Exercise 11: RAS and geometric multigrid

The purpose of this exercise is to sense the power of multilevel methods.

1. Implement a two-level restricted additive Schwarz (RAS) preconditioner for the 5-point stencil on a square grid (finite difference approximation of the Poisson equation with homogeneous Dirichlet boundary conditions on a square domain Ω). You should be able to reproduce some of the results at the end of Lecture 10. The RAS preconditioner with multiplicative coarse space correction is given by

   $\boldsymbol{r} = \boldsymbol{b} - A\boldsymbol{x}$;
   for j:=1 to d **do**
   $\quad\boldsymbol{x} = \boldsymbol{x} + R_j D_j (R_j^T A R_j)^{-1} R_j^T \boldsymbol{r}$;
   end for
   $\boldsymbol{x} = \boldsymbol{x} + R_0 (R_0^T A R_0)^{-1} R_0^T \boldsymbol{r}$;

   For R_0 coose the Nicolaides coarse space correction. See, slides 53 and 61 of Lecture 10. Choose the parameters of slide 63.

2. We have seen that smoothing and approximation properties are decisive for the quality of a multigrid method. The smoothing property states that

   $$\|S_h \boldsymbol{e}^h\|_{A_h}^2 \le \|\boldsymbol{e}^h\|_{A_h}^2 - \alpha \|A_h \boldsymbol{e}^h\|_{D^{-1}}^2 \quad \forall \boldsymbol{e}^h \in \Omega_h.$$

   while the approximation property hold if

   $$\min_{\boldsymbol{e}_H \in \Omega_H} \|\boldsymbol{e}^h - I_H^h \boldsymbol{e}^H\|_D^2 \le \beta \|\boldsymbol{e}^h\|_{A_h}^2.$$

   where both $\alpha > 0$ and $\beta > 0$ do *not* depend on the mesh size h. (See the lecture notes for the notation.)

   Show the approximation property for the 1-dimensional case when I_H^h denotes *linear interpolation*.

   Hint: Show the property for all eigenvectors $\boldsymbol{w}_k^h$ of $A_h = \text{tridiag}(-1, 2, -1)$. $\boldsymbol{w}_k^H$ is obtained from $\boldsymbol{w}_k^h$ by injection.

   Deduce that the same inequality also holds for the 2D case.

3. Implement a two-level multigrid preconditioner for the 5-point stencil on a square grid (finite difference approximation of the Poisson equation with homogeneous Dirichlet boundary conditions on a square domain Ω).

   Verify the results on the last page of Lecture 11. In the case of Jacobi-smoothing, compare the theoretical prediction with your results (number of iterations). The value for α is given in the lecture notes. You have determined a value for β in the previous exercise.

Please submit your solution via e-mail to Peter Arbenz (`arbenz@inf.ethz.ch`) by December 7, 2017. (12:00). Please specify the tag **FEM17** in the subject field.