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Exercise 11

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1 Minimum Enclosing Circle

Given a set P of n points in the plane, we want to find their *minimum enclosing circle* $C(P)$, i.e. the unique circle of minimum radius containing all points in P. You may assume the points are in general position, i.e. no four points lie on a circle.

The following fact is useful: for any set of points P, there exist three points $p_1, p_2, p_3 \in P$ such that the minimum enclosing circle of P is exactly the same circle as the minimum enclosing circle of p_1, p_2 and p_3. Thus, we also have that any circle that contains all of p_1, p_2 and p_3 must enclose all points in P.

You may assume basic geometric operations such as computing the enclosing circle of any given three points, and checking if a point is contained in a circle.

1. Give a $\mathcal{O}(|P|^4)$-time algorithm for finding $C(P)$.

2. Assume the following lemma holds:

   Lemma 1 *Suppose each point $p \in P$ has a positive weight $w(p)$, and a integer $r \geq 3$ is given. There is a randomized algorithm that, in $O(nr)$ time, constructs a set $R \subseteq P$ of up to r points from P such that the expected total weight of points in P not contained in the minimum enclosing circle of R satisfies*

   $$\mathop{\mathbb{E}}_{R}\left[\sum_{\substack{p \in P \\ p \notin C(R)}} w(p)\right] \leq \frac{3}{r+1} \cdot \sum_{p \in P} w(p).$$

   Using this lemma, give a randomized algorithm with expected running time $\mathcal{O}(n \log n)$ for finding the minimum enclosing circle.

   Hint: Use multiplicative weight updates. Select r to be a large constant, and recall that any circle not enclosing the entire point set must *not* contain at least one of p_1, p_2 or p_3.

3. Prove that selecting R by sampling r points weighted by w with replacement from P satisfies the equation in the above lemma.

   Hint: Suppose you sample points $x_1, x_2, \ldots, x_{r+1}$ independently from the same distribution of points. What is the probability that $x_{r+1} \notin C(x_1, x_2, \ldots, x_r)$?