Exercise 4

The goal of this exercise is to experiment with error rates.

1. **Construction of a test problem.** The two functions $g_1(x) = 1-x^2$ and $g_2(x) = x^3-x$ have both their root at $x_1 = -1$ and $x_2 = 1$. The function u can be constructed from $g_1(x)$ and $g_2(x)$ in the following way:

$$u(x,y) = g_2(x)g_1(y).$$

   It is easy to see that $u = 0$ holds on the boundary of the square $[-1,1]^2$. Determine f such that $-\Delta u = f$. (You may verify your solution in Matlab with `assempde`.)

2. **Energy error.** Show that if $u \in \mathcal{H}^1_E$ and $u_h \in S^h_E$ satisfy

$$a(u,v) = \ell(v) \qquad \text{for all } v \in \mathcal{H}^1_{E_0}$$

   and

$$a(u_h,v_h) = \ell(v_h) \qquad \text{for all } v_h \in S^h_0$$

   respectively, in the the case of zero Dirichlet data (i.e. $\mathcal{H}^1_E = \mathcal{H}^1_{E_0}$), then the error in energy satisfies

$$\|\mathbf{grad}\,(u-u_h)\|^2 = \|\mathbf{grad}\,u\|^2 - \|\mathbf{grad}\,u_h\|^2$$

   Hint: Use the Galerkin orthogonality (more than once).

3. We want to analyze the energy error for the problem of question 1. The energy is defined by

$$E_{\text{exact}} = a(u,u) = \int_\Omega \mathbf{grad}\,u \cdot \mathbf{grad}\,u\,dx$$

   If the exact solution is known E_{exact} can be computed.

   The finite element energy can be computed with the global stiffness matrix $\mathbf{A}$ and is given by $E_{\text{FE}} = \mathbf{u}^T\mathbf{A}\mathbf{u}$. To get a triangulation with controlled edge length use the command `initmesh(g,'hmax',h)` with the parameter option `'hmax'` or use the command `poimesh(g,n)` which generates a regular mesh on the rectangular geometry. Plot the energy error $|E_{\text{exact}} - E_{\text{FE}}| = \|\mathbf{grad}\,(u-u_h)\|^2$ against the resolution h. What convergence rate do you observe?

4. Do the same for the problem $-\Delta u = 1$ on the L-shaped domain $\Omega = [-1,1]^2 \setminus [-1,0]^2$ with homogeneous Dirichlet boundary conditions. Since the exact solution is not known, use a FE solution of high accuracy.

5. Solve the problem of the previous question with regular and with adaptive mesh refinement, starting from the same coarse mesh. How many grid points do you need in the two refinement strategies to get the energy error below 10^{-5}?

Please submit your solution via e-mail to Peter Arbenz (`arbenz@inf.ethz.ch`) by Thursday October 19, 2017. (12:00). Please specify the tag **FEM17** in the subject field.