FEM and Sparse Linear System Solving

Lecture 5, October 20, 2017: Beyond the Poisson problem

http://people.inf.ethz.ch/arbenz/FEM17

Peter Arbenz
Computer Science Department, ETH Zürich
E-mail: arbenz@inf.ethz.ch

- The finite element method
  - Introduction, model problems.
  - 1D problems. Piecewise polynomials in 1D.
  - 2D problems. Triangulations. Piecewise polynomials in 2D.
  - Variational formulations. Galerkin finite element method.
  - Theory of errors/error estimation.
  - Adaptive mesh refinement.
  - Some problems beyond the Poisson equation.
- Direct solvers for sparse systems.
- Iterative solvers for sparse systems.

Beyond the Poisson problem: Fluid Mechanics

- We consider some problems that are more complicated than the Poisson equation. The problems are taken from fluid dynamics.
- We start by reviewing the governing equations of mass and momentum balance and derive the Navier–Stokes equations.
- To that end we consider a fluid of density ρ moving in a three-dimensional domain Ω.

Suppose a particular small volume of fluid is at position $\boldsymbol{x}(t)$ at time t. Its velocity is given by

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{d\boldsymbol{x}}{dt}.$$

Each of the components of $\boldsymbol{u}$ is a function of space $\boldsymbol{x}$ and time t.

Beyond the Poisson problem: Fluid Mechanics (cont.)

Conservation of mass means that the rate of change of the mass in a volume D equals the amount of fluid flowing into D across ∂D.

In mathematical terms, this means that

$$\frac{d}{dt}\int_D \rho d\boldsymbol{x} = -\int_{\partial D} \rho\, \boldsymbol{u} \cdot \boldsymbol{n} ds = -\int_D \operatorname{div}(\rho \boldsymbol{u}) d\boldsymbol{x}. \qquad (1)$$

From (1) we get

$$\frac{d\rho}{dt} + \operatorname{div}(\rho \boldsymbol{u}) = 0.$$

Assuming a constant density ρ, this simplifies to

$$\operatorname{div} \boldsymbol{u} = 0.$$

Physically, this means that the volume of any small fluid particle dx does not change under deformation. Such fluids are said to be incompressible.

Beyond the Poisson problem: Fluid Mechanics (cont.)

Conservation of momentum means that the rate of change of the momentum of a fluid in a volume D equals the sum of the external forces. (Newton's law of motion)

In mathematical terms, this means that

$$\int_D \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \mathbf{grad}\,)\,\boldsymbol{u} \right) d\boldsymbol{x} = -\int_{\partial D} p\boldsymbol{n}\,d\boldsymbol{s} + \int_D \rho\,\boldsymbol{f}\,d\boldsymbol{x}, \quad (2)$$

The quantity $\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \mathbf{grad}\,)\,\boldsymbol{u}$ is the so-called convective derivative expressing the change of a quantity (vector of quantities) that is "following the fluid". Thus, the fluid acceleration is the convective derivative of the velocity.

In an ideal incompressible and homogeneous fluid, the only forces are pressure p and external body forces $\boldsymbol{f}$ like gravity.

Beyond the Poisson problem: Fluid Mechanics (cont.)

Using the equation

$$\int_{\partial D} p\, \boldsymbol{n}\, d\boldsymbol{s} = \int_D \mathbf{grad}\, p\, d\boldsymbol{x}$$

and taking into account that D is an arbitrary volume, we obtain the Euler equations for an ideal incompressible homogeneous fluid,

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \mathbf{grad})\, \boldsymbol{u} = -\frac{1}{\rho}\, \mathbf{grad}\, p + \boldsymbol{f}, \qquad \text{in } \Omega$$

$$\operatorname{div} \boldsymbol{u} = 0.$$

[Remember $\int\limits_D \partial_i u \cdot v \; d\boldsymbol{x} + \int\limits_D u \cdot \partial_i v \; d\boldsymbol{x} = \int\limits_{\partial D} u\, v\, n_i \; d\boldsymbol{s}$]

Beyond the Poisson problem: Fluid Mechanics (cont.)

For a "real" *viscous* fluid, each small volume of fluid is not only acted on by pressure forces (*normal stress*) but also by *tangential* or *shear* stresses. The Euler equations in this case have an additional term on the right,

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \mathbf{grad}\,)\,\boldsymbol{u} = -\frac{1}{\rho}\,\mathbf{grad}\,p + \nu\Delta\boldsymbol{u} + \boldsymbol{f}, \qquad \text{in } \Omega$$

$$\operatorname{div}\boldsymbol{u} = 0.$$

- The Laplacian Δ acts on all components of $\boldsymbol{u}$ *individually*.
- ν is called the kinematic viscosity.

Navier–Stokes et al.

Assuming *steady flow*, the temporal derivatives vanish. Thus we get the *Navier–Stokes equations* ($p \leftarrow p/\rho$)

$$-\nu\Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \mathbf{grad}\,)\,\boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f}, \qquad \text{in } \Omega$$
$$\text{div}\, \boldsymbol{u} = 0, \qquad \text{in } \Omega.$$

Removing the nonlinearity (low velocity flow) gives the *Stokes equations*

$$-\nu\Delta \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f}, \qquad \text{div}\, \boldsymbol{u} = 0.$$

Another linearization replaces $(\boldsymbol{u} \cdot \mathbf{grad}\,)\,\boldsymbol{u}$ by $(\boldsymbol{w} \cdot \mathbf{grad}\,)\,\boldsymbol{u}$ resulting in the *convection-diffusion equation*,

$$-\nu\Delta \boldsymbol{u} + (\boldsymbol{w} \cdot \mathbf{grad}\,)\,\boldsymbol{u} = \boldsymbol{f}, \qquad \text{div}\, \boldsymbol{u} = 0.$$

Convection-diffusion (or transport) equation

The weak form (of a scalar version) of the convection-diffusion equation is

Find $u \in \mathcal{H}^1_E(\Omega)$ such that

$$\nu \int_\Omega \mathbf{grad}\, u \cdot \mathbf{grad}\, v \; d\boldsymbol{x} + \int_\Omega (\boldsymbol{w} \cdot \mathbf{grad}\, u) v \; d\boldsymbol{x} = \int_\Omega f \; v \; d\boldsymbol{x} + \nu \int_{\partial\Omega_N} g_N \; v \; d\boldsymbol{s}, \quad \text{for all } v \in \mathcal{H}^1_{E_0}(\Omega)$$

If $\nu \ll 1$ then characteristics of this equation is very different from Poission equation. Nevertheless, the function spaces are the same. Two operators:

- $-\nu\Delta$ smears u proportionally to ν (diffusion)
- $\boldsymbol{w} \cdot \mathbf{grad}$ transports u in the direction of $\boldsymbol{w}$ (convection)

Convection-diffusion (or transport) equation (cont.)

As earlier, we choose finite dimensional vector spaces $S_0^h \subset \mathcal{H}^1_{E_0}(\Omega)$ and $S_E^h \subset \mathcal{H}^1_E(\Omega)$ consisting of piecewise polynomials.

We choose a basis span $\{\varphi_1, \ldots, \varphi_n\} \in S_0^h$ that we extend by additional functions $\varphi_{n+1}, \ldots, \varphi_{n+n_\partial}$ to satisfy the Dirichlet boundary conditions.
The matrix A corresponding to the FE discretization has elements

$$a_{ij} = \nu(\mathbf{grad}\,\varphi_i, \mathbf{grad}\,\varphi_j) + (\boldsymbol{w} \cdot \mathbf{grad}\,\varphi_j, \varphi_i).$$

It is nonsymmetric. Depending on the strength of the wind the problem tends to be more convective or more diffusive, i.e., more or less close to a Poisson problem.

Weak form of the Transport Equation

With these notation the weak form of the transport problem is

Find $u \in \mathcal{H}^1_E(\Omega)$ such that

$$a(u, v) = \ell(v) \qquad \text{for all } v \in H^1_{E_0}(\Omega)$$

where the bilinear and linear forms $a(\cdot, \cdot)$ and $\ell(\cdot)$ are

$$a(u, v) = \nu(\mathbf{grad}\, u, \mathbf{grad}\, v) + (\boldsymbol{w} \cdot \mathbf{grad}\, u, v)$$

$$\ell(v) = (f, v)$$

We can use piecewise linear elements, as before.

Standard Galerkin Finite Element Approximation

$S_E^h \subset \mathcal{H}_E^1(\Omega)$ is the space of continuous piecewise linear polynomials. The discrete problem is

Find $u_h \in S_E^h$ such that

$$a(u_h, v) = \ell(v) \qquad \text{for all } v \in S_E^h$$

The linear system for the unknown nodal values ξ_j of u_h is

$$A\boldsymbol{\xi} = \boldsymbol{b},$$

with $A_{ij} = \nu(\mathbf{grad}\,\varphi_j, \mathbf{grad}\,\varphi_i) + (\boldsymbol{w}\cdot\mathbf{grad}\,\varphi_j, \varphi_i)$,

$b_i = (f, \varphi_i), \qquad i, j = 1, \ldots, n_i, \; n_i = \#$ of interior nodes

The Galerkin Least Squares (GLS) FE Approximation

The transport equation $Lu = f$ with $L = -\nu\Delta + \boldsymbol{w}\cdot\textbf{grad}$ only weakly controls the derivatives of u (cf. Benzon & Larson, Ch. 10)

Find $u_h \in S_E^h$ such that

$$a_{\mathrm{sd}}(u_h, v) = \ell_{\mathrm{sd}}(v) \qquad \text{for all } v \in S_E^h$$

where the bilinear and linear forms $a_{\mathrm{sd}}(\cdot,\cdot)$ and $\ell_{\mathrm{sd}}(\cdot)$ are

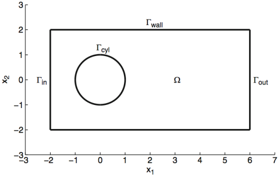
$$a_{\mathrm{sd}}(u, v) = a(u, v) + \delta(\boldsymbol{w}\cdot\textbf{grad}\, u, \boldsymbol{w}\cdot\textbf{grad}\, v)$$

$$\ell_{\mathrm{sd}}(v) = (f, v) + \delta(f, \boldsymbol{w}\cdot\textbf{grad}\, v)$$

The term $\delta(\boldsymbol{w}\cdot\textbf{grad}\, u, \boldsymbol{w}\cdot\textbf{grad}\, v)$ *stabilizes* the numerical method by adding diffusion proportional to δ along the streamlines. The GLS method is also referred to as the Streamline-Diffusion (SD) method.

Real-world application: Heat transfer in a fluid flow

Consider a heated object submerged into a channel with a flowing fluid. Fluid is flowing from left to right round a heated circle object.



Fluid flow is unaffected by temperature and given by velocity field

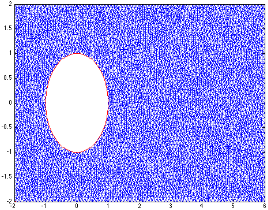
$$\boldsymbol{w}^T = U_\infty \left(1 - \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}, \frac{-2x_1x_2}{(x_1^2 + x_2^2)^2} \right)$$

where $U_\infty = 1$ is the free stream velocity of the fluid.

Heat transfer in a fluid flow: boundary conditions

- The cylinder is kept at constant temperature 1.
- The walls of the channel are insulated, no heat can flow across them. Means the normal heat flux $\boldsymbol{n} \cdot \boldsymbol{q}$ is zero on the walls, where $\boldsymbol{q}$ is given by Fourier's law $\boldsymbol{q} = -\nu\,\mathbf{grad}\,u + \boldsymbol{w}\,u$.
- At the outflow, ignore the diffusion $\boldsymbol{w}$, so $\nu\,\boldsymbol{n} \cdot \mathbf{grad}\,u = 0$.
- At the inflow, the fluid has zero temperature.

$$
\begin{aligned}
-\nu\Delta u + \boldsymbol{w} \cdot \mathbf{grad}\,u &= 0, \quad \text{in } \Omega \\
u &= 0, \quad \text{on } \Gamma_{\text{in}} \\
u &= 1, \quad \text{on } \Gamma_{\text{cyl}} \\
-\nu n \cdot \mathbf{grad}\,u &= 0, \quad \text{on } \Gamma_{\text{out}} \\
n \cdot (-\nu \mathbf{grad}\,u + \boldsymbol{w}\,u) &= 0, \quad \text{on } \Gamma_{\text{wall}}
\end{aligned}
\tag{3}
$$



In order to simplify the computer implementation, first approximate the Dirichlet conditions using the Robin conditions $-\nu \boldsymbol{n}\cdot\mathbf{grad}\, u = 10^6 u$ on $\Gamma_{\rm in}$ and $-\nu \boldsymbol{n}\cdot\mathbf{grad}\, u = 10^6(u-1)$ on $\Gamma_{\rm cyl}$. Multiplying the equation by test function v and integrating by parts both the diffusive and convective terms gives

$$
\begin{aligned}
0 &= \nu(\mathbf{grad}\, u, \mathbf{grad}\, v) - \nu(\boldsymbol{n}\cdot\mathbf{grad}\, u, v)_{L^2(\Gamma)} - (u, \boldsymbol{w}\cdot\mathbf{grad}\, v) + (\boldsymbol{n}\cdot\boldsymbol{w}\, u, v)_{L^2(\Gamma} \\
&= \nu(\mathbf{grad}\, u, \mathbf{grad}\, v) + 10^6(u, v)_{L^2(\Gamma_{\rm in})} + 10^6(u-1, v)_{L^2(\Gamma_{\rm cyl})} \\
&\quad - (u, \boldsymbol{w}\cdot\mathbf{grad}\, v) + (\boldsymbol{n}\cdot\boldsymbol{w}\, u, v)_{L^2(\Gamma_{\rm out})}
\end{aligned}
$$

The weak form

The weak form of the Eqs.(3)

$$\nu(\mathbf{grad}\, u, \mathbf{grad}\, v) + 10^6 (u, v)_{L^2(\Gamma_{\mathrm{in}})} + 10^6 (u, v)_{L^2(\Gamma_{\mathrm{cyl}})}$$
$$- (u, \boldsymbol{w} \cdot \mathbf{grad}\, v) + (\boldsymbol{n} \cdot \boldsymbol{w}\, u, v)_{L^2(\Gamma_{\mathrm{out}})} = 10^6 (1, v)_{L^2(\Gamma_{\mathrm{cyl}})}, \quad \forall v \in S_E^h.$$

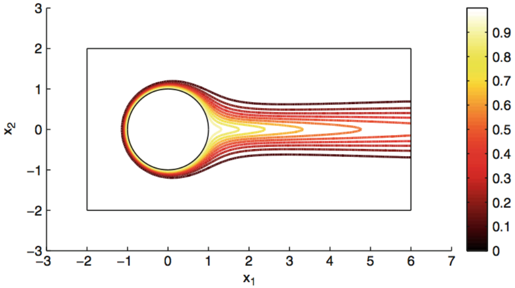
Adding the least squares term $\delta(\boldsymbol{w} \cdot \mathbf{grad}\, u, \boldsymbol{w} \cdot \mathbf{grad}\, v)$ to the weak form we obtain the GLS finite element approximation:
Find $u_h \in S_E^h$ such that

$$\nu(\mathbf{grad}\, u, \mathbf{grad}\, v) + 10^6 (u, v)_{L^2(\Gamma_{\mathrm{in}})} + 10^6 (u, v)_{L^2(\Gamma_{\mathrm{cyl}})}$$
$$- (u, \boldsymbol{w} \cdot \mathbf{grad}\, v) + (n \cdot \boldsymbol{w}\, u, v)_{L^2(\Gamma_{\mathrm{out}})}$$
$$+ \delta(\boldsymbol{w} \cdot \mathbf{grad}\, u, \boldsymbol{w} \cdot \mathbf{grad}\, v) = 10^6 (1, v)_{L^2(\Gamma_{\mathrm{cy}})}, \quad \forall v \in S_E^h.$$

The left hand side boundary terms can be written $(\kappa u, v)_{L^2(\Gamma)}$ with

$$\kappa = \begin{cases} 10^6, & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{cyl}} \\ \boldsymbol{w} \cdot n, & \text{on } \Gamma_{\text{out}} \\ 0, & \text{elsewhere} \end{cases}$$

Heat Transfer in a fluid flow: HeatFlowSolver2D.m is at
http://people.inf.ethz.ch/arbenz/FEM17/exercises/HeatFlowSolver2D.m



Contour plot of PDE solution

Stokes equations

The weak form of the Stokes equations is

Find $\boldsymbol{u} \in \mathcal{H}^1_E(\Omega)$ and $p \in L_2(\Omega)$ s.t.

$$\begin{aligned}
\int_\Omega \mathbf{grad}\,\boldsymbol{u} : \mathbf{grad}\,\boldsymbol{v}\; d\boldsymbol{x} - \int_\Omega p \,\mathrm{div}\, \boldsymbol{v}\; d\boldsymbol{x} &= \int_{\Gamma_N} \boldsymbol{s}\cdot\boldsymbol{v}\; d\boldsymbol{s} && \text{for all } \boldsymbol{v} \in \mathcal{H}^1_{E_0}(\Omega),\\
\int_\Omega div\boldsymbol{u}\, q\; d\boldsymbol{x} &= 0 && \text{for all } q \in L_2(\Omega),\\
\boldsymbol{u} &= \boldsymbol{w} && \text{on } \partial\Omega_D\\
\frac{\partial \boldsymbol{u}}{\partial n} - \boldsymbol{n}\, p &= \boldsymbol{s} && \text{on } \partial\Omega_N
\end{aligned}$$

Note:

$$\mathbf{grad}\,\boldsymbol{u} : \mathbf{grad}\,\boldsymbol{v} = \mathbf{grad}\, u_1 \cdot \mathbf{grad}\, v_1 + \mathbf{grad}\, u_2 \cdot \mathbf{grad}\, v_2 + \mathbf{grad}\, u_3 \cdot \mathbf{grad}\, v_3$$

Stokes equations (cont.)

In the Stokes equations we are looking for **two** functions at the same time. The three components of the first (vector) function $\boldsymbol{u}$ are in $\mathcal{H}^1_E(\Omega)$, so each of the three components of $\boldsymbol{u}$ can be discretized by piecewise linear finite element elements.
The pressure is only in $L_2(\Omega)$. Thus is requires less continuity. Piecewise constants are an option here.

Remark: The Stokes equations can be written as a so-called saddle point problem

$$\inf_{\boldsymbol{v}\in\mathcal{H}^1_{E_0}} \sup_{q\in L_2(\Omega)} \int_\Omega |\mathbf{grad}\,\boldsymbol{v}|^2 \, d\boldsymbol{x} - \int_\Omega q \,\text{div}\, \boldsymbol{v} \; d\boldsymbol{x} - \int_{\Gamma_N} \boldsymbol{s}\cdot\boldsymbol{v} \; d\boldsymbol{s}$$

Stokes equations (cont.)

Discretizing the Stokes equations leads to a matrix problem of the form

$$\begin{bmatrix} A & C \\ C^T & O \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix} \tag{4}$$

The matrix is symmetric but *indefinite*. A 'consists' of d copies of the Poisson matrix. C is the discrete divergence-free condition,

$$c_{ij} = (\psi_i, \operatorname{div} \boldsymbol{\varphi}_j).$$

The matrix in (4) does not admit a Cholesky factorization. If A is spd then

$$\begin{bmatrix} A & C \\ C^T & O \end{bmatrix} = \begin{bmatrix} I & \\ C^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & \\ & -C^T A^{-1} C \end{bmatrix} \begin{bmatrix} I & A^{-1} C \\ & I \end{bmatrix}.$$

Stokes equations (cont.)

- A in (4) often is spd. Then, there is a unique solution if C has maximal rank.
- If A is singular (e.g. symmetric positive semidefinite) then (4) has a unique solution if the intersection of the nullspace of A and of the nullspace of C^T is 'trivial',

$$\mathcal{N}(A) \cap \mathcal{N}(C^T) = \{\mathbf{0}\}.$$

- For a FE discretization to be stable the inf-sup condition

$$\min_{q_h \neq 0} \max_{\boldsymbol{v}_h \neq \mathbf{0}} \frac{|q_h \text{div } \boldsymbol{v}_h|}{\|q_h\|_{L_2(\Omega)} \|\boldsymbol{v}_h\|_{H^1(\Omega)}} \geq c > 0$$

has to be satisfied for all h, i.e., for all triangulations $\mathcal{T}_h$. This condition is also called Ladyzhenskaya-Babuška-Brezzi (LBB) stability condition.

Stokes equations (cont.)

- This condition is needed to show convergence of the finite element method.
- The $Q_2 - Q_1$ discretization on rectangular grids is stable.
- The LBB condition rules out simple choices like $Q_1 - P_0$.
- Stabilization procedures are used to make the zero (2,2) block in (4) 'more' negative definite.
- For details see Elman et al.

Exercise 5:

`http://people.inf.ethz.ch/arbenz/FEM17/pdfs/ex5.pdf`