

Formal Languages, Formally and Coinductively

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Abstract

Traditionally, formal languages are defined as sets of words. More recently, the alternative coalgebraic or coinductive representation as infinite tries, i.e., prefix trees branching over the alphabet, has been used to obtain compact and elegant proofs of classic results in language theory. In this paper, we study this representation in the Isabelle proof assistant. We define regular operations on infinite tries and prove the axioms of Kleene algebra for those operations. Thereby, we exercise corecursion and coinduction and confirm the coinductive view being profitable in formalizations, as it improves over the set-of-words view with respect to proof automation.

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1 Introduction

If we ask a computer scientist what a formal language is, the answer will most certainly be: a set of words. Here, we advocate another valid answer: an infinite trie. This is the coalgebraic approach to languages [21], viewed through the lens of a lazy functional programmer.

This paper shows how to formalize the coalgebraic or coinductive approach to formal languages in the Isabelle/HOL proof assistant in the form of a gentle introduction to corecursion and coinduction. Our interest in the coalgebraic approach to formal languages arose in the context of a larger formalization effort of coalgebraic decision procedures for regular languages [26, 27]. Indeed, we present here a reusable library modeling languages, which lies at the core of those formalized decision procedures. A lesson we have learned from this exercise and hope to convey here is that often it is worthwhile to look at well-understood objects from a different (in this case coinductive) perspective.

When programming with infinite structures in the total setting of a proof assistant, productivity must be ensured. Intuitively, a corecursive function is productive if it always eventually yields observable output, e.g., in form of a constructor. Functions that output exactly one constructor before proceeding corecursively are called primitively corecursive—a fragment dual to well-understood primitively recursive functions on inductive datatypes. Primitively corecursive functions are productive. Currently, the only form of corecursion supported by Isabelle is primitive corecursion. While sophisticated methods involving domain, measure, and category theory for handling more complex corecursive specifications have been proposed [4, 16], we explore here how far primitive corecursion can get us. Restricting ourselves to this fragment is beneficial in several ways. First, our constructions become mostly Isabelle independent, since primitive corecursion is supported by all coinduction-friendly proof assistants. Second, when working in the restricted setting, we quickly hit and learn to understand the limits. In fact, we will face some non-primitively corecursive specifications on infinite tries, which we reduce to a composition of primitively corecursive



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specifications. Those reductions are insightful and hint at a general pattern for handling certain non-primitively corecursive specifications.

Infinite data structures are often characterized in terms of observations. For infinite tries, which we define as a coinductive datatype or short codatatype (Section 2), we can observe the root, which in our case is labeled by a Boolean value. This label determines if the empty word is *accepted* by the trie. Moreover, we can observe the immediate subtrees of a trie, of which we have one for each alphabet letter. This observation corresponds to making transitions in an automaton or rather computing the left quotient of a language by a letter. Indeed, we will see that Brzozowski’s ingenious derivative operation [7], which mimics this computation recursively on the syntax regular expressions, arises very naturally when defining regular operations corecursively on tries (Section 3). To validate our definitions, we prove by coinduction that they satisfy the axioms of Kleene algebra (Section 4). After having presented our formalization, we step back and connect concrete intuitive notions (such as tries) with abstract coalgebraic terminology (Section 5). Furthermore, we discuss our formalization and its relation to other work on corecursion and coinduction with or without proof assistants (Section 6).

The material presented in this paper is based on the publicly available Isabelle/HOL formalization [25] and is partly described in the author’s Ph.D. thesis [27].

Preliminaries

Isabelle/HOL is a proof assistant for higher-order logic, built around a small trusted inference kernel. The kernel accepts only non-recursive type and constant definitions. High-level specification mechanisms, which allow the user to enter (co)recursive specifications, reduce this input to something equivalent but non-recursive. The original (co)recursive specification is later derived as a theorem. For a comprehensive introduction to Isabelle/HOL we refer to a recent textbook [17, Part I].

In Isabelle/HOL types τ are built from type variables α, β , etc., via type constructors κ written postfix (e.g., $\alpha \kappa$). Some special types are the product type $\alpha \times \beta$ and the function type $\alpha \rightarrow \beta$, for which the type constructors are written infix. Infix operators bind less tightly than the postfix or prefix ones. Other important types are the type of Booleans *bool* inhabited by exactly two different values \top (truth) and \perp (falsity) and the types α *list* and α *set* of lists and sets of elements of type α . For Boolean connectives and sets common mathematical notation is used. A special constant is equality $= :: \alpha \rightarrow \alpha \rightarrow \text{bool}$, which is polymorphic (it exists for any type, including the function type, on which it is extensional, i.e., $(\forall x. f x = g x) \longrightarrow f = g$). Lists are constructed from $[] :: \alpha \text{ list}$ and $\# :: \alpha \rightarrow \alpha \text{ list} \rightarrow \alpha \text{ list}$; the latter written infix. The notation $|xs|$ stands for the length of the list xs .

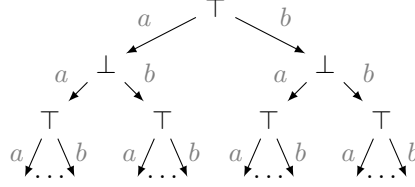
2 Languages as Infinite Tries

We define the type of formal languages as a codatatype of infinite tries, that is, (prefix) trees of infinite depth branching over the alphabet. We represent the alphabet by the type parameter α . Each node in a trie carries a Boolean label, which indicates whether the (finite) path to this node constitutes a word inside or outside of the language. The function type models branching: for each letter $x :: \alpha$ there is a subtree, which we call *x-subtree*.

$$\text{codatatype } \alpha \text{ lang} = \mathbf{L} (o : \text{bool}) (\delta : \alpha \rightarrow \alpha \text{ lang})$$

The `codatatype` command defines the type $\alpha \text{ lang}$ together with a *constructor* $\mathbf{L} :: \text{bool} \rightarrow (\alpha \rightarrow \alpha \text{ lang}) \rightarrow \alpha \text{ lang}$ and two *selectors* $o :: \alpha \text{ lang} \rightarrow \text{bool}$ and $\delta :: \alpha \text{ lang} \rightarrow \alpha \rightarrow \alpha \text{ lang}$

For a binary alphabet $\alpha = \{a, b\}$, the trie *even* shown in Figure 1 is an inhabitant of α lang. The label of its root is given by $o\ even = \top$ and its subtrees by another trie $odd = \delta\ even\ a = \delta\ even\ b$. Similarly, we have $o\ odd = \perp$ and $even = \delta\ odd\ a = \delta\ odd\ b$. Note that we could have equally written $even = \mathsf{L}\ \top\ (\lambda_.\ odd)$ and $odd = \mathsf{L}\ \perp\ (\lambda_.\ even)$ to obtain the same mutual characterization of *even* and *odd*.



■ **Figure 1** Infinite trie *even*

We gave our type the name α lang, to remind us to think of its inhabitants as formal languages. In the following, we use the terms language and trie synonymously.

Beyond defining the type and the constants, the `codatatype` command also exports a wealth of properties about them such as $o(\mathsf{L}\ b\ d) = b$, the injectivity of L , or more interestingly the coinduction rule. Informally, coinduction allows us to prove equality of tries which cannot be distinguished by finitely many selector applications.

Clearly, we would like to identify the trie *even* with the regular language of all words of even length $\{w \in \{a, b\}^* \mid |w| \bmod 2 = 0\}$, also represented by the regular expression $((a + b) \cdot (a + b))^*$. Therefore, we define the notion of word membership \in on tries by primitive (or structural) recursion on the word using Isabelle’s `primrec` command.

```
primrec  $\in :: \alpha\ list \rightarrow \alpha\ lang \rightarrow bool$  where
  []  $\in L = o\ L$ 
  (x # w)  $\in L = w \in \delta\ L\ x$ 
```

Using \in , each trie can be assigned a language in the traditional set of lists view.

```
definition out ::  $\alpha\ lang \rightarrow \alpha\ list\ set$  where
  out L = {w | w  $\in L$ }
```

With this definition, we obtain $out\ even = \{w \in \{a, b\}^* \mid |w| \bmod 2 = 0\}$.

3 Regular Operations on Tries

So far, we have only specified some concrete infinite tries informally. Formally, we will use primitive corecursion, which is dual to primitive recursion. Primitively recursive functions consume one constructor before proceeding recursively. Primitively corecursive functions produce one constructor whose arguments are allowed to be either non-recursive terms or a corecursive call (applied to arbitrary non-recursive arguments). The `primcorec` command reduces a primitively corecursive specification to a non-recursive definition, which is accepted by Isabelle’s inference kernel [3]. Internally, the reduction employs a dedicated combinator for primitive corecursion on tries generated by the `codatatype` command.

3.1 Primitively Corecursive Operations

We start with some simple examples: the languages of the base cases of regular expressions. Intuitively, the trie \emptyset representing the empty language is labeled with \perp everywhere and

the trie ε representing the empty word language is labeled with \top at its root and with \perp everywhere else. The trie $A a$ representing the singleton language of the one letter word a is labeled with \perp everywhere except for the root of its a -subtree. This intuition is easy to capture formally.

<pre> primcorec $\emptyset :: \alpha \text{ lang where}$ $\emptyset = \mathbf{L} \perp (\lambda x. \emptyset)$ primcorec $\varepsilon :: \alpha \text{ lang where}$ $\varepsilon = \mathbf{L} \top (\lambda x. \emptyset)$ </pre>	<pre> primcorec $A :: \alpha \rightarrow \alpha \text{ lang where}$ $o (A a) = \perp$ $\delta (A a) = \lambda x. \text{if } a = x \text{ then } \varepsilon \text{ else } \emptyset$ </pre>
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Among these three definitions only \emptyset is truly corecursive.

The specifications on the left differ syntactically from the one on the right. The constants \emptyset and ε are defined using the so called *constructor view*. It allows the user to enter equations of the form constant or function equals constructor, where the arguments of the constructor are restricted as described above (modulo some further syntactic conveniences, such as lambda abstractions, **case**-, and **if**-expressions). This kind of definitions should be familiar to any (lazy) functional programmer.

In contrast, the specification of A is expressed in the *destructor view*. Here, we specify the constant or function being defined by observations or experiments via *selector equations*. The allowed experiments on a trie are given by its selectors o and δ . We can observe the label at the root and the subtrees. Specifying the observation, again restricted to be either a non-recursive term or a direct corecursive call, for each selector yields a unique characterization of the function being defined.

It is straightforward to rewrite specifications in either of the views into the other one. The **primcorec** command performs this rewriting internally and outputs the theorems corresponding to the user's input specification in both views. The constructor view theorems serve as executable code equations. Isabelle's code generator [10] can use these equations to generate code which make sense in programming languages with lazy evaluation. In contrast, the destructor view offers safe simplification rules even when applied eagerly during rewriting as done by Isabelle's simplifier. Note that constructor view specifications such as $\emptyset = \mathbf{L} \perp (\lambda x. \emptyset)$ will cause the simplifier to loop when applied eagerly.

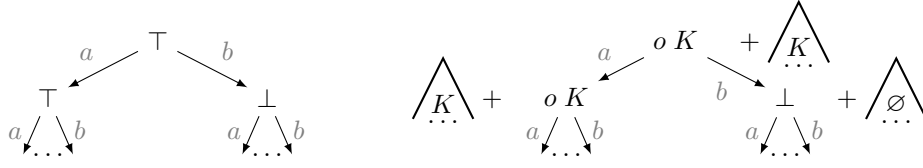
Now that the basic building blocks \emptyset , ε , and A are in place, we turn our attention to how to combine them to obtain more complex languages. We start with the simpler combinators for union, intersection, and negation, before moving to the more interesting concatenation and iteration. The union $+$ of two tries should denote set union of languages (i.e., $\text{out } (L + K) = \text{out } L \cup \text{out } K$ should hold). It is defined corecursively by traversing the two tries in parallel and computing for each pair of labels their disjunction. Intersection \cap is analogous. Negation \neg simply inverts every label.

```

primcorec  $+$   $:: \alpha \text{ lang} \rightarrow \alpha \text{ lang} \rightarrow \alpha \text{ lang where}$ 
   $o (L + K) = o L \vee o K$ 
   $\delta (L + K) = \lambda x. \delta L x + \delta K x$ 
primcorec  $\cap :: \alpha \text{ lang} \rightarrow \alpha \text{ lang} \rightarrow \alpha \text{ lang where}$ 
   $o (L \cap K) = o L \wedge o K$ 
   $\delta (L \cap K) = \lambda x. \delta L x \cap \delta K x$ 
primcorec  $\neg :: \alpha \text{ lang} \rightarrow \alpha \text{ lang where}$ 
   $o (\neg L) = \neg o L$ 
   $\delta (\neg L) = \lambda x. \neg \delta L x$ 

```

Let us look at the specifying selector equations which we have seen so far from a different perspective. Imagine L and K being not tries but instead syntactic regular expressions, A ,



■ **Figure 2** Tries for L (left) and the concatenation $L \cdot K$ (right)

$+$, \cap , and \neg constructors of a datatype for regular expressions, and o and δ two operations that we define recursively on this syntax. From that perspective, the operations are familiar: we have rediscovered Brzowski derivatives of regular expressions [7] and the empty word acceptance test on regular expressions in the destructor view equations for the selectors δ and o . There is an important difference, though: while Brzowski derivatives work with syntactic objects, our tries are semantic objects on which equality denotes language equivalence. For example, we will later prove $\emptyset + L = L$ for tries, whereas $\emptyset + L \neq L$ holds for regular expressions. The coinductive view reveals that derivatives and the acceptance test are the two fundamental ingredients that completely characterize regular languages and arise naturally even when only considering the semantics.

3.2 Reducing Corecursion Up-to to Primitive Corecursion

Concatenation \cdot is next on the list of regular operations that we want to define on tries. Thinking of Brzowski derivatives and the acceptance test, we would usually specify it by the following two equations.

$$\begin{aligned} o(L \cdot K) &= oL \wedge oK \\ \delta(L \cdot K) &= \lambda x. (\delta L x \cdot K) + (\text{if } oL \text{ then } \delta K x \text{ else } \emptyset) \end{aligned}$$

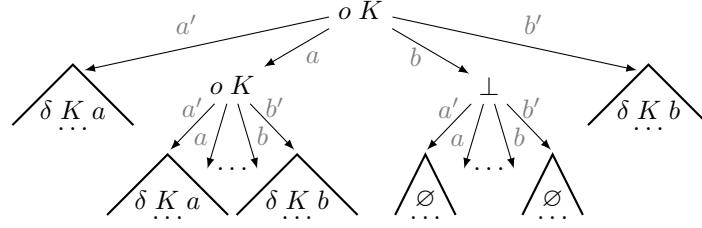
A difficulty arises here, since this specification is not primitively corecursive—the right hand side of the second equation contains a corecursive call but not at the topmost position (which is $+$ here). We call this kind of corecursion *up to* $+$.

Without tool support for corecursion up-to, concatenation must be defined differently—as a composition of other primitively corecursive operations. Intuitively, we would like to separate the above δ -equation into two along the $+$ and sum them up afterwards. Technically, the situation is more involved. Since the δ -equation is corecursive, we cannot just create two tries by primitive corecursion.

Figure 2 depicts the trie that should result from concatenating an arbitrary trie K to the concrete given trie L . Procedurally, the concatenation must replace every subtree T of L that has \top at the root (those are positions where words from L end) by the trie $U + K$ where U is the trie obtained from T by changing its root from \top to oK . For uniformity with the above δ -equation, we imagine subtrees F of L with label \perp at the root as also being replaced by $F + \emptyset$, which, as we will prove later, has the same effect as leaving F alone.

Figure 3 presents one way to bypass the restrictions imposed by primitive corecursion. We are not allowed to use $+$ *after* proceeding corecursively, but we may arrange the arguments of $+$ in a broader trie over a doubled alphabet formally modeled by pairing letters of the alphabet with a Boolean flag. In Figure 3 we write a for (a, \top) and a' for (a, \perp) . Because it defers the summation, we call this primitively corecursive procedure *deferred concatenation* $\hat{\cdot}$.

$$\begin{aligned} \text{primcorec } \hat{\cdot} &:: \alpha \text{ lang} \rightarrow \alpha \text{ lang} \rightarrow (\alpha \times \text{bool}) \text{ lang} \text{ where} \\ o(L \hat{\cdot} K) &= oL \wedge oK \\ \delta(L \hat{\cdot} K) &= \lambda(x, b). \text{ if } b \text{ then } \delta L x \hat{\cdot} K \text{ else if } oL \text{ then } \delta K x \hat{\cdot} \varepsilon \text{ else } \emptyset \end{aligned}$$



■ **Figure 3** Trie for deferred concatenation $L \hat{\cdot} K$

Note that unlike in the Figure 3, where we informally plug the trie $\delta K x$ as some x' -subtrees, the formal definition must be more careful with the types. More precisely, $\delta K x$ is of type $\alpha \text{ lang}$, while something of type $(\alpha \times \text{bool}) \text{ lang}$ is expected. This type mismatch is resolved by further concatenating ε to $\delta K x$ (again in a deferred fashion) without corrupting the intended semantics.

Once the trie for the deferred concatenation has been built, the desired trie for concatenation can be obtained by a second primitively corecursive traversal that sums the x - and x' -subtrees *before* proceeding corecursively.

primcorec $\hat{\oplus} :: (\alpha \times \text{bool}) \text{ lang} \rightarrow \alpha \text{ lang}$ where
 $o(\hat{\oplus} L) = o L$
 $\delta(\hat{\oplus} L) = \lambda x. \hat{\oplus}(\delta L(x, \top) + \delta L(x, \perp))$

Finally, we can define the concatenation as the composition of $\hat{\cdot}$ and $\hat{\oplus}$. The earlier standard selector equations for \cdot are provable for this definition.

definition $\cdot :: \alpha \text{ lang} \rightarrow \alpha \text{ lang} \rightarrow \alpha \text{ lang}$ where
 $L \cdot K = \hat{\oplus}(L \hat{\cdot} K)$

The situation with iteration is similar. The selector equations following the Brzozowski derivative of L^* yield a non-primitively corecursive specification: it is corecursive up to \cdot .

$o(L^*) = \top$
 $\delta(L^*) = \lambda x. \delta L x \cdot L^*$

As before, the restriction is circumvented by altering the operation slightly. We define the binary operation *deferred iteration* $L \hat{*} K$, whose language should represent $L \cdot K^*$ (although we have not defined $*$ yet). When constructing the subtrees of $L \hat{*} K$ we keep pulling copies of the second argument into the first argument before proceeding corecursively (the second argument itself stays unchanged).

primcorec $\hat{*} :: \alpha \text{ lang} \rightarrow \alpha \text{ lang} \rightarrow \alpha \text{ lang}$ where
 $o(L \hat{*} K) = o L$
 $\delta(L \hat{*} K) = \lambda x. (\delta(L \cdot (\varepsilon + K)) x) \hat{*} K$

Supplying ε as the first argument to $\hat{*}$ defines iteration for which the original selector equations hold.

definition $_{}^* :: \alpha \text{ lang} \rightarrow \alpha \text{ lang}$ where
 $L^* = \varepsilon \hat{*} L$

We have defined all the standard regular operations on tries. Later we will prove that those definitions satisfy the axioms of Kleene algebra, meaning that they behave as expected. Already now we can compose the operations to define new tries, for example the introductory $\text{even} = ((A a + A b) \cdot (A a + A b))^*$.

Beyond Regular Languages

Before we turn to proving, let us exercise one more corecursive definition. Earlier, we have assigned each trie a set of lists via the function `out`. Primitive corecursion enables us to define the converse operation.

```
primcorec in ::  $\alpha$  list set  $\rightarrow$   $\alpha$  lang where
  o (in L) = []  $\in$  L
   $\delta$  (in L) =  $\lambda x$ . in {w | x # w  $\in$  L}
```

The function `out` and `in` are both bijections. We show this by proving that their compositions (either way) are both the identity function. One direction, `out (in L) = L`, follows by set extensionality and a subsequent induction on words. The reverse direction requires a proof by coinduction, which is the topic of the next section.

Using `in` we can turn every (even undecidable) set of lists into a trie. This is somewhat counterintuitive, since, given a concrete trie, its word problem seems easily decidable (via the function `∈`). But of course in order to compute the trie `out` a list of sets L via in the word problem for L must be solved—we are reminded that higher-order logic is not a programming language where everything is executable, but a logic in which we write down specifications (and not programs). For regular operations from the previous section the situation is different. For example, Isabelle’s code generator can produce valid Haskell code that evaluates $[a, b, a, a] \in (A a \cdot (A a + A b))^*$ to \top .

4 Proving Equalities on Tries

We have seen the definitions of many operations, justifying their meaningfulness by appealing to the reader’s intuition. This is often not enough to guarantee correctness, especially if we have a theorem prover at hand. So let us formally prove in Isabelle that the regular operations on tries form a Kleene algebra by proving Kozen’s famous axioms [13] as (in)equalities on tries.

Codatatypes are equipped with the perfect tool for proving equalities: the coinduction principle. Intuitively, this principle states that the existence of a relation R that is closed under the codatatype’s observations (given by selectors) implies that elements related by R are equal. In other words, if we cannot distinguish elements of a codatatype by (finite) observations, they must be equal. Formally, for our codatatype α lang we obtain the following coinduction rule.

$$\frac{R L K \quad \forall L_1 L_2. R L_1 L_2 \longrightarrow (o L_1 = o L_2 \wedge \forall x. R (\delta L_1 x) (\delta L_2 x))}{L = K}$$

The second assumption of this rule formalizes the notion of being closed under observations: If two tries are related then their root’s labels are the same and all their subtrees are related. A relation that satisfies this assumption is called a *bisimulation*. Thus, proving an equation by coinduction amounts to exhibiting a bisimulation witness that relates the left and the right hand sides.

Let us observe the coinduction rule, called *coinduct_{lang}* below, in action. Figure 4 shows a detailed proof of the Kleene algebra axiom that the empty language is the left identity of union that is accepted by Isabelle. After applying the coinduction rule backwards (line 2), the proof has three parts. First, we supply a definition of a witness relation R (line 3). Second, we prove that R relates the left and the right hand side (line 4). Third, we prove that R is a bisimulation (lines 5–8). Proving $R (\emptyset + L) L$ for our particular definition of R

```

1  theorem  $\emptyset + L = L$ 
2  proof (rule coinductlang)
3    def  $R L_1 L_2 = (\exists K. L_1 = \emptyset + K \wedge L_2 = K)$ 
4    show  $R (\emptyset + L) L$  by simp
5    fix  $L_1 L_2$ 
6    assume  $R L_1 L_2$ 
7    then obtain  $K$  where  $L_1 = \emptyset + K$  and  $L_2 = K$  by simp
8    then show  $o L_1 = o L_2 \wedge \forall x. R (\delta L_1 x) (\delta L_2 x)$  by simp
9  qed

```

■ **Figure 4** A detailed proof by coinduction

is trivial after instantiating the existentially quantified K with L . Proving the bisimulation property is barely harder: for two tries L_1 and L_2 related by R we need to show $o L_1 = o L_2$ and $\forall x. R (\delta L_1 x) (\delta L_2 x)$. Both properties follow by simple calculations rewriting L_1 and L_2 in terms of a trie K (line 7), whose existence is guaranteed by $R L_1 L_2$, and simplifying with the selector equations for $+$ and \emptyset .

$$\begin{aligned}
o L_1 &= o (\emptyset + K) = (o \emptyset \vee o K) = (\perp \vee o K) = o K = o L_2 \\
R (\delta L_1 x) (\delta L_2 x) &= R (\delta (\emptyset + K) x) (\delta K x) \\
&= R (\delta \emptyset x + \delta K x) (\delta K x) = R (\emptyset + \delta K x) (\delta K x) \\
&= (\exists K'. \emptyset + \delta K x = \emptyset + K' \wedge \delta K x = K') = \top
\end{aligned}$$

The last step is justified by instantiating K' with $\delta K x$.

So in the end, the only part that required ingenuity was the definition of the witness R . But was it truly ingenious? It turns out that in general, when proving a conditional equation $P \bar{x} \longrightarrow l \bar{x} = r \bar{x}$ by coinduction, where \bar{x} denotes a list of variables occurring freely in the equation, the canonical choice for the bisimulation witness is $R a b = (\exists \bar{x}. a = l \bar{x} \wedge b = r \bar{x} \wedge P \bar{x})$. There is no guarantee that this definition yields a bisimulation. However, after performing a few proofs by coinduction, this particular pattern emerges as a natural first choice to try. Indeed, the choice is so natural, that it was worth to automate it in the *coinduction* proof method [3]. This method instantiates the coinduction rule *coinduct_{lang}* with the canonical bisimulation witness constructed from the goal, where the existentially quantified variables must be given explicitly using the *arbitrary* modifier. Then it applies the instantiated rule in a resolution step, discharges the first assumption, and unpacks the existential quantifiers from R in the remaining subgoal (the **obtain** step in the above proof). Many proofs collapse to an automatic one-line proof using this convenience, including the one above. Some examples follow.

```

theorem  $\emptyset + L = L$  by (coinduction arbitrary: L) auto
theorem  $L + L = L$  by (coinduction arbitrary: L) auto
theorem  $(L_1 + L_2) + L_3 = L_1 + L_2 + L_3$  by (coinduction arbitrary: L1 L2 L3) auto
theorem in (out L) = L by (coinduction arbitrary: L) auto
theorem in (L  $\cup$  K) = in L + in K by (coinduction arbitrary: L K) auto

```

As indicated earlier, sometimes the *coinduction* method does not succeed. It is instructive to consider one example where this is the case: $o L \longrightarrow \varepsilon + L = L$.

If we attempt to prove the above statement by coinduction instantiated with the canonical bisimulation witness $R \ L_1 \ L_2 = (\exists K. L_1 = \varepsilon + K \wedge L_2 = K \wedge o K)$, after some simplification we are stuck with proving that R is a bisimulation.

$$\begin{aligned} R (\delta L_1 x) (\delta L_2 x) &= R (\delta (\varepsilon + K) x) (\delta K x) \\ &= R (\delta \varepsilon x + \delta K x) (\delta K x) = R (\emptyset + \delta K x) (\delta K x) \\ &= R (\delta K x) (\delta K x) = \exists K'. \delta K x = \varepsilon + K' \wedge \delta K x = K' \wedge o K' \end{aligned}$$

The remaining goal is not provable: we would need to instantiate K' with $\delta K x$, but then, we are left to prove $o (\delta K x)$, which we cannot deduce (we only know $o K$). If we, however, instantiate the coinduction rule with $R^\bar{\ } L_1 \ L_2 = R \ L_1 \ L_2 \vee L_1 = L_2$, we are able to finish the proof. This means that our canonical bisimulation witness R is not a bisimulation, but $R^\bar{\ }$ is. In such cases R is called a *bisimulation up to equality*.

Instead of modifying the *coinduction* method to instantiate the rule $coinduct_{lang}$ with $R^\bar{\ }$, it is more convenient to capture this improvement directly in the coinduction rule. This results in the following rule which we call *coinduction up to equality* or $coinduct_{lang}^\bar{\ }$.

$$\frac{R \ L \ K \quad \forall L_1 \ L_2. R \ L_1 \ L_2 \longrightarrow (o L_1 = o L_2 \wedge \forall x. R^\bar{\ } (\delta L_1 x) (\delta L_2 x))}{L = K}$$

Note that $coinduct_{lang}^\bar{\ }$ is not just an instance of $coinduct_{lang}$, with R replaced by $R^\bar{\ }$. Instead, after performing this replacement, the first assumption is further simplified to $R \ L \ K$ —we would not use coinduction in the first place, if we could prove $R^\bar{\ } L \ K$ by reflexivity. Similarly, the reflexivity part in the occurrence on the left of the implication in the second assumption is needless and therefore eliminated. Using this coinduction up to equality rule, we have regained the ability to write one-line proofs.

theorem $o L \longrightarrow \varepsilon + L = L$ by (*coinduction arbitrary: L rule: $coinduct_{lang}^\bar{\ }$ auto*)

This brings us to the next hurdle. Suppose that we already have proved the standard selector equations for concatenation \cdot . (This requires finding some auxiliary properties of $\hat{\cdot}$ and $\hat{\oplus}$ but is an overall straightforward usage of coinduction up to equality.) Next, we want to reason about \cdot . For example, we prove its distributivity over $+$: $(L + K) \cdot M = (L \cdot M) + (K \cdot M)$. As before, we are stuck proving that the canonical bisimulation witness $R \ L_1 \ L_2 = (\exists L' \ K' \ M'. L_1 = (L' + K') \cdot M' \wedge L_2 = (L' \cdot M') + (K' \cdot M'))$ is a bisimulation (and this time even for up to equality).

$$\begin{aligned} R^\bar{\ } (\delta L_1 x) (\delta L_2 x) &= R^\bar{\ } (\delta ((L' + K') \cdot M') x) (\delta ((L' \cdot M') + (K' \cdot M')) x) \\ &= \begin{cases} R^\bar{\ } ((\delta L' x + \delta K' x) \cdot M') & \\ \quad ((\delta L' x \cdot M') + (\delta K' x \cdot M')) & \text{if } \neg o L' \wedge \neg o K' \\ R^\bar{\ } ((\delta L' x + \delta K' x) \cdot M' + \delta M' x) & \\ \quad ((\delta L' x \cdot M' + \delta M' x) + (\delta K' x \cdot M')) & \text{if } o L' \wedge \neg o K' \\ R^\bar{\ } ((\delta L' x + \delta K' x) \cdot M' + \delta M' x) & \\ \quad ((\delta L' x \cdot M') + (\delta K' x \cdot M' + \delta M' x)) & \text{if } \neg o L' \wedge o K' \\ R^\bar{\ } ((\delta L' x + \delta K' x) \cdot M' + \delta M' x) & \\ \quad ((\delta L' x \cdot M' + \delta M' x) + (\delta K' x \cdot M' + \delta M' x)) & \text{if } o L' \wedge o K' \end{cases} \\ &= \begin{cases} \top & \text{if } \neg o L' \wedge \neg o K' \\ R^\bar{\ } ((\delta L' x + \delta K' x) \cdot M' \ \color{gray}{+} \ \delta M' x) & \\ \quad ((\delta L' x \cdot M' + \delta K' x \cdot M') \ \color{gray}{+} \ \delta M' x) & \text{otherwise} \end{cases} \end{aligned}$$

The remaining subgoal cannot be discharged by the definition of R . In principle it could be discharged by equality (the two tries are equal), but this is almost exactly the property we originally started proving, so we have not simplified the problem by coinduction but rather are going in circles here. However, if our relation could somehow split its arguments across the outermost $+$ highlighted in gray, we could prove the left pair being related by R and the right pair by $=$. The solution is easy: we allow the relation to do just that. Accordingly, we define the congruence closure R^+ of a relation R under $+$ inductively by the following rules.

$$\frac{L = K}{R^+ L K} \quad \frac{R L K}{R^+ L K} \quad \frac{R^+ L K}{R^+ K L} \quad \frac{R^+ L_1 L_2 \quad R^+ L_2 L_3}{R^+ L_1 L_3} \quad \frac{R^+ L_1 K_1 \quad R^+ L_2 K_2}{R^+ (L_1 + L_2) (K_1 + K_2)}$$

The closure R^+ is then used to strengthen the coinduction rule, just as the earlier reflexive closure $R^=$ strengthening. Note that the closure $R^=$ can also be viewed as inductively generated by the first two of the above rules. In summary, we obtain *coinduction up to congruence of $+$* , denoted by $\text{coinduct}_{\text{lang}}^+$.

$$\frac{R L K \quad \forall L_1 L_2. R L_1 L_2 \longrightarrow (o L_1 = o L_2 \wedge \forall x. R^+ (\delta L_1 x) (\delta L_2 x))}{L = K}$$

As intended this rule makes the proof of distributivity into another automatic one-liner. This is because our new proof principle, coinduction up to congruence of $+$, matches exactly the definitional principle of corecursion up to $+$ used in the selector equations of \cdot .

theorem $(L + K) \cdot M = (L \cdot M) + (K \cdot M)$
 by (*coinduction arbitrary: $L K M$ rule: $\text{coinduct}_{\text{lang}}^+$*) *auto*

For properties involving iteration $*$, whose selector equations are corecursive up to \cdot , we will need a further strengthening of the coinduction rule (using the congruence closure under \cdot). Overall, the most robust solution to keep track of the different rules is to maintain a coinduction rule up to all previously defined operations on tries: we define R^\bullet to be the congruence closure of R under $+$, \cdot , and $*$ and then use the following rule for proving.

$$\frac{R L K \quad \forall L_1 L_2. R L_1 L_2 \longrightarrow (o L_1 = o L_2 \wedge \forall x. R^\bullet (\delta L_1 x) (\delta L_2 x))}{L = K}$$

We will not spell out all axioms of Kleene algebra [13] and their formal proofs [25] here. Most proofs are automatic; some require a few manual hints in which order to apply the congruence rules. Note that the axioms also contain some inequalities, such as $\varepsilon + L \cdot L^* \leq L^*$, and even conditional inequalities, such as $L + M \cdot K \leq M \longrightarrow L \cdot K^* \leq M$. However, $L \leq K$ is defined as $L + K = K$, such that proofs by coinduction also are applicable here.

5 Coalgebraic Foundations

We briefly connect the formalized but still intuitive notions, such as tries, from earlier sections with the key coalgebraic concepts and terminology that is usually used to present the coalgebraic view on formal languages. Thereby, we explain how particularly useful abstract objects gave rise to concrete tools in Isabelle/HOL. More theoretical and detailed introductions to coalgebra can be found elsewhere [12, 22].

Given a functor F an (F -)coalgebra is a *carrier* object A together with a map $A \rightarrow F A$ —the *structural map* of a coalgebra. In the context of higher-order logic—that is in the category of types which consists of types as objects and of functions between types as arrows—a functor is a type constructor F together with a map function $\text{map}_F :: (\alpha \rightarrow \beta) \rightarrow \alpha F \rightarrow \beta F$

$$\begin{array}{ccc}
\alpha & \xrightarrow{s} & \alpha F \\
f \downarrow & & \downarrow \text{map}_F f \\
\beta & \xrightarrow{t} & \beta F
\end{array}$$

■ **Figure 5** Commutation property of a coalgebra morphism

$$\begin{array}{ccc}
\tau & \xrightarrow{s} & \tau D \\
\text{corec } s \downarrow & & \downarrow \text{map}_D (\text{corec } s) \\
\alpha \text{ lang} & \xrightarrow{\langle o, \delta \rangle} & \alpha \text{ lang } D
\end{array}$$

■ **Figure 6** Unique morphism $\text{corec } s$ to the final coalgebra $(\alpha \text{ lang}, \langle o, \delta \rangle)$

that preserves identity and composition: $\text{map}_F \text{id} = \text{id}$ and $\text{map}_F (f \circ g) = \text{map}_F f \circ \text{map}_F g$. An F -coalgebra in HOL is therefore simply a function $s :: \alpha \rightarrow \alpha F$. A function $f :: \alpha \rightarrow \beta$ is a coalgebra *morphism* between two coalgebras $s :: \alpha \rightarrow \alpha F$ and $t :: \beta \rightarrow \beta F$ if it satisfies the commutation property $t \circ f = \text{map}_F f \circ s$, also depicted by the commutative diagram in Figure 5.

An (F -)coalgebra to which there exists a unique morphism from any other coalgebra is called a *final (F -)coalgebra*. Not all functors F admit a final coalgebra. Two different final coalgebras are necessarily isomorphic. Final coalgebras correspond to codatatypes in Isabelle/HOL. Isabelle’s facility for defining codatatypes maintains a large class of functors—bounded natural functors [28]—for which a final coalgebra does exist. Moreover, for any bounded natural functor F , Isabelle can construct its final coalgebra with the codatatype CF as the carrier and define a bijective constructor $C_F :: CF F \rightarrow CF$ and its inverse, the destructor $D_F :: CF \rightarrow CF F$. The latter takes the role of the structural map of the coalgebra.

$$\text{codatatype } CF = C_F (D_F : CF F)$$

Finally, we are ready to connect these abstract notions to our tries. The codatatype of tries $\alpha \text{ lang}$ is the final coalgebra of the functor $\beta D = \text{bool} \times (\alpha \rightarrow \beta)$ with the associated map function $\text{map}_D g = \text{id} \times (\lambda f. g \circ f)$, where $(f \times g) (x, y) = (f x, g y)$. The structural map of this final coalgebra is the function $D_D = \langle o, \delta \rangle$, where $\langle f, g \rangle x = (f x, g x)$.

The finality of $\alpha \text{ lang}$ gives rise to the definitional principle of primitive corecursion. In Isabelle this principle is embodied by the primitive corecursor $\text{corec} :: (\tau \rightarrow \tau D) \rightarrow \tau \rightarrow \alpha \text{ lang}$, that assigns to the given D -coalgebra the unique morphism from itself to the final coalgebra. In other words, the primitive corecursor allows us to define functions of type $\tau \rightarrow \alpha \text{ lang}$ by providing a D -coalgebra on τ , i.e., a function of type $\tau \rightarrow \text{bool} \times (\alpha \rightarrow \tau)$ that essentially describes a deterministic (not necessarily finite) automaton without an initial state. To clarify this automaton analogy, it is customary to present the F -coalgebra s as two functions $s = \langle o, d \rangle$ with τ being the states of the automaton, $o : \tau \rightarrow \text{bool}$ denoting accepting states, and $d : \alpha \rightarrow \tau \rightarrow \tau$ being the transition function. From a given s , we uniquely obtain the function $\text{corec } s$ that assigns to a separately given initial state $t : \tau$ the language $\text{corec } s t : \alpha \text{ lang}$ and makes the diagram in Figure 6 commute.

The `primcorec` command [3] reduces a user given specification to a non-recursive definition using the corecursor. For example, the union operation $+$ is internally defined as $\lambda L K. \text{corec } (\lambda (L, K). (o L \wedge o K, \lambda a. (\delta L a, \delta K a))) (L, K)$. The D -coalgebra given as the argument to `corec` resembles the right hand sides of the selector equations for $+$ (with the corecursive calls stripped away). As end users, most of the time we are happy being able to write the high-level corecursive specifications, without having to explicitly supply coalgebras.

It is worth noting that the final coalgebra $\alpha \text{ lang}$ itself corresponds to the automaton, whose states are languages, acceptance is given by $\circ L = o L$, and the transition function by $d a L = \delta L a$. For these definitions, we obtain $\text{corec } \langle \circ, d \rangle (L : \alpha \text{ lang}) = L$. For regular languages this automaton corresponds to the minimal automaton (since equality on tries corresponds to language equivalence), which is finite by the Myhill–Nerode theorem. This correspondence is not very practical though, since we typically label states of automata with something finite, in particular not with languages (represented by infinite tries).

A second consequence of the finality of $\alpha \text{ lang}$ is the coinduction principle that we have seen earlier. It follows from the fact that final coalgebras are quotients by bisimilarity, where bisimilarity is defined as the existence of a bisimulation relation.

6 Discussion and Related Work

Our development is a formalized counterpart of Rutten’s introduction to the coalgebraic view on languages [21]. In this section we discuss further related work.

Adding Further Operations

With the coinductive representation adding new operations corresponds to defining a new corecursive function on tries. Compared with adding a new constructor to the inductive datatype of regular expressions and extending all previously defined recursive functions on regular expressions to account for this new case, this is a rather low-cost library extension. Wadler called this tension between extending syntactic and semantic objects the Expression Problem [29]. Note that codatatypes alone are not the solution to the Expression Problem—they just populate the other side of the spectrum with respect to datatypes. In fact, adding new selectors to our tries would be as painful as adding new constructors to the datatype of regular expressions. Rendel et al. [20] outline how automatic conversions between the inductive and the coinductive view can help solving the Expression Problem.

We have extended our library with the regular shuffle product operation on languages and an operation that transforms a context free grammar in weak Greibach normal form into a trie with the same language [25]. Both operations are corecursive up to $+$ (just as \cdot is).

Coalgebraic View on Formal Languages

The coalgebraic approach to languages has recently received some attention. Landmark results in language theory were rediscovered and generalized. Silva’s recent survey [24] highlights some of those results including the proofs of correctness of Brzozowski’s subtle deterministic finite automaton minimization algorithm [5]. The coalgebraic approach yields some algorithmic advantages too. Bonchi and Pous present a coinductive algorithm for checking equivalence of non-deterministic automata that outperforms all previously known algorithms by one order of magnitude [6]. Another recent development is our formally verified coalgebraic algorithm for deciding weak monadic second-order logic of one successor (WS1S) [26]. This formalization employs the Isabelle library presented here.

Tutorials on Coinduction

The literature is abound with tutorials on coinduction. So why bother writing yet another one? First, because we finally can do it in Isabelle/HOL, which became a coinduction-friendly proof assistant recently [3]. Earlier studies of coinduction in Isabelle had to engage

in tedious manual constructions just to define a codatatype [18]. Second, coinductive techniques are still not as widespread as they could be (and we believe should be, since they constitute a convenient proof tool for questions about semantics). Third, many tutorials [8, 9, 11, 12, 14, 23], with or without a theorem prover, exercise streams to the extent that one starts to believe having seen every single stream example one can imagine. Also streams are often not that convincing for codatatype-skeptics since they are easily modeled by sequences, i.e., functions from natural numbers to elements. In contrast, Rutten [21] demonstrates that it is entirely feasible to start with a structure slightly more complicated than streams, but familiar to everybody with a computer science degree. Our work puts Rutten’s exposition in the context of a proof assistant.

Non-Primitive Corecursion in Proof Assistants

Automation for corecursion in proof assistants is much less developed than its recursive counterpart. Currently, Isabelle/HOL provides only a command to handle primitively corecursive specifications. The Coq proof assistant can do slightly more: it supports corecursion up to constructors [8]. Looking at our examples, however, this means that Coq will not be able to prove productivity of the natural concatenation and iteration specifications automatically, since both go beyond up-to constructors. Instead, our reduction to primitive corecursion can be employed to bypass Coq’s productivity checker.

Agda’s combination of copatterns (i.e., destructor view) and sized types [1,2] is the most advanced implemented support for corecursion in proof assistants to date. However, using sized types often means that one has to encode proofs of productivity manually in the type of the defined function. Thus, it should be possible to define concatenation and iteration using their natural specifications in Agda when we accept the need for heavier type annotations.

Recently, we proposed a general framework for reducing corecursion up to so called friendly operations to primitive corecursion in Isabelle/HOL [4]. An operation is friendly if, under lazy evaluation, it does not peek too deeply into its arguments, before producing at least one constructor. For example, the friendly operation $L + K = \mathbf{L} (o L \vee o K) (\lambda x. \delta L x + \delta K x)$ destructs only one layer of constructors, in order to produce the topmost \mathbf{L} . Since $+$ is friendly, and \cdot is corecursive up to $+$, using this framework will allow us to use the natural specifications for \cdot without changing any types. (The same applies for $*$.) In contrast, the primitively corecursive equation $\mathbf{deep} L = \mathbf{L} (o L) (\lambda x. \mathbf{deep} (\delta (\delta L x) x))$ destructs two layers of constructors (via δ) before producing one and is therefore not friendly. Indeed, we will not be able to reduce the equation $\mathbf{bad} = \mathbf{L} \top (\lambda _ . \mathbf{deep} \mathbf{bad})$ (which is corecursive up-to \mathbf{deep}) to a primitively corecursive specification. And there is a reason for it: \mathbf{bad} is not uniquely specified by the above equation, or in other words not productive.

Support for friendly operations will advance Isabelle over its competitors, once fully implemented. To achieve the reduction to primitive corecursion the framework follows and abstract, category theory inspired construction. Yet, what this reduction yields in practice is relatively close to our manual construction for concatenation. (In contrast, the iteration case takes some shortcuts, which the abstract view does not offer.)

Formal Languages in Proof Assistants

Such a basic thing as the traditional set-of-words view on formal languages is formalized in most proof assistants. In contrast, we are not aware of any other formalization of the coalgebraic view on formal languages in a proof assistant.

Here, we want to compare our formalization with the Isabelle incarnation of the set-of-words view developed by Krauss and Nipkow for the correctness proof of their regular ex-

pression equivalence checker [15]. Both libraries are comparably concise. In 500 lines Krauss and Nipkow prove almost all axioms of Kleene algebra and the characteristic equations for the left quotients (the δ -specifications in our case). They reuse Isabelle’s libraries for sets and lists, which come with carefully tuned automation setup. Still, their proofs tend to require additional induction proofs of auxiliary lemmas, especially when reasoning about iteration. Our formalization is 700 lines long. We prove all axioms of Kleene algebra and connect our representation to the set-of-words view via the bijections `out` and `in`. Except for those bijections our formalization does not rely on any other library. Moreover, when we changed our 5000 lines long formalization of a coalgebraic decision procedure for WS1S [26] to use the infinite tries instead of the set-of-words view, our proofs about WS1S became approximately 300 lines shorter. Apparently, a coalgebraic library is a good fit for a coalgebraic procedure.

Paulson presents a concise formalization of automata theory based on hereditarily finite sets [19]. For the semantics he reuses Krauss and Nipkow’s set-of-words formalization.

7 Conclusion

We have presented a particular Formal Structure for Computation and Deduction: infinite tries modeling formal languages. Although this representation is semantic and infinite, it is suitable for computation—in particular we obtain a matching algorithm for free on tries constructed by regular operations. Deduction does not come short either: coinduction is the convenient reasoning tool for infinite tries. Coinductive proofs are concise, especially for (in)equational theorems such as the axioms of Kleene algebra.

Codatatypes might be just the right tool for thinking algorithmically about semantics. We hope to have contributed to their dissemination by outlining some of their advantages.

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