $\tilde{\lambda}_j$ in $L_1^{\text{down}}(X)$. Then

$$\begin{aligned} My &= (L_1^{\text{up}}(X) + \beta L_1^{\text{down}}(X))y \\ &= 0 + \beta\tilde{\lambda}_j y = (\beta\tilde{\lambda}_j)y. \end{aligned}$$

We want $\lambda(X) = \min_i \lambda_i$ to be the minimum eigenvalue of M, hence it must hold for all j that

$$\min_i \lambda_i = \lambda(X) \leq \beta\tilde{\lambda}_j.$$

Therefore

$$\beta \geq \frac{\lambda(X)}{\min_j \tilde{\lambda}_j} = \frac{\lambda(X)}{\lambda_{\text{down}}(X)}.$$

Choosing β as small as possible we set

$$\beta := \frac{\lambda(X)}{\lambda_{\text{down}}(X)},$$

which proves the Proposition. □

5.2 An Upper Bound on the Boundary Operator

5.2.1 Introduction and General Bounds

Recall Proposition 5.1 from Chapter 5.

Proposition 5.1

$$\left(1 - \frac{\langle \partial_1 f, \partial_1 f\rangle}{\lambda_{\text{down}}(X)\langle f, f\rangle}\right)\lambda(X) \leq h(X).$$

The next step is to bound $\frac{\langle \partial_1 f, \partial_1 f \rangle}{\lambda_{\mathrm{down}}(X)\langle f,f \rangle}$ by a constant. In this section we focus on finding an upper bound for $\langle \partial_1 f, \partial_1 f \rangle$. We will see that to achieve a reasonable upper bound we need some assumptions on the underlying 1-skeleton G of our simplicial complex.

In a first step we want to derive a general bound on $\langle \partial_1 f, \partial_1 f \rangle$. Let $p = \frac{d_{av}}{|V|-1}$, where $d_{av} = \frac{1}{|V|} \sum_{v \in V} \deg(v)$ is the average vertex degree.

Consider a vertex $u_i \in A_i$ for some $i \in \{0, 1, 2\}$. For $j \neq i$ one can always express $|e(u_i, A_j)|$ by

$$|e(u_i, A_j)| = p|A_j| + \varepsilon_j(u_i)$$

for some error term $\varepsilon_j(u_i) \in \mathbb{R}$. Note that for complexes with the complete graph as 1-skeleton we have $p = 1$ and $\varepsilon_j(u_i) = 0$.

This notation is useful to obtain a general upper bound on $\langle \partial_1 f, \partial_1 f \rangle$ as the following lemma shows.

Lemma 5.6 *For any* 2*-simplicial complex it holds that*

$$\begin{aligned}
\langle \partial_1 f, \partial_1 f \rangle \leq 2 \sum_{u \in A_0} &\left[(\varepsilon_2(u)|A_1|)^2 + (\varepsilon_1(u)|A_2|)^2 \right] \\
&+ 2 \sum_{u \in A_1} \left[(\varepsilon_0(u)|A_2|)^2 + (\varepsilon_2(u)|A_0|)^2 \right] \\
&+ 2 \sum_{u \in A_2} \left[(\varepsilon_1(u)|A_0|)^2 + (\varepsilon_0(u)|A_1|)^2 \right].
\end{aligned}$$

Proof Using the above notation we can express $\langle \partial_1 f, \partial_1 f \rangle$ by

$$\begin{aligned}
\langle \partial_1 f, \partial_1 f \rangle &= \sum_{u \in V} (\partial_1 f(u))^2 \\
&= \sum_{u \in V} \left(\sum_{\{u,v\} \in E} [\{u,v\} : u] f(\{u,v\}) \right)^2 \\
&= \sum_{u \in A_0} (|e(u, A_2)||A_1| - |e(u, A_1)||A_2|)^2 \\
&\quad + \sum_{u \in A_1} (|e(u, A_0)||A_2| - |e(u, A_2)||A_0|)^2 \\
&\quad + \sum_{u \in A_2} (|e(u, A_1)||A_0| - |e(u, A_0)||A_1|)^2 \\
&= \sum_{u \in A_0} (p|A_2||A_1| + \varepsilon_2(u)|A_1| - (p|A_1||A_2| + \varepsilon_1(u)|A_2|))^2 \\
&\quad + \sum_{u \in A_1} (p|A_0||A_2| + \varepsilon_0(u)|A_2| - (p|A_2||A_0| + \varepsilon_2(u)|A_0|))^2
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{u \in A_2} \left(p|A_1||A_0| + \varepsilon_1(u)|A_0| - (p|A_0||A_1| + \varepsilon_0(u)|A_1|)\right)^2 \\
&= \sum_{u \in A_0} \left(\varepsilon_2(u)|A_1| - \varepsilon_1(u)|A_2|\right)^2 \\
&\quad + \sum_{u \in A_1} \left(\varepsilon_0(u)|A_2| - \varepsilon_2(u)|A_0|\right)^2 \\
&\quad + \sum_{u \in A_2} \left(\varepsilon_1(u)|A_0| - \varepsilon_0(u)|A_1|\right)^2 \\
&\leq 2 \sum_{u \in A_0} \left[\left(\varepsilon_2(u)|A_1|\right)^2 + \left(\varepsilon_1(u)|A_2|\right)^2\right] \\
&\quad + 2 \sum_{u \in A_1} \left[\left(\varepsilon_0(u)|A_2|\right)^2 + \left(\varepsilon_2(u)|A_0|\right)^2\right] \\
&\quad + 2 \sum_{u \in A_2} \left[\left(\varepsilon_1(u)|A_0|\right)^2 + \left(\varepsilon_0(u)|A_1|\right)^2\right],
\end{aligned}
$$

where the last step follows because $2ab \leq a^2 + b^2$, for $a, b \in \mathbb{R}$. $\square$

Consider the following obvious bound.

$$
\begin{aligned}
\langle \partial_1 f, \partial_1 f \rangle &= \sum_{u \in V} \left(\sum_{(v,u) \in E} f(v,u) \right)^2 \\
&= \sum_{u \in A_0} \left(\sum_{\substack{(v,u) \in E \\ v \in A_2}} |A_1| - \sum_{\substack{(v,u) \in E \\ v \in A_1}} |A_2| \right)^2 \\
&\quad + \sum_{u \in A_1} \left(\sum_{\substack{(v,u) \in E \\ v \in A_0}} |A_2| - \sum_{\substack{(v,u) \in E \\ v \in A_2}} |A_0| \right)^2 \\
&\quad + \sum_{u \in A_2} \left(\sum_{\substack{(v,u) \in E \\ v \in A_1}} |A_0| - \sum_{\substack{(v,u) \in E \\ v \in A_0}} |A_1| \right)^2 \\
&\leq \sum_{u \in A_0} \left(|A_1||A_2|\right)^2 + \sum_{u \in A_1} \left(|A_2||A_0|\right)^2 + \sum_{u \in A_2} \left(|A_0||A_1|\right)^2 \\
&= |A_0||A_1||A_2| \left(|A_0||A_1| + |A_1||A_2| + |A_2||A_0|\right).
\end{aligned}
$$

Remember we must find a constant upper bound for

$$
\frac{\langle \partial_1 f, \partial_1 f \rangle}{\lambda_{\text{down}}(X) \langle f, f \rangle}.
$$

By Lemma 5.4 we know that

$$\langle f, f\rangle \geq |V||F^{\partial}(A_0, A_1, A_2)|.$$

Since $|F^{\partial}(A_0, A_1, A_2)|$ can be of order $|A_0||A_1||A_2|$ and by the Lemma 5.2 $\lambda_{\text{down}}(X) \in (0, 2\max_{v\in V}\deg(v)]$, we see that this obvious bound is of no use.

5.2.2 An Estimate of the Boundary Operator Using Jumbledness Condition

To obtain a non-trivial upper bound for $\langle \partial_1 f, \partial_1 f\rangle$ we will find bounds on the error terms $\varepsilon_j(u)$. We assume G is (p,α)-jumbled, which is defined as follows.

Definition 5.7 *A graph $G = (V, E)$ is called (p,α)-*jumbled *if for all $U, W \subseteq V$ and $U \cap W = \emptyset$*

$$|e(U,W) - p|U||W|| \leq \alpha\sqrt{|U||W|}.$$

One can assume that $\alpha = \mathcal{O}(\sqrt{np})$, which is the best case possible. For a thorough introduction to (p,α)-jumbled graphs see [18].

In a first step we show that it is not possible for every vertex in A_i to have maximal error term given by (p,α)-jumbledness of the graph. Recall that the error term $\varepsilon_j(u)$ of a vertex $u \in A_i$, $i \neq j$ was defined by $\varepsilon_j(u) = |e(u, A_j)| - p|A_j|$. Hence we will show that it is not possible that

$$|\varepsilon_j(u)| = \alpha\sqrt{|A_j|},$$

for all $u \in A_i$. We will use this to our advantage to obtain a better upper bound on $\langle \partial_1 f, \partial_1 f\rangle$.

For $i \neq j \in \{0,1,2\}$ and $c \in (0,1]$ let

$$\begin{aligned}
A_i^j(c^+) &= \left\{u \in A_i \colon |e(u, A_j)| \geq p|A_j| + c\alpha\sqrt{|A_j|}\right\}\\
&= \left\{u \in A_i \colon \varepsilon_j(u) \geq c\alpha\sqrt{|A_j|}\right\} \text{ and}\\
A_i^j(c^-) &= \left\{u \in A_i \colon e(u, A_j) \leq p|A_j| - c\alpha\sqrt{|A_j|}\right\}\\
&= \left\{u \in A_i \colon \varepsilon_j(u) \leq c\alpha\sqrt{|A_j|}\right\}.
\end{aligned}$$

For a fixed c we want to know how many vertices at most there can be, which have an error term at least the c-fraction of the maximal possible error term. How large can $|A_i^j(c^+)|$ and $|A_i^j(c^-)|$ be? By definition of $A_i^j(c^+)$

$$\begin{aligned}|e(A_i^j(c^+), A_j)| &\geq \left(p|A_j| + c\alpha\sqrt{|A_j|}\right)|A_i^j(c^+)| \\ &= p|A_i^j(c^+)||A_j| + c\alpha|A_i^j(c^+)|\sqrt{|A_j|}.\end{aligned}$$

On the other hand, by (p,α)-jumbledness we know

$$|e(A_i^j(c^+), A_j)| \leq p|A_i^j(c^+)||A_j| + \alpha\sqrt{|A_i^j(c^+)||A_j|}.$$

Hence

$$\begin{aligned}p|A_i^j(c^+)||A_j| + c\alpha|A_i^j(c^+)|\sqrt{|A_j|} \leq p|A_i^j(c^+)||A_j| + \alpha\sqrt{|A_i^j(c^+)||A_j|} \Leftrightarrow \\ |A_i^j(c^+)| \leq \frac{1}{c^2}.\end{aligned} \tag{5.4}$$

The same way we find $|A_i^j(c^-)| \leq \frac{1}{c^2}$. This shows that for example there can be at most four vertices for which the error term is at least $\frac{1}{2}\alpha\sqrt{|A_j|}$.

Using the above idea to our advantage one can improve the obvious bound of $\langle \partial_1 f, \partial_1 f\rangle$ to the following.

Proposition 5.8 *For a* 2*-complex with an* α*-jumbled* 1*-skeleton and* $|A_0|, |A_1|, |A_2| \geq 17$ *it holds that*

$$\begin{aligned}\langle \partial_1 f, \partial_1 f\rangle \leq 6\alpha^2\Big(&\ln(|A_0|)|A_2||A_1|^2 + \ln(|A_0|)|A_1||A_2|^2 \\ &+ \ln(|A_1|)|A_0||A_2|^2 + \ln(|A_1|)|A_2||A_0|^2 \\ &+ \ln(|A_2|)|A_1||A_0|^2 + \ln(|A_2|)|A_0||A_1|^2\Big).\end{aligned}$$

Please note that the condition $|A_i| \geq 17$ is a technical one, needed to get rid of a constant error term.

Proof of Proposition 5.8 Consider the result from Lemma 5.6, i.e., that

$$\begin{aligned}\langle \partial_1 f, \partial_1 f\rangle \leq 2&\sum_{u\in A_0}\left[(\varepsilon_2(u)|A_1|)^2 + (\varepsilon_1(u)|A_2|)^2\right] \\ &+ 2\sum_{u\in A_1}\left[(\varepsilon_0(u)|A_2|)^2 + (\varepsilon_2(u)|A_0|)^2\right] \\ &+ 2\sum_{u\in A_2}\left[(\varepsilon_1(u)|A_0|)^2 + (\varepsilon_0(u)|A_1|)^2\right].\end{aligned}$$

For simplicity reasons consider only one of the error terms, say

$$\sum_{u \in A_0} (\varepsilon_2(u)|A_1|)^2 .$$

The following estimates will work for the other five terms equivalently. Let

$$\begin{aligned} A_0^+ &= \{u \in A_0 \colon \varepsilon_2(u) > 0\} \text{ and} \\ A_0^- &= \{u \in A_0 \colon \varepsilon_2(u) < 0\}. \end{aligned}$$

Then

$$\sum_{u \in A_0} (\varepsilon_2(u)|A_1|)^2 = \sum_{u \in A_0^+} (\varepsilon_2(u)|A_1|)^2 + \sum_{u \in A_0^-} (\varepsilon_2(u)|A_1|)^2 . \tag{5.5}$$

Again for simplicity reasons consider only the first sum over A_0^+. Since the error term $\varepsilon_2(u)$ only appears in its square the calculation for the second term will work equivalently.

Now we want to make use of equation (5.4), i.e.,

$$|A_i^j(c^+)| \leq \frac{1}{c^2}.$$

Let $c_k = \frac{1}{k}$ for $k \in \mathbb{N}$. Then we can rewrite our sum as

$$\sum_{u \in A_0^+} (\varepsilon_2(u)|A_1|)^2 = \sum_{k=1}^{\infty} \sum_{u \in A_0^2(c_k^+) \setminus A_0^2(c_{k-1}^+)} (\varepsilon_2(u)|A_1|)^2, \tag{5.6}$$

where $A_0^2(c_0) := \emptyset$.

Observations.

1. The above sum is maximized if the bound is tight on $|A_0^2(c_k^+)|$ for small k, i.e., if $|A_0^2(c_k^+)| = \frac{1}{c_k^2}$. Then

$$|A_0^2(c_k^+) \setminus A_0^2(c_{k-1}^+)| = \frac{1}{c_k^2} - \frac{1}{c_{k-1}^2} = k^2 - (k-1)^2 = 2k - 1.$$

2. For $k \geq 2$ and $u \in A_0^2(c_k^+) \setminus A_0^2(c_{k-1}^+)$ it holds that

$$\varepsilon_2(u) \leq c_{k-1}\alpha\sqrt{|A_2|}.$$

3. If we assume worst case, i.e., the case where (5.6) is maximized, we do obviously not need to sum up to infinity but summing up to $\lceil \sqrt{|A_0|} \rceil$ suffices since we only have $|A_0|$ vertices for which we need to estimate the error term. For $k = \lceil \sqrt{|A_0|} \rceil$ we have

$$|A_0^2(c_k)| = \frac{1}{c_k^2} = (\lceil \sqrt{|A_0|} \rceil)^2 \geq |A_0|.$$

Hence if we sum up from 1 to $\lceil \sqrt{|A_0|} \rceil$ in the worst case scenario, we have estimated the error for all vertices in A_0 from above.

From now on without loss of generality for simplicity of notation, assume that $\sqrt{|A_0|}$ is a natural number. The reader is welcome to assure himself that the calculations for $\sqrt{|A_0|}$ not a natural number work in the same manner.

Plugging observation 1., 2. and 3. into equation (5.6) we obtain the following upper bound.

$$\begin{aligned}
\sum_{u \in A_0^+} (\varepsilon_2(u)|A_1|)^2 &\leq \left(\alpha\sqrt{|A_2|}|A_1|\right)^2 + \sum_{k=2}^{\sqrt{|A_0|}} \left(\frac{1}{c_k^2} - \frac{1}{c_{k-1}^2}\right)\left(c_{k-1}\alpha\sqrt{|A_2|}|A_1|\right)^2 \\
&= \alpha^2|A_2||A_1|^2 \left(\left(\sum_{k=2}^{\sqrt{|A_0|}} (k^2 - (k-1)^2)\frac{1}{(k-1)^2}\right) + 1\right) \\
&= \alpha^2|A_2||A_1|^2 \left(\left(\sum_{k=2}^{\sqrt{|A_0|}} \frac{2k-1}{(k-1)^2}\right) + 1\right) \\
&= 2\alpha^2|A_2|A_1|^2 \left(\left(\sum_{k=1}^{\sqrt{|A_0|}-1} \frac{1}{k}\right) + \left(\sum_{k=1}^{|A_0|-1} \frac{1}{2k^2}\right) + 1\right).
\end{aligned}$$

Using the well known results

$$\begin{aligned}
\sum_{k=1}^{m} \frac{1}{k} &< \ln(m) + 1 \text{ and} \qquad (5.7) \\
\sum_{k=1}^{\infty} \frac{1}{k^2} &= \frac{\pi^2}{6}
\end{aligned}$$

we get

$$\begin{aligned}
\sum_{u \in A_0^+} (\varepsilon_2(u)|A_1|)^2 &< 2\alpha^2|A_2||A_1|^2 \left(\ln\left(\sqrt{|A_0|} - 1\right) + 1 + \frac{\pi^2}{12} + 1\right) \\
&\leq 2\alpha^2|A_2||A_1|^2 \left(\frac{1}{2}\ln(|A_0|) + \frac{\pi^2}{12} + 2\right)
\end{aligned}$$

$$
\begin{aligned}
&< 2\alpha^2|A_2||A_1|^2\left(\frac{1}{2}\ln\left(|A_0|\right)+\ln(17)\right)\\
&\leq 3\alpha^2|A_2||A_1|^2\ln(|A_0|),
\end{aligned}
$$

where the last step holds for $|A_0| \geq 17$.

Plugging in into equation (5.5) we obtain

$$\sum_{u\in A_0}(\varepsilon_2(u)|A_1|)^2 < 6\alpha^2|A_2||A_1|^2\ln(|A_0|).$$

Hence by Lemma 5.6

$$
\begin{aligned}
\langle\partial_1 f,\partial_1 f\rangle \leq 6\alpha^2(&\ln(|A_0|)|A_2||A_1|^2+\ln(|A_0|)|A_1||A_2|^2\\
&+\ln(|A_1|)|A_0||A_2|^2+\ln(|A_1|)|A_2||A_0|^2\\
&+\ln(|A_2|)|A_1||A_0|^2+\ln(|A_2|)|A_0||A_1|^2),
\end{aligned}
$$

which proves Proposition 5.8. □

The following example shows why up to a constant factor, the result of Proposition 5.8 is the best we can hope for.

Example. Let A_0, A_1, A_2 have the following properties (where c_k is defined as above).

- For $k = 2,\ldots,\frac{1}{4}\sqrt{|A_0|}$, let $|A_0^2(c_k^+)| = \frac{1}{c_k^2}$, $|A_0^2(c_1^+)| = 0$. (Without loss of generality assume that $\frac{1}{4}\sqrt{|A_0|} \in \mathbb{N}$).
- For all remaining vertices $u \in A_0$ suppose $\varepsilon_2(u) = 0$.
- For all vertices $u \in A_0$ assume $\varepsilon_1(u) = 0$.

Note that it is easily possible to construct a simplicial complex that satisfies the above properties and the condition that the partition A_0, A_1, A_2 mimimizes the function $h(X)$. However this does not show that the underlying graph is (p,α)-jumbled. We will only show that any such underlying graph satisfies (p,α)-jumbledness property between the blocks A_0 and A_2 and the size of $\langle\partial_1 f,\partial_1 f\rangle$ is of the same order as the bound given in Proposition 5.8.

Since by definition of (p,α)-jumbledness

$$|e(A_0,A_2)| \leq p|A_0||A_2| + \alpha\sqrt{|A_0||A_2|},$$

for (p,α)-jumbledness property to hold between the sets A_0 and A_2, the following must hold.

$$\sum_{u\in A_0}\varepsilon_2(u) \leq \alpha\sqrt{|A_0||A_2|}.$$

We now show that by our choice of the graph above this condition is not violated. By plugging in the above assumptions we obtain

$$\begin{aligned}
\sum_{u \in A_0} \varepsilon_2(u) &= \sum_{k=2}^{\frac{1}{4}\sqrt{|A_0|}} \sum_{u \in A_0^2(c_k) \setminus A_0^2(c_{k-1})} \varepsilon_2(u) \\
&\leq \sum_{k=2}^{\frac{1}{4}\sqrt{|A_0|}} \sum_{u \in A_0^2(c_k) \setminus A_0^2(c_{k-1})} c_{k-1}\alpha\sqrt{|A_2|} \\
&= \alpha\sqrt{|A_2|} \sum_{k=2}^{\frac{1}{4}\sqrt{|A_0|}} (2k-1)\frac{1}{k-1} \\
&= \alpha\sqrt{|A_2|} \sum_{k=1}^{\frac{1}{4}\sqrt{|A_0|}-1} (2k+1)\frac{1}{k} \\
&\overset{(5.7)}{\leq} \alpha\sqrt{|A_2|}\left(2\left(\frac{1}{4}\sqrt{|A_0|}-1\right) + \ln\left(\frac{1}{4}\sqrt{|A_0|}-1\right) + 1\right) \\
&\leq \alpha\sqrt{|A_0||A_2|},
\end{aligned}$$

where the last step holds for $|A_0|$ large enough. Hence there is no contradiction to the (p, α)-jumbledness condition between the sets A_0 and A_2.

Now it remains to show that $\langle \partial_1 f, \partial_1 f \rangle$ is of the same order as the bound given in Proposition 5.8. As seen in the proof of Lemma 5.6 we have

$$\langle \partial_1 f, \partial_1 \rangle \geq \sum_{u \in A_0} \left(\varepsilon_2(u)|A_1| - \varepsilon_1(u)|A_2|\right)^2.$$

By our assumption on the error terms $\varepsilon_1(u) = 0$ for all $u \in A_0$, so this yields

$$\begin{aligned}
\langle \partial_1 f, \partial_1 \rangle &\geq \sum_{u \in A_0} \left(\varepsilon_2(u)|A_1|\right)^2 \\
&= \sum_{k=2}^{\frac{1}{4}\sqrt{|A_0|}} \sum_{u \in A_0^2(c_k) \setminus A_0^2(c_{k-1})} \left(\varepsilon_2(u)|A_1|\right)^2 \\
&\geq \sum_{k=2}^{\frac{1}{4}\sqrt{|A_0|}} \sum_{u \in A_0^2(c_k) \setminus A_0^2(c_{k-1})} \left(c_k\alpha\sqrt{|A_2|}|A_1|\right)^2 \\
&= \alpha^2|A_1|^2|A_2| \sum_{k=2}^{\frac{1}{4}\sqrt{|A_0|}} (2k-1)\frac{1}{k^2} \\
&\geq 2\alpha^2|A_1|^2|A_2| \sum_{k=2}^{\frac{1}{4}\sqrt{|A_0|}} \frac{1}{k}
\end{aligned}$$

$$\geq 2\alpha^2|A_1|^2|A_2|\left(\ln\left(\frac{1}{4}\sqrt{|A_0|}+1\right)-1\right)$$
$$\geq \alpha^2 \ln(|A_0|)|A_1|^2|A_2|,$$

where again the last step holds for $|A_0|$ large enough and we used the fact that

$$\ln(m+1) \leq \sum_{k=1}^{m}\frac{1}{k}.$$

Up to a constant this is of the same order as the bound given in Proposition 5.8, hence this is the best bound one can hope for.

5.3 The Bound

Recall the following results from the previous sections.

Proposition 5.1

$$\left(1-\frac{\langle\partial_1 f,\partial_1 f\rangle}{\lambda_{\text{down}}(X)\langle f,f\rangle}\right)\lambda(X)\leq h(X).$$

Propsition 5.8 *For a* 2*-complex with an* α*-jumbled* 1*-skeleton and* $|A_0|,|A_1|,|A_2|\geq 17$ *it holds that*

$$\begin{aligned}\langle\partial_1 f,\partial_1 f\rangle \leq 6\alpha^2(&\ln(|A_0|)|A_2||A_1|^2+\ln(|A_0|)|A_1||A_2|^2\\ &+\ln(|A_1|)|A_0||A_2|^2+\ln(|A_1|)|A_2||A_0|^2\\ &+\ln(|A_2|)|A_1||A_0|^2+\ln(|A_2|)|A_0||A_1|^2).\end{aligned}$$

The goal in this Chapter is to bound $\frac{\langle\partial_1 f,\partial_1 f\rangle}{\lambda_{\text{down}}(X)\langle f,f\rangle}$ by using the two results above and the right assumptions on the simplicial complex X. The main result is the following theorem.

Theorem 5.9 *Let X be a* 2*-complex with* (p,α)*-jumbled* 1*-skeleton. Let* $0<\varepsilon<1$. *If there exists* $0<\varepsilon_1<1$ *and* $a_1>0$ *such that for all* $i\in\{0,1,2\}$

1. $\lambda_{down}(X)\geq a_1\alpha^2$,
2. $|A_i|\geq\frac{\alpha}{\varepsilon_1 p}$,
3. $|A_i|\geq 17$,
4. $\sqrt{|A_i|}\geq\frac{36}{(1-\varepsilon_1)\varepsilon a_1 p}$,

then $(1-\varepsilon)\lambda(X)\leq h(X)$.

Note that for fixed p and n large enough condition 2 implies condition 3 and 4.

Remember that by Lemma 5.2 we know that $\lambda_{\text{down}}(X) \in (0, 2\text{maxdeg(v)}]$ and since $\alpha = \mathcal{O}(\sqrt{np})$, the assumption on the eigenvalue $\lambda_{\text{down}}(X)$ is legitimate. Unfortunately, to use this theorem we need to know the sizes of the A_i's, which are usually hard to compute. Also, the bound $(1-\varepsilon)\lambda(X) \leq h(X)$ is weaker than the bound $\lambda(X) \leq h(X)$ from Theorem 4.1.

To prove Theorem 5.9 by Proposition 5.1 it is enough to show that

$$\frac{\langle \partial_1 b, \partial_1 b\rangle}{\lambda_{\text{down}}(X)\langle f, f\rangle} \leq \varepsilon,$$

with the assumptions of Theorem 5.9. This requires a few preparatory lemmas.

Lemma 5.10 *Let X be a 2-dimensional simplicial complex with (p,α)-jumbled 1-skeleton G and $|A_0|, |A_1|, |A_2| \geq 17$. Then*

$$\begin{aligned}\frac{\langle \partial_1 f, \partial_1 f\rangle}{\lambda_{down}(X)\langle f, f\rangle} \leq \frac{6\alpha^2}{\lambda_{down}(X)}\Bigg(&\frac{\ln(|A_1|)|A_2| + \ln(|A_2|)|A_1|}{|e(A_1, A_2)|} \\ &+ \frac{\ln(|A_2|)|A_0| + \ln(|A_0|)|A_2|}{|e(A_2, A_0)|} \\ &+ \frac{\ln(|A_0|)|A_1| + \ln(|A_1|)|A_0|}{|e(A_0, A_1)|}\Bigg).\end{aligned}$$

Proof By Proposition 5.8 we know that

$$\begin{aligned}\langle \partial_1 b, \partial_1 b\rangle \leq 6\alpha^2\Big(&\ln(|A_0|)|A_2||A_1|^2 + \ln(|A_0|)|A_1||A_2|^2 \\ &+ \ln(|A_1|)|A_0||A_2|^2 + \ln(|A_1|)|A_2||A_0|^2 \\ &+ \ln(|A_2|)|A_1||A_0|^2 + \ln(|A_2|)|A_0||A_1|^2\Big) \\ = 6\alpha^2\Big(&|A_0|^2(\ln(|A_1|)|A_2| + \ln(|A_2|)|A_1|) \\ &+ |A_1|^2(\ln(|A_2|)|A_0| + \ln(|A_0|)|A_2|) \\ &+ |A_2|^2(\ln(|A_0|)|A_1| + \ln(|A_1|)|A_0|)\Big).\end{aligned} \tag{5.8}$$

By definition of f

$$\langle f, f\rangle = |e(A_0, A_1)||A_2|^2 + |e(A_1, A_2)||A_0|^2 + |e(A_2, A_0)||A_2|^2. \tag{5.9}$$

Observe that for any $b_1, b_2, b_3 \geq 0$ and $d_1, d_2, d_3 > 0$

$$\frac{b_1 + b_2 + b_3}{d_1 + d_2 + d_3} = \frac{b_1}{d_1 + d_2 + d_3} + \frac{b_2}{d_1 + d_2 + d_3} + \frac{b_3}{d_1 + d_2 + d_3}$$

$$\leq \frac{b_1}{d_1} + \frac{b_2}{d_2} + \frac{b_3}{d_3}. \tag{5.10}$$

Therefore by plugging (5.8) and (5.9) into $\frac{\langle \partial f, \partial f \rangle}{\lambda_{\text{down}}(X)\langle f, f \rangle}$ and using observation (5.10) we obtain

$$\begin{aligned}
\frac{\langle \partial_1 b, \partial_1 b \rangle}{\lambda_{\text{down}}(X)\langle f, f \rangle} &\leq \frac{6\alpha^2}{\lambda_{\text{down}}(X)} \Bigg(\frac{|A_0|^2(\ln(|A_1|)|A_2| + \ln(|A_2|)|A_1|)}{|e(A_1, A_2)||A_0|^2} \\
&\quad + \frac{|A_1|^2(\ln(|A_2|)|A_0| + \ln(|A_0|)|A_2|)}{|e(A_2, A_0)||A_1|^2} \\
&\quad + \frac{|A_2|^2(\ln(|A_0|)|A_1| + \ln(|A_1|)|A_0|)}{|e(A_0, A_1)||A_2|^2} \Bigg) \\
&= \frac{6\alpha^2}{\lambda_{\text{down}}(X)} \Bigg(\frac{\ln(|A_1|)|A_2| + \ln(|A_2|)|A_1|}{|e(A_1, A_2)|} \\
&\quad + \frac{\ln(|A_2|)|A_0| + \ln(|A_0|)|A_2|}{|e(A_2, A_0)|} \\
&\quad + \frac{\ln(|A_0|)|A_1| + \ln(|A_1|)|A_0|}{|e(A_0, A_1)|} \Bigg)
\end{aligned}$$

as desired. □

For further calculations we only consider the first term

$$\frac{\ln(|A_1|)|A_2| + \ln(|A_2|)|A_1|}{|e(A_1, A_2)|},$$

the results for the other two terms follow equivalently.

Lemma 5.11 *For $i \in \{0, 1, 2\}$ assume $|A_i| \geq \frac{\alpha}{\varepsilon_1 p}$ for some $0 < \varepsilon_1 < 1$. Then $|e(A_1, A_2)| \geq (1 - \varepsilon_1)p|A_1||A_2|$.*

Proof Since $|A_i| \geq \frac{\alpha}{\varepsilon_1 p}$ we observe

$$\alpha \leq \varepsilon_1 p \min_i |A_i| \leq \varepsilon_1 p \sqrt{|A_1||A_2|}.$$

Using (p, α)-jumbledness and this observation we obtain

$$\begin{aligned}
|e(A_1, A_2)| &\geq p|A_1||A_2| - \alpha\sqrt{|A_1||A_2|} \\
&\geq p|A_1||A_2| - \varepsilon_1 p\sqrt{|A_1||A_2|}\sqrt{|A_1||A_2|} \\
&= (1 - \varepsilon_1)p|A_1||A_2|.
\end{aligned}$$

□

Lemma 5.12 *If for $i \in \{0,1,2\}$ we have $\sqrt{|A_i|} \geq \frac{36}{(1-\varepsilon_1)\varepsilon a_1 p}$ and $\lambda_{down}(X) \geq a_1\alpha$ for some $a_1 > 0$, then*

$$\ln(|A_1|)|A_2| + \ln(|A_2|)|A_1| \leq \frac{\varepsilon a_1}{18}(1-\varepsilon_1)p|A_1||A_2|.$$

Proof Since for all $i \in \{0,1,2\}$, $\sqrt{|A_i|} \geq \frac{36}{(1-\varepsilon_1)\varepsilon a_1 p}$, it follows that

$$a_1 \geq \frac{36}{(1-\varepsilon_1)\varepsilon p \min_i \sqrt{|A_i|}}.$$

Hence,

$$\begin{aligned}
\frac{\varepsilon a_1}{18}(1-\varepsilon_1)p|A_1||A_2| &\geq \frac{36}{(1-\varepsilon_1)\varepsilon p \min_i \sqrt{|A_i|}}\frac{\varepsilon}{18}(1-\varepsilon_1)p|A_1||A_2| \\
&= 2\frac{|A_1||A_2|}{\min_i \sqrt{|A_i|}} \\
&\geq \frac{|A_1||A_2|}{\sqrt{|A_1|}} + \frac{|A_1||A_2|}{\sqrt{|A_2|}} \\
&\geq \ln(|A_1|)|A_2| + \ln(|A_2|)|A_1|,
\end{aligned}$$

where the last step follows from the inequality $ln(x) \leq \sqrt{x}$ for $x \geq 1$. □

Note that this last estimate may seem bad at first. As we will see Lemma 5.11 and Lemma 5.12 are both needed to prove Theorem 5.9. Since the condition $|A_i| \geq \frac{\alpha}{\varepsilon_1 p}$ in Lemma 5.11 is generally stronger than the assumption in Lemma 5.12, there is no use in trying to find a tighter estimate.

Using Lemma 5.10, 5.11 and 5.12 we are now able to prove Theorem 5.9.

Proof of Theorem 5.9. By Lemma 5.10 we know that

$$\begin{aligned}
\frac{\langle\partial b, \partial b\rangle}{\lambda_{\text{down}}(X)\langle f, f\rangle} \leq \frac{6\alpha^2}{\lambda_{\text{down}}(X)}\Bigg(&\frac{\ln(|A_1|)|A_2| + \ln(|A_2|)|A_1|}{|e(A_1, A_2)|} \\
&+ \frac{\ln(|A_2|)|A_0| + \ln(|A_0|)|A_2|}{|e(A_2, A_0)|} \\
&+ \frac{\ln(|A_0|)|A_1| + \ln(|A_1|)|A_0|}{|e(A_0, A_1)|}\Bigg).
\end{aligned} \tag{5.11}$$

By our assumption that $\lambda_{\text{down}}(X) \geq a_1\alpha^2$ and by Lemma 5.11 we obtain

$$\frac{6\alpha^2(\ln(|A_1|)|A_2| + \ln(|A_2|)|A_1|)}{\tilde{\lambda}_{min}|e(A_1, A_2)|} \leq \frac{6(\ln(|A_1|)|A_2| + \ln(|A_2|)|A_1|)}{a_1(1-\varepsilon_1)p|A_1||A_2|}.$$

Using Lemma 5.12 this is

$$\leq \frac{\frac{\varepsilon a_1}{3}(1-\varepsilon_1)p|A_1||A_2|}{(1-\varepsilon_1)a_1 p|A_1||A_2|} = \frac{\varepsilon}{3}.$$

The same bound holds for the other two factors of inequality (5.11). Hence

$$\left(1 - \frac{\langle \partial b, \partial b \rangle}{\lambda_{\text{down}}(X)\langle f, f \rangle}\right) \geq 1 - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = 1 - \varepsilon$$

and by Proposition 5.1

$$(1-\varepsilon)\lambda(X) \leq \left(1 - \frac{\langle \partial f, \partial f \rangle}{\lambda_{\text{down}}(X)\langle f, f \rangle}\right)\lambda(X) \leq h(X)$$

as desired. □

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