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Exercise 3

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1 MAX-SAT

Consider a conjunctive normal form (CNF) formula on Boolean variables $x_1, x_2, \ldots, x_n$, that is, a formula defined as AND of a number m of clauses, each of which is the OR of some literals appearing either positively as x_i or negated as $\tilde{x_i}$. Suppose that each clause c_j has some weight w_j. The goal is to find an assignment of TRUE/FALSE values to the literals with the objective of maximizing the total weight of the satisfied clauses.

1. Consider a clause that has $k \geq 1$ literals. Argue that the simple randomized algorithm that sets each variable at random (true or false, each with probability half) satisfied this clause with probability $(1 - 2^{-k})$. Argue that this gives a randomized algorithm that outputs a solution that is in expectation 1/2-approximation for our problem. Then turn this algorithm into one that gives 0.49-approximation with probability at least 99%.

2. Consider the linear program with objective

$$\text{maximize } \sum_{j=1}^{m} w_j z_j$$

$$\text{subject to } \forall j \in \{1, 2, \ldots, m\} : \sum_{i \in S_j^+} y_i + \sum_{i \in S_j^-} (1 - y_i) \geq z_j$$

$$\forall j \in \{1, 2, \ldots, m\} : z_j \in [0, 1]$$

$$\forall i \in \{1, 2, \ldots, n\} : y_i \in [0, 1]$$

   Here, S_j^+ denotes the set of variables that appear in the j-th clause positively, and S_j^- denotes the set of variables that appear in the j-th clause in a negated form. Explain how this is a relaxation of an integer linear program for our objective. Moreover, let (y^*, z^*) denote the optimal solution of this LP. Show that the natural randomized rounding algorithm that sets each x_i to be True with probability y_i^* provides the following guarantee: if the j-th clause has k literals, it is satisfied with probability at least $(1 - (1 - 1/k)^k) z_j^*$.

   As in the previous exercise, argue that this gives an algorithm that in expectation provides $1 - 1/e$-approximation for our problem.

3. Notice that the first algorithm handles well large clauses and the second algorithm handles well the smaller clauses. Put the two together to get a 3/4 approximation algorithm for the MAX-SAT problem.

4. Find a CNF such that there is a 3/4 gap between the value of the solution of LP described in part two and the optimal Boolean assignment to the variables. Hint: find a CNF with 4 clauses, each of weight 1, such that the LP has value 4 but any assignment satisfies at most 3 clauses. This implies that the 3/4 factor is the integrability gap of this LP formulation and no rounding technique for it will give an approximation better than 3/4.

5. (optional) In part two, we considered a linear randomized rounding process. Consider a non-linear rounding which rounds variable x_i to be true with probability $f(y_i^*)$ where $f : [0, 1] \to [0, 1]$ is an arbitrary function such that $f(y) \in [1 - 4^{-y}, 4^{y-1}]$. Prove that this non-linear rounding directly gives a 3/4 approximation algorithm.