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| **Advanced Algorithms 2024** | 23/09, 2024 |

Exercise 01

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## 1 Monotone Submodular Maximization

Consider a set $\mathcal{U}$ of $n$ elements that we can buy and a function $f : 2^{\mathcal{U}} \to \mathbb{R}^+$, where for each subset $S \subseteq \mathcal{U}$, the value $f(S)$ determines our profit if we buy exactly the elements of set $S$.

We assume two properties about this profit function: (A) Function $f$ is monotone in the sense that $f(S) \leq f(T)$ for any two sets $S, T$ such that $S \subseteq T$, and (B) Function $f$ is submodular in the sense that $f(S \cup i) - f(S) \geq f(T \cup i) - f(T)$ for any $i \in \mathcal{U}$ and any two sets $S, T$ such that $S \subseteq T$. In simple words, the submodularity means that the marginal gain that we have by adding $i$ to our purchase set diminishes as we move from one purchase set $S$ to a superset of it $T$. That is, roughly speaking, the more that we already have in the purchase set, the less extra gain by adding an element to it.

Devise an algorithm that purchases a set $S$ of (approximately) maximum profit, subject to the constraint that $|S| \leq k$, for some given value $k \in \{1, 2, 3, \ldots, n\}$. What approximation factor do you get?

## 2 2-Approximation for Knapsack (Vazirani 8.2)

In the knapsack problem (discussed in the class), discard all elements that are larger than the budget $B$, and then sort the remaining elements by decreasing ratio of profit to size, let this order be $a_1, a_2, \ldots, a_n$. Let $k$ be the smallest number such that the total size of the first $k$ elements $a_1, a_2, \ldots, a_k$ exceeds the budget $B$. Pick the more profitable of the following two options: $\{a_1, a_2, \ldots, a_{k-1}\}$ and $\{a_k\}$. Prove that this gives a 2-approximation for the most profitable set that fits in the knapsack.

## 3 Bin Covering (Vazirani 9.7)

Given $n$ items with sizes $a_1, a_2, \ldots, a_n \in [c, 1]$ for some fixed constant $c \in (0, 1)$, give a Polynomial-Time Approximation Scheme (PTAS) for the problem of maximizing the number of bins, subject to the constraint that each bin has items with total size at least 1.