6.2.1 *General strategies for preconditioning*

For the Stokes problem, discretization error is measured in the energy norm for velocities and in the L_2 norm for pressure (see Section 5.4). Therefore, the natural matrix norm is $\|\mathrm{e}^{(k)}\|_E$ where

$$E = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & Q \end{bmatrix} \tag{6.14}$$

(see (5.151), (5.152) and Problem 6.1). Since $K\mathrm{e}^{(k)} = \mathbf{r}^{(k)}$, in terms of the residual this is

$$\|\mathbf{e}^{(k)}\|_E^2 = \langle EK^{-1}\mathbf{r}^{(k)}, K^{-1}\mathbf{r}^{(k)}\rangle = \|\mathbf{r}^{(k)}\|_{K^{-1}EK^{-1}}^2 .$$

For the Stokes problem, with coefficient matrix (6.1), the relevant matrix is

$$\begin{aligned} K^{-1}EK^{-1} = (KE^{-1}K)^{-1} &= \left(\begin{bmatrix} \mathbf{A} & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{A} & B^T \\ B & -C \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \mathbf{A} + B^TQ^{-1}B & B^T - B^TQ^{-1}C \\ B - CQ^{-1}B & B\mathbf{A}^{-1}B^T + CQ^{-1}C \end{bmatrix}^{-1} . \end{aligned} \tag{6.15}$$

Since it is $\|\mathbf{r}^{(k)}\|_{M^{-1}}$ that is reduced by the MINRES method, it would appear that a good choice of preconditioner is the positive-definite matrix

$$M = \begin{bmatrix} \mathbf{A} + B^TQ^{-1}B & B^T - B^TQ^{-1}C \\ B - CQ^{-1}B & B\mathbf{A}^{-1}B^T + CQ^{-1}C \end{bmatrix} . \tag{6.16}$$

For uniformly stabilized approximation ($C = 0$), this has the form

$$M = \begin{bmatrix} \mathbf{A} + B^TQ^{-1}B & B^T \\ B & B\mathbf{A}^{-1}B^T \end{bmatrix} . \tag{6.17}$$

Here, the presence of the two relevant Schur complements for stability (see (6.9) and (6.10)) are evident. Notice that $B\mathbf{A}^{-1}B^T$ is a discrete operator representing $\nabla \cdot (\nabla^2)^{-1}\nabla$, and $B^TQ^{-1}B$ represents $\nabla(\nabla \cdot)$ on the vector of velocity components. It is clear, however, that these matrix operators are not suitable as components of a preconditioner for the Stokes system, since they do not satisfy the requirement concerning ease of solution of systems of the form $M\mathbf{z} = \mathbf{r}$.

We will now derive some effective and practical preconditioners in this setting. In Section 6.2.3 we will also show that under appropriate circumstances, the resulting strategies are in fact essentially as good as (6.16)–(6.17) with respect to the norm being minimized by MINRES.

The form of the Galerkin matrix (6.1) and the desired norm based on the matrix (6.14) suggests that it is important to take account of the block structure when preconditioning. We thus consider block diagonal preconditioning matrices of the form

$$M = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & T \end{bmatrix}, \tag{6.18}$$

where both $\mathbf{P} \in \mathbb{R}^{n_u \times n_u}$ and $T \in \mathbb{R}^{n_p \times n_p}$ are symmetric and positive-definite. The convergence bound (6.7) then indicates that the speed of MINRES convergence depends on the eigenvalues λ of the generalized eigenvalue problem

$$\begin{bmatrix} \mathbf{A} & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{P} & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix}. \tag{6.19}$$

It is readily seen that if $\mathbf{P} = \mathbf{A}$, then $\lambda = 1$ is an eigenvalue of multiplicity at least $n_u - n_p$ corresponding to any eigenvector $[\mathbf{u}^T, \mathbf{0}^T]^T$ with $B\mathbf{u} = 0$. The multiplicity comes simply from the size of the right null space of the rectangular matrix B; thus if B is of full rank, n_p, then the multiplicity is exactly $n_u - n_p$. In the uniformly stable case ($C = 0$), if also $T = B\mathbf{A}^{-1}B^T$, then the remaining eigenvalues satisfy,

$$(1-\lambda)\mathbf{A}\mathbf{u} = -B^T\mathbf{p} \quad \text{and} \quad B\mathbf{u} = \lambda B\mathbf{A}^{-1}B^T\mathbf{p}$$

or by eliminating $\mathbf{u}$,

$$(\lambda^2 - \lambda - 1)B\mathbf{A}^{-1}B^T\mathbf{p} = 0.$$

Thus, since the assumed *inf–sup* stability in this case ensures that $B\mathbf{A}^{-1}B^T$ is positive-definite, we deduce that $\lambda = 1/2 \pm \sqrt{5}/2$ are the remaining eigenvalues, each with multiplicity n_p. This is an ideal situation from the point of view of convergence of MINRES — since the preconditioned matrix has only three distinct eigenvalues, there is a cubic polynomial with these three roots, and the convergence bound (6.7) will be zero for $k = 3$. That is, MINRES will terminate with the exact solution after three iterations irrespective of the size of the discrete problem.

This is an idealized situation, since the preconditioning operation with (6.18) requires the action of the inverses of $\mathbf{A}$ and of the Schur complement $B\mathbf{A}^{-1}B^T + C$. Three iterations require three such computations. The operation with the Schur complement is completely impractical since this is a full matrix. Note, moreover, that the congruence transform (6.2) would allow *direct solution* of this problem with two such operations with $\mathbf{A}$ and one with the Schur complement. However, this special choice of M suggests what is really needed, namely, a suitably chosen $\mathbf{P}$ that approximates $\mathbf{A}$, and a suitable T to approximate the Schur complement $B\mathbf{A}^{-1}B^T + C$.

We continue to consider the uniformly stable case for the moment, though our analysis in Section 6.2.2 covers also the stabilized case. The key to handling the Schur complement is provided by the stability condition (6.9) together with the boundedness condition (6.11): the sparse pressure mass matrix Q is spectrally equivalent to the dense matrix $B\mathbf{A}^{-1}B^T$ so that there is a lot to gain