Advanced Algorithms 2024 18.11, 2024

Exercise 09

Lecturer: Bernhard Haeupler *Teaching Assistant: Jakob Nogler*

1 Tree embedding in cycle

In this exercise, we consider the tree embedding for the simple case where the graph is just a cycle on n vertices. That is, the vertices of our graph are numbers $1, 2, \ldots, n$ and there is an edge between each i ($1 \le i \le n-1$) and $i+1$, as well as between n and 1. Recall that we wish to approximate the metric induced on our graph by shortest paths (that is, distance between $i < j$ is $d_G(i,j) = \min(|j-i|, |n-j+i|)$) by a metric d_T induced by shortest paths on a weighted tree T with the same vertex set as in our cycle.

In the next two parts, first, we argue that we cannot hope for a good deterministic solution. Then, we show that for the case of the cycle we can achieve even a constant stretch *in expectation.*

1. Show that for any tree T with nonegative lengths of edges that satisfies $\forall i, j : d_G(i,j) \le d_T(i,j)$, there exist two indices i, j such that $d_T(i,j) \ge (n-1) \cdot d_G(i,j)$.

   Hint 1:

   Gradually turn T into a path wrapping clockwise or anticlockwise around the cycle.

   Each change to T maintains its stretch, while the sum of edge lengths of T drops.

   Hint 2:

   Imagine three vertices $u < v < w$ such that T contains an edge $\{u, w\}$

   of length $w - u$ and an edge $\{v, w\}$ of length $w - v$.

   Change $\{u, w\}$ to $\{u, v\}$ in this particular case.

2. The tree embedding algorithm from the lecture shows that there is a distribution over trees with average stretch $O(\log n)$, i.e., for any i, j we have $\mathrm{Exp}[d_T(i,j)] \le O(\log n) \cdot d_G(i,j)$.

   Show that in the case of the cycle there is actually a distribution over trees that achieves the stretch 2.

   Hint:

   Leaving a random edge out of the cycle gives you a path.

2 Steiner Forest

Given edge weighted graph G, and a set of pairs of terminals $(s_1, t_1), \ldots, (s_k, t_k)$, consider problem of finding minimal cost E' such that s_i, t_i are connected in $G[E']$. Use the tree embedding algorithm from the lecture to build $\mathcal{O}(\log n)$ approximate algorithm in expectation to this problem.

Hint:

Sample single tree T and solve the problem in T. Project solution back to G.

3 Analyze the Ball-Carving with Exponential Clocks

Consider a following process of Ball-Carving:

- every vertex v pick radius r_v according to *exponential distribution*, that is with probability density $\text{EXP}(x) = \beta \cdot e^{-\beta x}$ for $x \geq 0$.
- every vertex u picks as its ball-center the vertex $v = \arg\max_x (r_x - d(u, x))$ (we say that $u \in B_v$)

1. Give a reasonable upper bound on the diameter of each B_v that holds w.h.p.
2. Show that for every $v \in V(G)$ and for all $u \in B_v$, all the vertices on the shortest path from u to v are also in B_v.

   Hint:

   If w lies on the shortest path between u and v, but $w \in B_z$, shouldn't u also be in B_z?

3. What is the probability of two neighboring nodes u and v being in different balls, i.e., that they picked different ball centers?