A generalization of crossing families

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Abstract

For a set $\mathcal{P}$ of points in the plane, a *crossing family* for $\mathcal{P}$ is a set $\mathcal{C}$ of line segments, each joining two of the points from $\mathcal{P}$, such that any two line segments from $\mathcal{C}$ cross. We investigate the following generalization of crossing families: a *spoke set* for $\mathcal{P}$ is a set of lines such that each unbounded region of the induced line arrangement contains at least one point of $\mathcal{P}$.

We show that every point set of size n has a spoke set of size $\sqrt{\frac{n}{8}}$. We also characterize the matchings obtained by selecting exactly one point in each unbounded region and connecting every such point to the point in the antipodal unbounded region.

1 Introduction

Let $\mathcal{P}$ be a finite point set in *general position* (i.e., no three points on a line). Throughout this paper, we assume all point sets to be in general position. A *crossing family* for $\mathcal{P}$ is a set $\mathcal{C}$ of line segments, each joining two of the points from $\mathcal{P}$, such that any two line segments from $\mathcal{C}$ cross (i.e., intersect in their interior). Crossing families were introduced by Aronov et al. [1], who have shown that any set of n points in general position has a crossing family of size $\sqrt{\frac{n}{12}}$. Since then, there have been several results about crossing families [3, 4], but even though it is conjectured that any point set in general position has a crossing family of linear size [1], the bound of Aronov et al. is still the best known result.

A point set $\mathcal{A}$ *avoids* a point set $\mathcal{B}$ if no line through two points in $\mathcal{A}$ intersects the convex hull of $\mathcal{B}$. Note that this means that every point in $\mathcal{B}$ sees the points in $\mathcal{A}$ in the same rotational order. If $\mathcal{B}$ also avoids $\mathcal{A}$, the two sets are called *mutually avoiding*. The bound in [1] on the size of the largest crossing family is proven in two steps: first it is shown that two mutually avoiding sets $\mathcal{A}$ and $\mathcal{B}$, each of size k, induce a crossing family of size k. Then it is shown that every set of n points in general position contains two mutually avoiding subsets of size $\sqrt{\frac{n}{12}}$. In this paper we will follow the same approach, but for a generalization of crossing families.

Bose et al. [2] have introduced the following generalization of crossing families: A *spoke set of size k* for $\mathcal{P}$ is a set $\mathcal{S}$ of k pairwise non-parallel lines such that in each unbounded region of the arrangement defined by the lines in $\mathcal{S}$ there lies at least one point of $\mathcal{P}$. Note that it is easy to obtain a spoke set from a crossing family by slightly rotating the supporting lines of the line segments in the crossing family. Then each endpoint of a line segment in the crossing family lies in a different unbounded region. We will show that every set of n points in general position contains a spoke set of size $\sqrt{\frac{n}{8}}$. To this end, we first translate the notion of spoke sets to the dual setting in Section 2. In Section 3 we then use the dual version to construct large spoke sets for the union of two point sets $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ avoids $\mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ can be separated by a line. Finally, we show that every point set contains such point sets $\mathcal{A}$ and $\mathcal{B}$ and give bounds on their sizes.

The motivation for the introduction of spoke sets in [2] is the fact that with a spoke set of size k for $\mathcal{P}$, one can construct a covering of the edge set of the complete geometric graph drawn on $\mathcal{P}$ with $n-k$ crossing-free spanning trees. The result in this paper thus also improves the previous upper bound of $n-\sqrt{\frac{n}{12}}$ for this problem. However, the original question from [2], whether there is always a spoke set of linear size, remains open.

Another interesting question is whether it is always possible to find a crossing family of size linear in the size of the largest spoke set. Theorem 6 is a first step in this direction as it characterizes the matchings obtained from spoke sets and shows that even though they might not all be crossing families, they still satisfy a number of conditions.

For space reasons, we will not be able to give all proofs. Instead, we refer the interested reader to full version [6].

Preliminaries

Let $\mathcal{S}$ be a spoke set of size k for $\mathcal{P}$. Consider the ordering of $\mathcal{S}=\{\ell_1,\dots,\ell_k\}$ by increasing slope. Let U_i^+ be the unbounded region that lies below $\ell_1,\dots,\ell_i$ and above $\ell_{i+1},\dots\ell_k$. Similarly let U_i^- be the unbounded region that lies above $\ell_1,\dots,\ell_i$ and below $\ell_{i+1},\dots\ell_k$. We call the regions U_i^+ and U_i^- *antipodal.*

Let $\mathcal{Q}$ be a subset of $\mathcal{P}$ that has exactly one point in each unbounded region. Note that then each line of $\mathcal{S}$ is a halving line for $\mathcal{Q}$. The *spoke matching* of $\mathcal{Q}$ is the matching obtained by drawing a straight line segment from each point p in $\mathcal{Q}$ to the unique point

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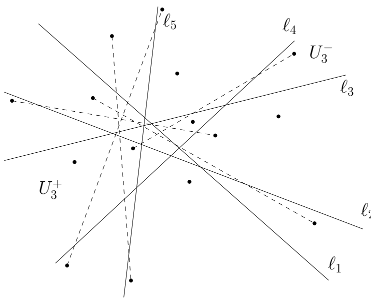

Figure 1: A spoke set and a spoke matching (dashed).

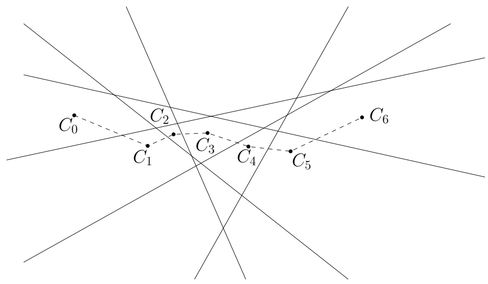

Figure 2: A line-monotone AB-semialternating cell-path of length 6.

q in $\mathcal{Q}$ that lies in the antipodal unbounded region of the spoke set. See Figure 1 for an example. Note that in a spoke matching, each edge intersects every line of the spoke set. In Section 4 we characterize the geometric matchings that are spoke matchings.

2 Spoke sets under duality

In this section we will translate the properties of spoke sets into the dual setting, that is under the point-line duality. For this we start with some definitions.

Given an arrangement $\mathcal{A}$ of lines, without loss of generality none of them horizontal or vertical, a *cell-path* R is a sequence of cells such that consecutive cells share an edge. If the edge shared by two consecutive cells is a subset of some line a_i of $\mathcal{A}$, we say that R *crosses* a_i. The *length* of a cell-path is one less than the number of cells involved. We call a cell-path *line-monotone* if it crosses each line of $\mathcal{A}$ at most once.

If $\mathcal{A}'$ is an arrangement induced by a subset of the lines of $\mathcal{A}$, then R *restricted to* $\mathcal{A}'$ is the cell path obtained by replacing each cell C of $\mathcal{A}$ in R by the cell C' in $\mathcal{A}'$ with $C \subseteq C'$ and deleting consecutive multiples.

Finally, for a cell-path $R = (C_0, C_1, \ldots, C_k)$, let a_i be the line in $\mathcal{A}$ that contains the edge shared by C_i and C_{i+1}. We call the pair (a_{2j}, a_{2j+1}) *AB-alternating*, if C_{2j+1} either lies above both a_{2j} and a_{2j+1} or below both. We call a cell path $P = (C_0, C_1, \ldots, C_{2k})$ *AB-semialternating* if for every $j < k$ the pair (a_{2j}, a_{2j+1}) is AB-alternating. See Figure 2 for an example.

We now have all the vocabulary that is necessary to describe the dual of spoke sets: given an arrangement $\mathcal{A}$ of lines, a *spoke path* $(R, \mathcal{A}')$ is a cell-path R together with an arrangement $\mathcal{A}'$ induced by a subset of the lines of $\mathcal{A}$, such that R restricted to $\mathcal{A}'$ is line-monotone and AB-semialternating. The length of a spoke path $(R, \mathcal{A}')$ is the length of R restricted to $\mathcal{A}'$. Note that all the definitions generalize to x-monotone pseudoline arrangements.

Lemma 1 *Let $\mathcal{P}$ be a point set and $\mathcal{P}^*$ its dual line arrangement. Then $\mathcal{P}$ contains a spoke set of size k if and only if $\mathcal{P}^*$ contains a spoke path of length $2k$.*

For a proof we refer to the full version. It is worth mentioning that for a spoke path $(R, \mathcal{A}')$, the primal of $\mathcal{A}'$ corresponds to a subset of $\mathcal{P}$ that has exactly one point in each unbounded region. The fact that all the points in the primal of $\mathcal{A}'$ are in unbounded regions follows from the line-monotonicity of R restricted to $\mathcal{A}'$. The AB-semialternation implies that two lines a_{2j} and a_{2j+1} correspond to endpoints of the spoke matching in the primal.

3 Finding large spoke sets

In this section, we will construct large spoke sets by constructing long spoke paths in the dual arrangement.

Lemma 2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two disjoint point sets of size k such that $\mathcal{A}$ avoids $\mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ can be separated by a line. Let $\mathcal{P} = \mathcal{A} \cup \mathcal{B}$. Then the dual arrangement $\mathcal{P}^*$ contains a spoke path of length $k+2$, if k is even, or $k+3$, if k is odd.*

For a full proof we again refer to the full version. But we will briefly sketch the main steps of the construction.

Step 1: Let $\mathcal{A}^*$ and $\mathcal{B}^*$ denote the duals of $\mathcal{A}$ and $\mathcal{B}$, respectively. Draw $\mathcal{B}^*$ as a wiring diagram in color red. As $\mathcal{A}$ avoids $\mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ can be separated by a line, all lines of $\mathcal{A}^*$ cross the lines of $\mathcal{B}^*$ in the same order, so we can draw $\mathcal{A}^*$ as pseudolines that are straight and vertical in the region where they cross the red pseudolines and get a pseudoline arrangement that is isomorphic to $\mathcal{P}^*$. We call such a drawing an *extended diagram.* See Figure 3 for an example of an extended diagram. Let r_1 be the bottommost pseudoline at left infinity of the wiring diagram of $\mathcal{B}^*$. For some directed pseudoline g, we define the *color sequence* $c(g)$ of g by moving along g and writing for each crossing with another pseudoline an r or a b if the crossed pseudoline is red or blue, respectively. In particular, $c(r_1)$ denotes the color sequence defined by moving along r_1 from left infinity to right infinity.

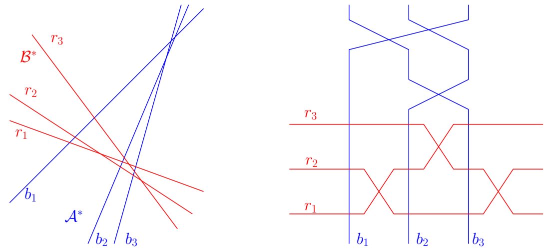

Figure 3: A line arrangement and its extended diagram.

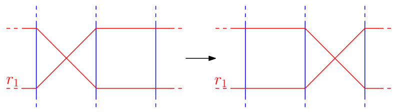

Figure 4: A right single crossing move.

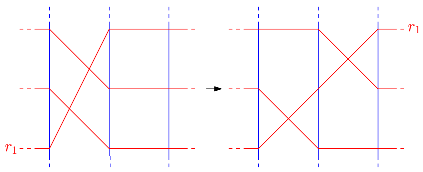

Figure 5: A right split crossing move.

Step 2: We modify the extended diagram using a sequence of moves. We use two different types of moves. For an illustration of the moves, see Figures 4 and 5. The *right (left) single crossing move* can be used if $c(r_1) = \dots brbb \dots$ ($c(r_1) = \dots bbrb \dots$). We move the crossing with the red pseudoline to the right (left), changing the color sequence of r_1 to $c(r_1) = \dots bbrb \dots$ ($c(r_1) = \dots brbb \dots$). The *right (left) split crossing move* can be used if there is more than one crossing with red pseudolines between two blue pseudolines, i.e., if $c(r_1) = \dots brr \dots rrbb \dots$ ($c(r_1) = \dots bbrr \dots rrb \dots$). We split the last of these crossings off and move it to the right (left), changing the color sequence of r_1 to $c(r_1) = \dots brr \dots rbrb \dots$ ($c(r_1) = \dots brbr \dots rrb \dots$). The same moves can also be defined if $c(r_1)$ starts with rbb or $r \dots rbb$ (ends with bbr or $bbr \dots r$). We do these moves until we reach a goal diagram in which r_1 has the color sequence $c(r_1) = brbrbr \dots brb$ (note that $c(r_1)$ has length $2k-1$). As in a split crossing move we split two consecutive r's and no move joins two r's, we can conclude that among the moves we need to reach the goal diagram, at most $k-2$ are split crossing moves. The goal diagram is of course not isomorphic to $\mathcal{P}^*$ anymore.

Step 3: We draw two new directed pseudolines g_1 and g_2 in the goal drawing, representing cell paths given by the cells they intersect. Let C_0 be the unbounded cell that is under all red pseudolines and left of all blue pseudolines. Both g_1 and g_2 start in C_0 and end in the antipodal cell of C_0, but g_1 crosses r_1 first and then always stays at a small distance to it, whereas g_2 always stays at small distance to r_1 and crosses it at the very end. Then g_1 and g_2 have the color sequences $c(g_1) = rbrbrbr \dots brb$ and $c(g_2) = brbrbr \dots brbr$, see Figure 6 for an illustration. For any color sequence we call a subsequence $x_1, \dots, x_j$ *semialternating* if j is even, i.e., $j = 2m$, and for every $i \leq m$ we have that $x_{2i-1} = r \Leftrightarrow x_{2i} = b$. By $\phi(g_1)$ and $\phi(g_2)$ we denote the length of the longest semialternating subsequences of $c(g_1)$ and $c(g_2)$, respectively. Note that by our construction of g_1 and g_2 we have that $\phi(g_1) = \phi(g_2) = 2k$.

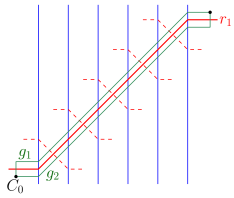

Figure 6: The goal diagram with the pseudolines g_1 and g_2.

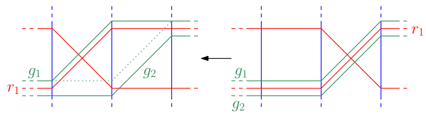

Figure 7: Reversing a single crossing move.

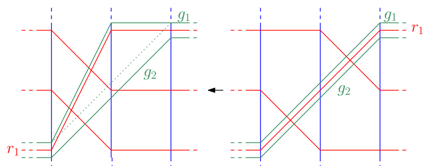

Figure 8: Reversing a split crossing move.

Step 4: We reverse the moves to get back to our initial extended diagram. While doing so, we change g_1 and g_2 only if one of them crosses r_1 more than once. In that case we just delete the part between the newly introduced crossings and replace it with a pseudoline segment that stays at a small distance to r_1. For an illustration see Figures 7 and 8. In each step we only need to change either g_1 or g_2, but never both. Also, $\phi(g_1)$ or $\phi(g_2)$ only changes when we reverse a split crossing move, where it decreases by 2 only for the pseudoline that was modified.

Step 5: We reach the initial extended diagram, but with two additional directed pseudolines g_1 and g_2, representing cell paths. For both of these directed pseudolines, the longest semialternating subsequence of the color sequence represents a line-monotone AB-semialternating cell-path, i.e., a spoke path of length $\phi(g_1)$ or $\phi(g_2)$, respectively. In the goal diagram we had $\phi(g_1) + \phi(g_2) = 4k$. While reversing the moves, this sum has only changed by the term -2 when we reversed a split crossing move. As we used at most $k-2$ split crossing moves, for the initial diagram we have $\phi(g_1) + \phi(g_2) \geq 4k - (k-2) \cdot 2 = 2k + 4$. The result now follows from the pigeonhole principle and the fact that by definition $\phi(g)$ is always even.

Corollary 3 *If a point set $\mathcal{P}$ contains two subsets $\mathcal{A}$ and $\mathcal{B}$ of size k, such that $\mathcal{A}$ avoids $\mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ can be separated by a line, then $\mathcal{P}$ contains a spoke set of size $\lceil \frac{k}{2} \rceil + 1$.*

Proof. Combine Lemma 1 and Lemma 2. ☐

Modifying the proof of Aronov et al. [1] for finding mutually avoiding sets in a point set, we can prove the following theorem:

Theorem 4 *Every point set of size n contains two point sets $\mathcal{A}$ and $\mathcal{B}$ of size $\lfloor \sqrt{\frac{n}{2}+1} - 1 \rfloor$ such that $\mathcal{A}$ avoids $\mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ can be separated by a line.*

A proof of this can be found in the full version.

Corollary 5 *Every point set $\mathcal{P}$ of size n allows a spoke set of size at least $\sqrt{\frac{n}{8}}$.*

Proof. By Theorem 4, $\mathcal{P}$ contains two subsets $\mathcal{A}$ and $\mathcal{B}$ of size $\lfloor \sqrt{\frac{n}{2}+1} - 1 \rfloor$ such that $\mathcal{A}$ avoids $\mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ can be separated by a line. Thus, by Corollary 3, the point set contains a spoke set of size

$$\left\lceil \frac{\lfloor \sqrt{\frac{n}{2}+1} - 1 \rfloor}{2} \right\rceil + 1 \geq \left\lceil \sqrt{\frac{n}{8} + \frac{1}{4}} - 1 \right\rceil + 1 \geq \sqrt{\frac{n}{8}}.$$

☐

It is worth mentioning that there are point sets that have no mutually avoiding subsets of size larger than $\mathcal{O}(\sqrt{n})$ [7]. However, it is not clear whether this still holds if we only insist that one of the subsets avoids the other one. So while there is no hope of finding larger crossing families by finding larger mutually avoiding subsets, it might still be possible to find larger spoke sets with this approach.

4 Spoke matchings

In this section we characterize a family of geometric matchings that arise from spoke sets. For this we need a few definitions:

Let e and f be two line segments and let s be the intersection of their supporting lines. If s lies in both e and f, we say that e and f *cross*. If s lies in f but not in e, we say that e *stabs* f and we call the vertex of e that is closer to s the *stabbing vertex* of e. If s lies neither in e nor in f, or if the supporting lines of e and f do not meet, we say that e and f are *parallel*.

A *stabbing chain* in a geometric matching are three edges, e, f and g, where e stabs f and f stabs g. We call f the *middle edge* of the stabbing chain.

Theorem 6 *A geometric matching M is a spoke matching if and only if it satisfies the following three conditions:*

(a) *no two edges are parallel,*

(b) *if an edge e stabs two other edges f and g, then the respective stabbing vertices of e lie inside the convex hull of f and g, and*

(c) *if there is a stabbing chain, then the stabbing vertex of the middle edge lies inside the convex hull of the other two edges.*

For a proof we refer to the full version. Note that the fact that every crossing family of size k induces a spoke set of size k can also be derived from this result, as it shows that every crossing family is a spoke matching. However, the family of spoke matchings also contains matchings that are not crossing families. In fact, it is even possible to construct a crossing-free spoke matching. In [5], it has been shown that there are sets of n points in general position that do not allow any matching satisfying conditions (a), (b) and (c) of size larger than $\frac{9}{20}n$. Hence we get the following corollary:

Corollary 7 *There are point sets of n points in general position that do not admit a spoke set of size larger than $\frac{9}{20}n$.*

References

[1] B. Aronov, P. Erdős, W. Goddard, D. J. Kleitman, M. Klugerman, J. Pach, and L. J. Schulman. Crossing families. In *Proceedings of the Seventh Annual Symposium on Computational Geometry, North Conway, NH, USA, June 10-12, 1991*, pages 351–356, 1991.

[2] P. Bose, F. Hurtado, E. Rivera-Campo, and D. R. Wood. Partitions of complete geometric graphs into plane trees. *Computational Geometry*, 34(2):116–125, 2006.

[3] R. Fulek and A. Suk. On disjoint crossing families in geometric graphs. *Electronic Notes in Discrete Mathematics*, 38:367–375, 2011.

[4] J. Pach and J. Solymosi. Halving lines and perfect cross-matchings. *Advances in Discrete and Computational Geometry*, 223:245–249, 1999.

[5] P. Schnider. Partitions and packings of complete geometric graphs with plane spanning double stars and paths. Master's thesis, ETH Zürich, 2015.

[6] P. Schnider. A generalization of crossing families. *CoRR*, abs/1702.07555, 2017.

[7] P. Valtr. On mutually avoiding sets. In *The mathematics of Paul Paul Erdős, II (R. L. Graham and J. Nešetřil, eds.) Algorithms and Combin. 14*, pages 324–332, 1997.