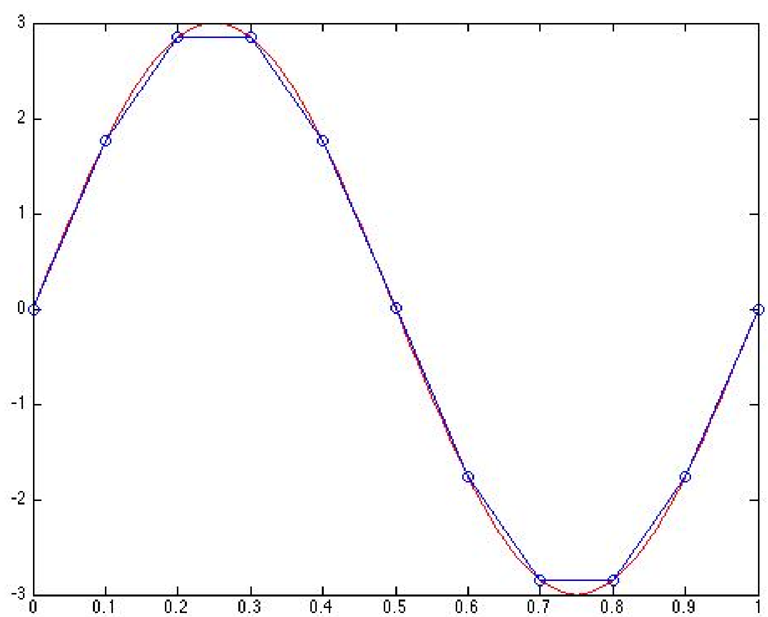
Exercise 1 – Solutions

Assigment 1.1

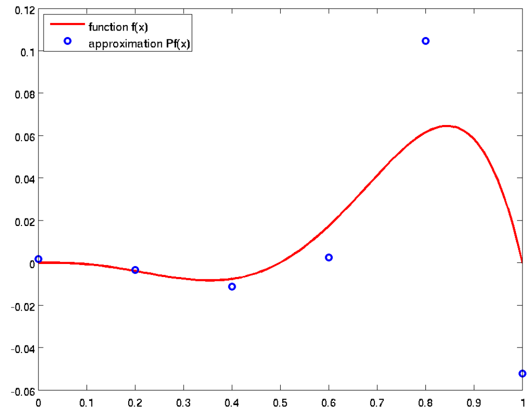
```
% FEM16 - Exercise-1.1
% fq = interp1(x,y,xq) returns interpolated values of a 1-D function
% at specific query points using linear interpolation.
% Vector x contains the sample points,
% Vector y contains the corresponding values, y=f(x).
% Vector xq contains the coordinates of the query points.

f = @(x) 3*sin(2*pi*x);     % Define f(x)

x = 0:0.01:1;   % sample points
y = f(x);

h = 0.1;        % length of subinterval
xq = 0:h:1;     % query points
yq = interp1(x,y,xq);
plot(x,y,'r-',xq,yq,'o-')
```



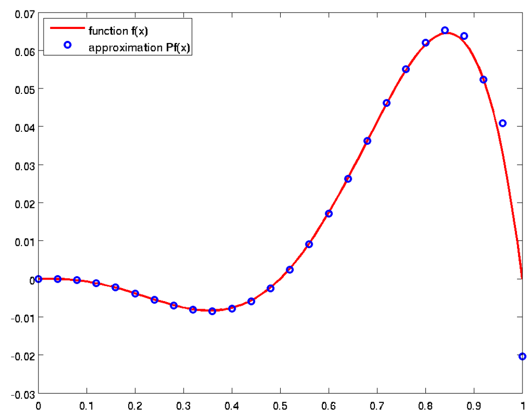
Assigment 1.2

```
% FEM16 - Exercise-1.2
% L2-projection : piecewise polynomial approximation
% We use a uniform mesh in the interval I = [0; 1]
% with n = 5, 25 and 100 subintervals.

function L2Projector1D()
  n = 5; %25,100            % number of subintervals
  h = 1/n;                  % length of subinterval
  x = 0:h:1;                % node points (query points)
  f = @(x) x.^3.*(x-1).*(1-2*x);  % define f(x)
  M = MassAssembler1D(x);
  b = LoadAssembler1D(x,f);
  Pf = M\b;                  % Solve linear system of equations

  xq = 0:0.01:1;
  plot (xq, f(xq),'r', x, Pf, 'o', 'LineWidth', 2)
  legend ('function f(x)','approximation Pf(x)','Location','NorthWest')

function M = MassAssembler1D(x)
  n = length(x)-1;
  M = zeros(n+1, n+1);
  for i= 1:n
    h = x(i+1) - x(i);
    M(i,i) = M(i,i) + h/3;
    M(i,i+1) = M(i,i+1) + h/6;
    M(i+1,i) = M(i+1,i) + h/6;
    M(i+1,i+1) = M(i+1,i+1) + h/3;
end

function b = LoadAssembler1D(x,f)
  n = length(x)-1;
  b = zeros(n+1, 1);
  for i= 1:n
    h = x(i+1) - x(i);
    b(i) = b(i) + f(x(i))*h/2;
    b(i+1) = b(i+1) + f(x(i+1))*h/2;
end
```

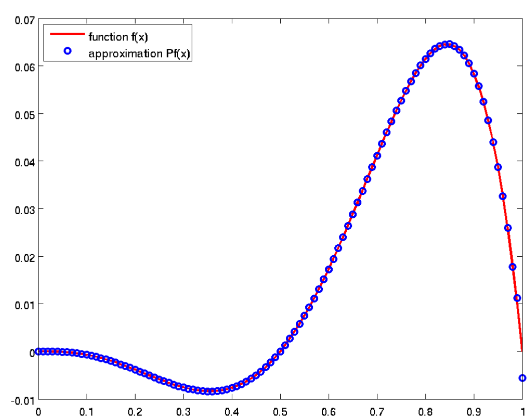
Solution for $n = 5$



Solution for $n = 25$



Solution for $n = 100$



Assigment 1.3

The difference of the function f to be approximated and the interpolating piecewise linear polynomial πf is

$$(\pi f - f)(x) = \begin{cases} 0, & 0 \leq \frac{1}{2} \text{ and } \frac{1}{2} + h < x \leq 1, \\ 1 - \frac{1}{h}(x - \frac{1}{2}), & \frac{1}{2} < x \leq \frac{1}{2} + h. \end{cases}$$

So,

$$\begin{aligned} \|\pi f - f\|_2^2 = \int_0^1 (\pi f(x) - f(x))^2 \, dx &= \int_{\frac{1}{2}}^{\frac{1}{2}+h} (1 - \frac{1}{h}(x - \frac{1}{2}))^2 dx \\ &= \int_0^h (1 - \frac{x}{h})^2 dx = \frac{-1}{3h^2}(h - x)^3\Big|_0^h = \frac{h}{3}, \end{aligned}$$

and

$$\|(\pi f - f)'\|_2^2 = \int_0^h (\frac{d}{dx}(1 - \frac{x}{h}))^2 dx = \frac{1}{h^2} \int_0^h dx = \frac{1}{h}.$$

Therefore, πf converges very slowly to f, i.e., $\|\pi f - f\|_2 = \mathcal{O}(\sqrt{h})$, while $\|(\pi f - f)'\|_2$ even diverges as $h \to 0$. This does not contradict the theorem on Slide I-23 since the second derivative of f is not square integrable.