Foundational, Compositional (Co)datatypes for Higher-Order Logic

Category Theory Applied to Theorem Proving

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Technische Universität München

Outline

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Isabelle/HOL

- LCF philosophy

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  Small inference kernel

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- Foundational approach

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  Weaker than ZF

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- HOL = simply typed set theory with ML-style polymorphism

  Restrictive logic

  Weaker than ZF

- Datatype specification

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\begin{aligned}
\texttt{datatype}\ \alpha\ \text{list} &= \text{Nil} \mid \text{Cons}\ \alpha\ (\alpha\ \text{list}) \\
\texttt{datatype}\ \alpha\ \text{tree} &= \text{Node}\ \alpha\ (\alpha\ \text{tree list})
\end{aligned}
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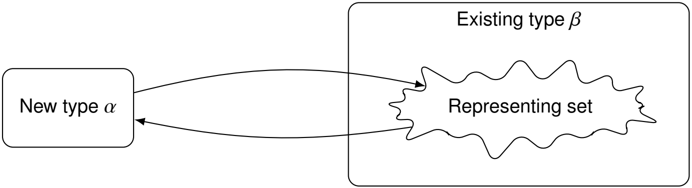
- Primitive type definitions



The traditional approach

Melham 1989, Gunter 1994

- Fragment of ML (non-co)datatypes

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\texttt{and}\quad \alpha\ \mathsf{tree\_list} & = & \mathsf{Nil} \mid \mathsf{Cons}\ (\alpha\ \mathsf{tree})\ (\alpha\ \mathsf{tree\_list})
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- Implemented in Isabelle by Berghofer & Wenzel 1999

Limitations

Berghofer & Wenzel 1999

1. noncompositionality
2. no codatatypes
3. no non-free structures

~~Limitations~~

LICS 2012

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Datatypes in HOL—State of the Art

Bounded Natural Functors

(Co)datatypes

(Co)nclusion

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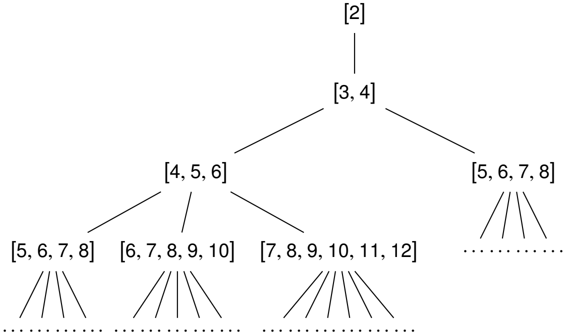
- `P n = print n; for i = 1 to n do P (n + i);`

`datatype` α list = Nil | Cons α (α list)

`codatatype` α tree = Node α (α tree list)

- `P n = print n; for i = 1 to n do P (n + i);`
- evaluation tree for `P 2`



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- Compositionality = no unfolding

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- Compositionality = no unfolding
- Need abstract interface

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- Compositionality = no unfolding
- Need abstract interface
- What interface?

Type constructors are not just operators on types!

The interface: **bounded natural functor**

type constructor F

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Fmap

} functor

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type constructor F Fmap	} functor
Fset	} natural transformation

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The interface: **bounded natural functor**

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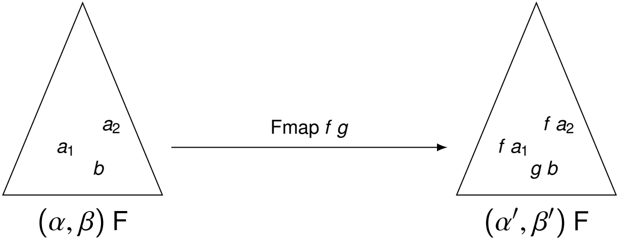
BNF = type constructor + polymorphic constrants + assumptions

Type constructors are functors

$$\text{Fmap} : (\alpha \to \alpha') \to (\beta \to \beta') \to (\alpha, \beta)\ \text{F} \to (\alpha', \beta')\ \text{F}$$



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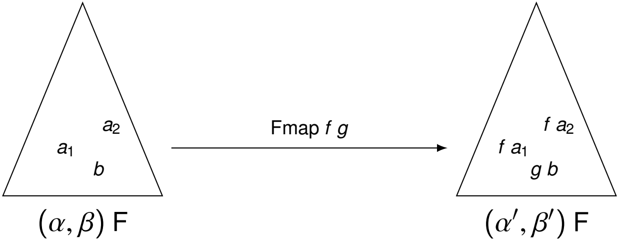
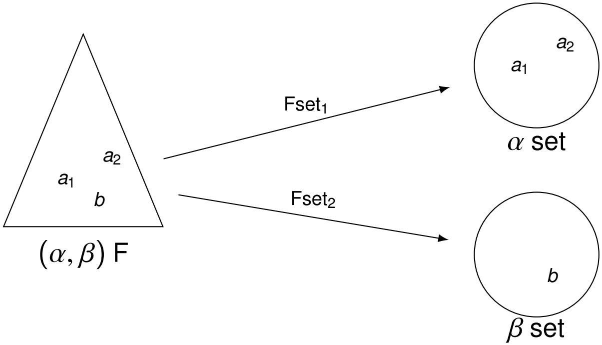


$$\begin{aligned} \text{Fmap id id} &= \text{id} \\ \text{Fmap}\ f_1\ f_2 \circ \text{Fmap}\ g_1\ g_2 &= \text{Fmap}\ (f_1 \circ g_1)\ (f_2 \circ g_2) \end{aligned}$$

Type constructors are containers

$$\mathsf{Fset}_1 : (\alpha, \beta)\ \mathsf{F} \rightarrow \alpha\ \mathsf{set}$$

$$\mathsf{Fset}_2 : (\alpha, \beta)\ \mathsf{F} \rightarrow \beta\ \mathsf{set}$$



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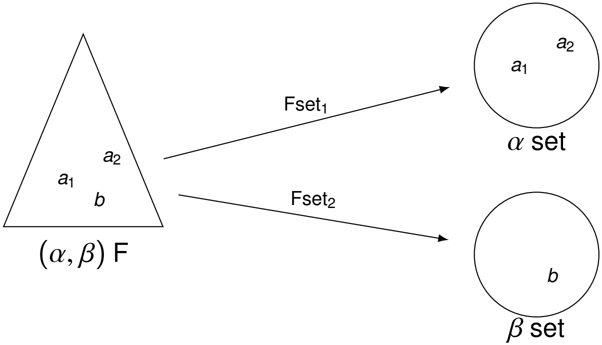

$$\mathsf{Fset}_1 \circ \mathsf{Fmap}\ f_1\ f_2 = \mathsf{image}\ f_1 \circ \mathsf{Fset}_1$$

$$\mathsf{Fset}_2 \circ \mathsf{Fmap}\ f_1\ f_2 = \mathsf{image}\ f_2 \circ \mathsf{Fset}_2$$

Further BNF assumptions

$$\left.\begin{array}{l}\forall x \in \mathrm{Fset}_1\ z.\ f_1\ x = g_1\ x \\ \forall x \in \mathrm{Fset}_2\ z.\ f_2\ x = g_2\ x\end{array}\right\} \quad \Rightarrow \quad \mathrm{Fmap}\ f_1\ f_2\ z = \mathrm{Fmap}\ g_1\ g_2\ z$$

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(F, Fmap) preserves weak pullbacks

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- admit final coalgebras (codatatypes)
- are closed under initial algebras and final coalgebras
- make initial algebras and final coalgebras expressible in HOL

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From user specifications to (co)datatypes

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$$\mathsf{iter} : (\mathsf{unit} + \alpha \times \alpha\ \mathsf{list} \to \beta) \to \alpha\ \mathsf{list} \to \beta$$

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7. Prove that list is a BNF (enables nested recursion)

From user specifications to (co)datatypes

Given

$$\texttt{codatatype}\ \alpha\ \mathsf{llist} = \mathsf{LNil} \mid \mathsf{LCons}\ \alpha\ (\alpha\ \mathsf{llist})$$

1. Abstract to $\beta = \mathsf{unit} + \alpha \times \beta$
2. Prove that $(\alpha, \beta)\ \mathsf{F} = \mathsf{unit} + \alpha \times \beta$ is a BNF
3. Define F-coalgebras
4. Construct final coalgebra

$$(\alpha\ \mathsf{llist}, \mathsf{unf} : \alpha\ \mathsf{llist} \to \mathsf{unit} + \alpha \times \alpha\ \mathsf{llist})$$

5. Define coiterator

$$\mathsf{coiter} : (\beta \to \mathsf{unit} + \alpha \times \alpha\ \mathsf{llist}) \to \beta \to \alpha\ \mathsf{llist}$$

6. Prove characteristic theorems (e.g. coinduction)
7. Prove that llist is a BNF (enables nested corecursion)

Induction

$\beta = (\alpha, \beta)$ F

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- Given $\varphi : \alpha$ IF $\to$ bool
- Abstract induction principle

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- Given $\varphi : \alpha\ \mathsf{list} \to \mathsf{bool}$
- Case distinction on z

$$\frac{\begin{array}{r}(\forall ys \in \emptyset.\ \varphi\ ys)\ \Rightarrow\ \varphi\ (\mathsf{fld}\ (\mathsf{Inl}\ ()))\\ \forall x\ xs.\ (\forall ys \in \{xs\}.\ \varphi\ ys)\ \Rightarrow\ \varphi\ (\mathsf{fld}\ (\mathsf{Inr}\ (x, xs)))\end{array}}{\forall xs.\ \varphi\ xs}$$

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- Concrete induction principle

$$\frac{\begin{array}{lcl} & & \varphi\ (\text{fld}\ (\text{Inl}\ ())) \\ \forall x\ xs. \qquad \varphi\ xs & \Rightarrow & \varphi\ (\text{fld}\ (\text{Inr}\ (x, xs))) \end{array}}{\forall xs.\ \varphi\ xs}$$

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- Given $\varphi : \alpha\ \text{list} \to \text{bool}$
- In constructor notation

$$\frac{\begin{array}{ll} & \varphi\ \text{Nil} \\ \forall x\ xs.\quad \varphi\ xs \Rightarrow & \varphi\ (\text{Cons}\ x\ xs) \end{array}}{\forall xs.\ \varphi\ xs}$$

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- Given $\psi : \alpha\ \mathsf{JF} \to \alpha\ \mathsf{JF} \to \mathsf{bool}$
- Abstract coinduction principle

$$\frac{\forall x\ y.\ \psi\ x\ y \Rightarrow \mathsf{Fpred}\ \mathsf{Eq}\ \psi\ (\mathsf{unf}\ x)\ (\mathsf{unf}\ y)}{\forall x\ y.\ \psi\ x\ y \Rightarrow x = y}$$

Example

$\texttt{codatatype}\ \alpha$ tree $=$ Node (lab: α) (sub: α tree fset)

Example

$\texttt{codatatype}\ \alpha\ \text{tree} = \text{Node}\ (\text{lab}: \alpha)\ (\text{sub}: \alpha\ \text{tree fset})$

$\texttt{corec}\ \text{tmap} : (\alpha \to \beta) \to \alpha\ \text{tree} \to \beta\ \text{tree}\ \texttt{where}$

$$\begin{aligned} \text{lab}\ (\text{tmap}\ f\ t) &= f\ (\text{lab}\ t) \\ \text{sub}\ (\text{tmap}\ f\ t) &= \text{image}\ (\text{tmap}\ f)\ (\text{sub}\ t) \end{aligned}$$

Example

`codatatype` α tree $=$ Node (lab: α) (sub: α tree fset)

`corec` tmap : $(\alpha \to \beta) \to \alpha$ tree $\to \beta$ tree `where`

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`lemma` tmap $(f \circ g)\; t =$ tmap f (tmap $g\; t$)

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$\texttt{corec}\ \text{tmap} : (\alpha \to \beta) \to \alpha\ \text{tree} \to \beta\ \text{tree}\ \texttt{where}$

$$\begin{aligned} \text{lab}\ (\text{tmap}\ f\ t) &= f\ (\text{lab}\ t) \\ \text{sub}\ (\text{tmap}\ f\ t) &= \text{image}\ (\text{tmap}\ f)\ (\text{sub}\ t) \end{aligned}$$

$\texttt{lemma}\ \text{tmap}\ (f \circ g)\ t = \text{tmap}\ f\ (\text{tmap}\ g\ t)$

$\texttt{by}\ (\texttt{intro}\ \text{tree\_coinduct}[\texttt{where}\ \psi = \lambda t_1\ t_2.\ \exists t.\ t_1 = \text{tmap}\ (f \circ g)\ t \wedge t_2 = \text{tmap}\ f\ (\text{tmap}\ g\ t)])\ \texttt{force+}$

Outline

Datatypes in HOL—State of the Art

Bounded Natural Functors

(Co)datatypes

(Co)nclusion

Foundational, Compositional (Co)datatypes for Higher-Order Logic

Category Theory Applied to Theorem Proving

Foundational, Compositional (Co)datatypes for Higher-Order Logic

Category Theory Applied to Theorem Proving

- Framework for defining types in HOL

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Thank you for your attention!

Foundational, Compositional (Co)datatypes for Higher-Order Logic

Category Theory Applied to Theorem Proving

Dmitriy Traytel Andrei Popescu Jasmin Blanchette

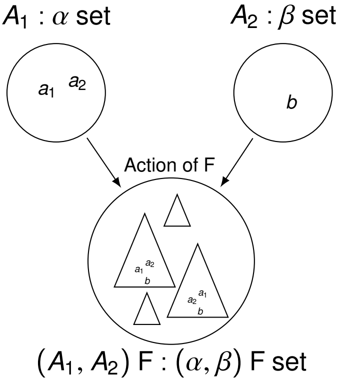
Technische Universität München

Outline

Backup slides

Type constructors act on sets

$$(A_1, A_2)\ \mathsf{F} = \{z \mid \mathsf{Fset}_1\ z \subseteq A_1 \wedge \mathsf{Fset}_2\ z \subseteq A_2\}$$



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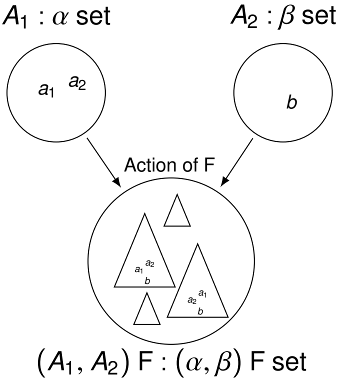
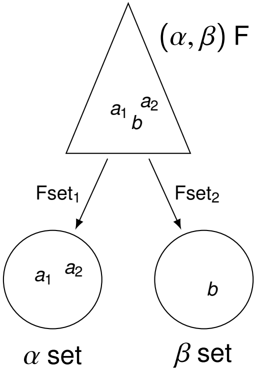


$$(\forall i \in \{1,2\}.\ \forall x \in \mathsf{Fset}_i\ z.\ f_i\ x = g_i\ x) \quad \Rightarrow \quad \mathsf{Fmap}\ f_1\ f_2\ z = \mathsf{Fmap}\ g_1\ g_2\ z$$

Type constructors are bounded

Fbd: infinite cardinal



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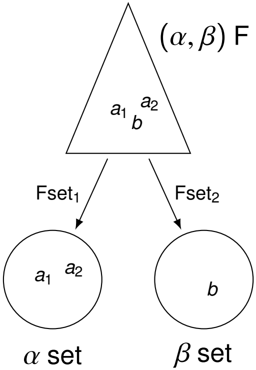

$$|\mathsf{Fset}_i\ z| \quad \leq \quad \mathsf{Fbd}$$

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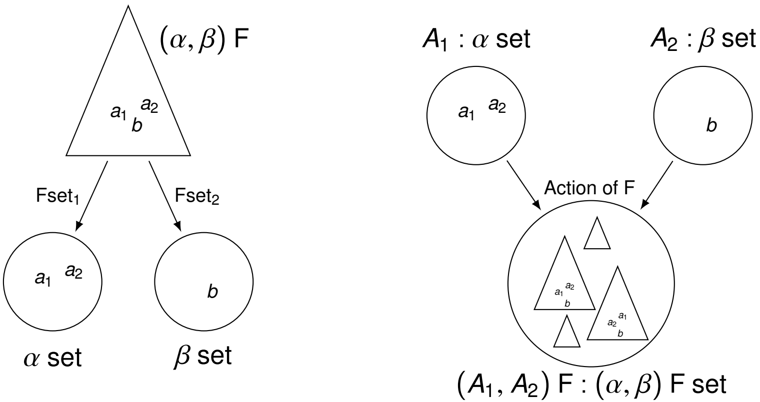

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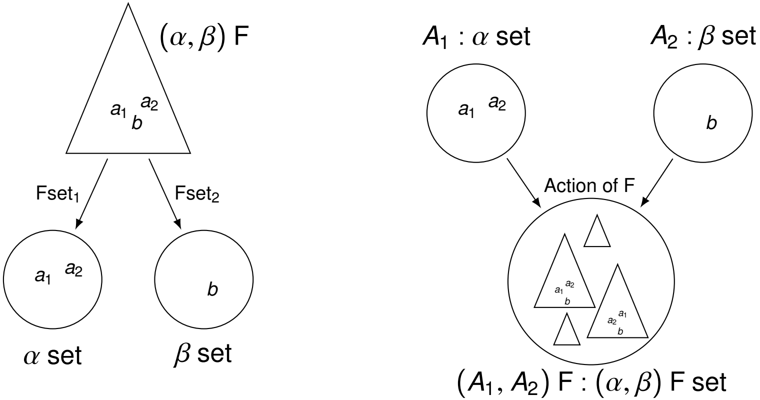


$$
\begin{aligned}
|\mathsf{Fset}_i\ z| &\leq \mathsf{Fbd} \\
|(A_1, A_2)\ \mathsf{F}| &\leq (|A_1| + |A_2| + 2)^{\mathsf{Fbd}}
\end{aligned}
$$

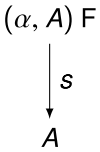
Algebras, Coalgebras & Morphisms

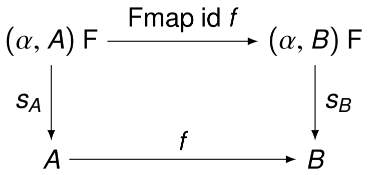
$\beta = (\alpha, \beta)\,\mathsf{F}$

$$
\begin{array}{c}
(\alpha, A)\,\mathsf{F} \\
\Big\downarrow s \\
A
\end{array}
$$

Algebras, Coalgebras & Morphisms

$\beta = (\alpha, \beta)$ F





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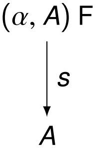

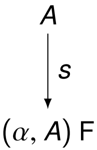

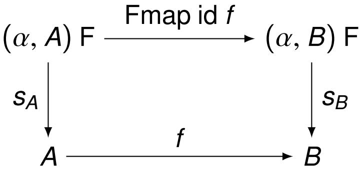

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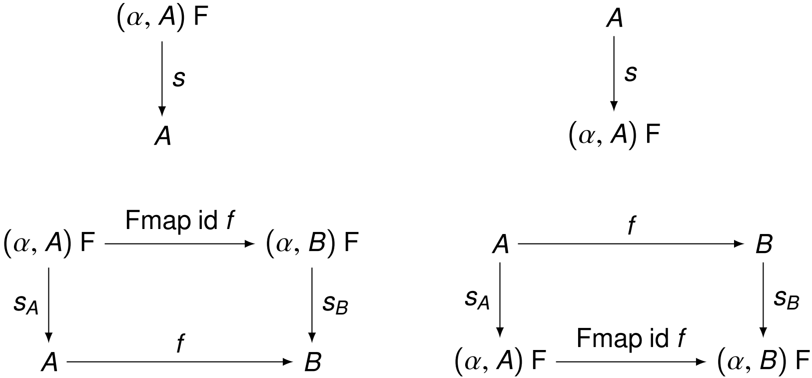

Initial Algebras & Final Coalgebras

$\beta = (\alpha, \beta)$ F

weakly initial:	exists morphism to any other algebra
initial:	exists *unique* morphism to any other algebra
weakly final:	exists morphism from any other coalgebra
final:	exists *unique* morphism from any other coalgebra

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- Product of all algebras is weakly initial
- Suffices to consider algebras over types of certain cardinality
- Minimal subalgebra of weakly initial algebra is initial

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- Minimal subalgebra of weakly initial algebra is initial
- Construct minimal subalgebra from below by transfinite recursion

$\Rightarrow$ Have a bound for its cardinality

$$\Rightarrow (\alpha\ \mathsf{IF}, \mathsf{fld} : (\alpha, \alpha\ \mathsf{IF})\ \mathsf{F} \to \alpha\ \mathsf{IF})$$

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- Sum of all coalgebras is weakly final
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$\Rightarrow$ Have a bound for its cardinality

$\Rightarrow (\alpha$ IF, fld : $(\alpha, \alpha$ IF$)$ F $\rightarrow \alpha$ IF$)$

- Sum of all coalgebras is weakly final
- Suffices to consider coalgebras over types of certain cardinality
- Quotient of weakly final coalgebra to the greatest bisimulation is final
- Use concrete weakly final coalgebra (elements are tree-like structures)

$\Rightarrow$ Have a bound for its cardinality

$\Rightarrow (\alpha$ JF, unf : α JF $\rightarrow (\alpha, \alpha$ JF$)$ F$)$

Iteration & Coiteration

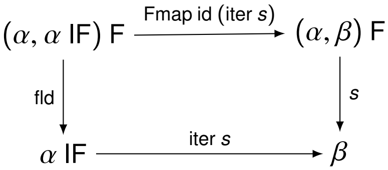
$\beta = (\alpha, \beta)\ \mathsf{F}$

- Given $s : (\alpha, \beta)\ \mathsf{F} \to \beta$

Iteration & Coiteration

$\beta = (\alpha, \beta)$ F

- Given $s : (\alpha, \beta)\ \mathsf{F} \to \beta$
- Obtain unique morphism iter s from (α IF, fld) to (U_β, s)



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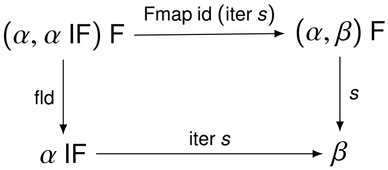

- Given $s : \beta \to (\alpha, \beta)\ \mathsf{F}$

Iteration & Coiteration

$\beta = (\alpha, \beta)\,\mathsf{F}$

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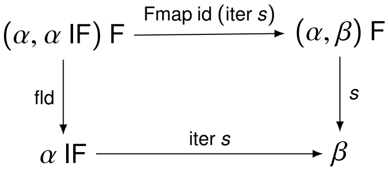
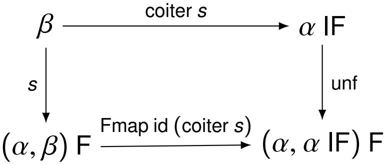
- Given $s : \beta \to (\alpha, \beta)\,\mathsf{F}$
- Obtain unique morphism coiter s from (U_β, s) to $(\alpha\ \mathsf{JF}, \mathsf{unf})$





Preservation of BNF Properties

$\beta = (\alpha, \beta)$ F

- IFmap f = iter (fld ∘ Fmap f id)
- IFset = iter collect, where

collect z=$\mathsf{Fset}_1\ z \cup \bigcup \mathsf{Fset}_2\ z$

Preservation of BNF Properties

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Theorem

$(\mathsf{IF}, \mathsf{IFmap}, \mathsf{IFset}, 2^{\mathsf{Fbd}})$ *is a BNF*

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- JFmap f = coiter (Fmap f id ∘ unf)
- JFset $x = \bigcup_{i \in \mathbb{N}}$ collect$_i$ x, where

$$\text{collect}_0\ x = \emptyset$$

$$\text{collect}_{i+1}\ x = \text{Fset}_1\ (\text{unf}\ x) \cup \bigcup_{y \in \text{Fset}_2\ (\text{unf}\ x)} \text{collect}_i\ y$$

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(IF, IFmap, IFset, 2^{Fbd}) *is a BNF*

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$$\mathsf{collect}\ z = \mathsf{Fset}_1\ z \cup \bigcup \mathsf{Fset}_2\ z$$

- $\mathsf{JFmap}\ f = \mathsf{coiter}\ (\mathsf{Fmap}\ f\ \mathsf{id} \circ \mathsf{unf})$
- $\mathsf{JFset}\ x = \bigcup_{i \in \mathbb{N}} \mathsf{collect}_i\ x$, where

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Theorem

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Theorem

$(\mathsf{JF}, \mathsf{JFmap}, \mathsf{JFset}, \mathsf{Fbd}^{\mathsf{Fbd}})$ *is a BNF*