Exercise 10: Preconditioning

1. **Preconditioned GMRES.** We want to solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$ with the GMRES algorithm with three kinds of preconditioning. Here, A is a nonsymmetric real sparse 1600×1600 matrix and $\boldsymbol{b}$ is defined so that the true solution is a vector of all ones. You can create them in Matlab by the command

   ```
   A = gallery('neumann',1600) + 0.001*speye(1600); b = sum(A,2);
   ```

   (i) In the first version, we use split preconditioning with the incomplete LU factors of the matrix L and U. So, instead of solving $A\boldsymbol{x} = \boldsymbol{b}$, we solve the equation

   $$L^{-1}AU^{-1}\boldsymbol{z} = L^{-1}\boldsymbol{b}, \qquad U^{-1}\boldsymbol{z} = \boldsymbol{x}.$$

   (ii) In the second version, we use left preconditioning with $M = LU$.

   (iii) In the third version, we use right preconditioning with $M = LU$.

   Use Matlab's incomplete LU factorization provided by the command `ilu` with different parameter settings:

   - With `setup.type = 'nofill'` we get the often used ILU(0) preconditioner.
   - To get more fill-in experiment with
     `setup.type = 'crout'; setup.droptol = xx;`
     Set `xx = .01, .1, .2`. What do you observe?

   For all instances provide number of nonzeros of L and U and iteration count.

   Note: Since `gmres` only provides left preconditioning some of the preconditioners have to be implemented by functions.

2. **Preconditioning with stationary iterations.** We want to solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$ with symmetric positive-definite A by the conjugate gradient algorithm. We have an SPD preconditioner M available that we use to determine the preconditioned polynomial from $M\boldsymbol{z} = \boldsymbol{r}$.

   (i) Show that solving $M\boldsymbol{z} = \boldsymbol{r}$ is actually one step of a stationary iteration for solving $A\boldsymbol{z} = \boldsymbol{r}$ with preconditioner M.

   (ii) What would be the preconditioner if we executed two steps of this stationary iteration? Is it symmetric positive-definite?

3. **Stokes equation.** The finite element discretization of the Stokes equations (cf. Slide 21 of Lecture 5) leads to a matrix system of the form

   $$\mathcal{A}\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{p}\end{bmatrix} \equiv \begin{pmatrix}A & B^T\\ B & 0\end{pmatrix}\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{p}\end{bmatrix} = \begin{bmatrix}\boldsymbol{f}\\ \boldsymbol{0}\end{bmatrix},$$

   where A is SPD and B has full rank. Let S be the Schur complement of A in $\mathcal{A}$, $S = BA^{-1}B^T$. We choose as a preconditioner of $\mathcal{A}$ the matrix

   $$\mathcal{M} = \begin{pmatrix}A & O\\ O & S\end{pmatrix}.$$

Please submit your solution via e-mail to Peter Arbenz (`arbenz@inf.ethz.ch`) by November 30, 2017. (12:00). Please specify the tag **FEM17** in the subject field.

(a) Show that $\mathcal{M}^{-1}\mathcal{A}$ only has the three eigenvalues $1, \frac{1}{2} \pm \frac{\sqrt{5}}{2}$. How many of each exist?

(b) $\mathcal{M}$ is not a feasible preconditioner. Why? How could one use the above information to arrive at a good preconditioner.

(c) Experiment with MATLAB's `minres` (with preconditioner $\mathcal{M}$) to show that only 2 iteration steps are needed until convergence. For A use a (small) Poisson matrix and for B a random matrix (`sprand`). Make sure it has full rank!

Hints:

(1) Investigate first the case where $\boldsymbol{u}$ is in the null space of B to get the eigenvalue 1.

(2) Then eliminate $\boldsymbol{u}$ from the first equation to arrive at a system for $\boldsymbol{p}$ only. Get the other two eigenvalues from there.

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