Solution of Exercise 10.2

We want to solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$ with symmetric positive-definite A by the conjugate gradient algorithm. We have an SPD preconditioner M available that we use to determine the preconditioned polynomial from $M\boldsymbol{z} = \boldsymbol{r}$.

(i) Show that solving $M\boldsymbol{z} = \boldsymbol{r}$ is actually one step of a stationary iteration for solving $A\boldsymbol{z} = \boldsymbol{r}$ with preconditioner M.

(ii) What would be the preconditioner if we executed two steps of this stationary iteration? Is it symmetric positive-definite?

Solution: For question (i) see Slide 18 of Lecture 10.
Regarding question (ii) let M be a symmetric positive definite preconditioner for A. This means that all eigenvalues of the corresponding iteration matrix G have modulus below one. So, if $(\lambda_i, \boldsymbol{u}_i)$ is an eigenpair of G, then

$$G\boldsymbol{u}_i = \boldsymbol{u}_i - M^{-1}A\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i, \qquad |\lambda_i| \leq \rho < 1. \tag{1}$$

Thus,

$$M\boldsymbol{u}_i - A\boldsymbol{u}_i = \lambda_i M \boldsymbol{u}_i. \tag{2}$$

Let $\boldsymbol{x}_k$ be an approximation of the solution of the linear system $A\boldsymbol{x} = \boldsymbol{b}$. We execute two steps of the stationary iteration with preconditioner M.

$$\boldsymbol{x}' = \boldsymbol{x}_k + M^{-1}\boldsymbol{r}_k, \qquad \boldsymbol{x}_{k+1} = \boldsymbol{x}' + M^{-1}\boldsymbol{r}',$$

with $\boldsymbol{r}_k = \boldsymbol{b} - A\boldsymbol{x}_k$ and $\boldsymbol{r}' = \boldsymbol{b} - A\boldsymbol{x}'$, respectively. Then

$$\begin{aligned}
\boldsymbol{x}_{k+1} &= \boldsymbol{x}' + M^{-1}\boldsymbol{r}' \\
&= \boldsymbol{x}_k + M^{-1}\boldsymbol{r}_k + M^{-1}(\boldsymbol{b} - A(\boldsymbol{x}_k + M^{-1}\boldsymbol{r}_k)) \\
&= \boldsymbol{x}_k + M^{-1}\boldsymbol{r}_k + M^{-1}\boldsymbol{r}_k - M^{-1}AM^{-1}\boldsymbol{r}_k \\
&= \boldsymbol{x}_k + (2M^{-1} - M^{-1}AM^{-1})\boldsymbol{r}_k \\
&= \boldsymbol{x}_k + M^{-1}(2M - A)M^{-1}\boldsymbol{r}_k
\end{aligned}$$

So, in our notation, the 2-step preconditioner is $M_2 = M(2M - A)^{-1}M$ which is evidently symmetric. From eq. (2) we see that

$$2M\boldsymbol{u}_i - A\boldsymbol{u}_i = (\lambda_i + 1)M\boldsymbol{u}_i$$

with $\lambda_i + 1 > 1 - \rho > 0$. Therefore,

$$\boldsymbol{u}_i^T M_2 \boldsymbol{u}_i = \boldsymbol{u}_i^T M(2M - A)^{-1}M\boldsymbol{u}_i = \frac{1}{1 + \lambda_i}\boldsymbol{u}_i^T M \boldsymbol{u}_i \geq \frac{1}{1 - \rho}\boldsymbol{u}_i^T M \boldsymbol{u}_i > 0.$$

Since this inequality holds for all eigenvectors, it holds for all vectors.