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Sample Solutions 02

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1 Rounding for bin packing

Let $x_i = \frac{1}{2}$ for all i. Then, for any $\epsilon > 0$, packing items with sizes $(1+\epsilon)x_i = \frac{1}{2} + \frac{\epsilon}{2}$ takes n bins, versus the $n/2$ of the original items: $\alpha = 2$. This shows that the same, simple rounding as in the FPTAS for knapsack (rounding item values to a power of $(1+\epsilon)$) cannot be used to achieve a PTAS, and a more complicated rounding approach is required.

2 Target shooting

1. Let $X = \sum X_i$. The standard Chernoff bound, selecting $(1+\varepsilon)$ as the scaling parameter, gives
$$P(|X - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X]) \leq 2\exp(-\varepsilon^2 \mathbb{E}[X]/3) = 2\exp(-\varepsilon^2 \frac{|T|}{|S|} m/3)$$
then, selecting m as suggested gives a probability of $O(\delta)$ of multiplicative error of $(1+\varepsilon)$ or more.

2. A single sample is from T with probability $\frac{|T|}{|S|}$. Thus, as long as $|T| \leq \frac{1}{2}|S|$, the probability none of $O(\frac{|S|}{|T|})$ samples is from T is
$$\left(1 - \frac{|T|}{|S|}\right)^{O\left(\frac{|S|}{|T|}\right)} = \left(\left(1 - \frac{|T|}{|S|}\right)^{\frac{|S|}{|T|}}\right)^{O(1)} \geq 4^{-O(1)}$$
which is constant.

3. Call the algorithm $O(\log 1/\delta)$ times, and return the median result. This only fails if more than half of the returned values are less than $(1-\varepsilon)\text{OPT}$ or more than half are more than $(1+\varepsilon)\text{OPT}$. But the probability of landing outside the correct range is at most $\frac{1}{3}$, and that for correct range at least $\frac{2}{3}$. Thus, probability of failure with at least $O(\log 1/\delta)$ calls is at most $2^{-O(\log 1/\delta)} = O(\delta)$, as desired.

3 Counting satisfying assignments

We can propose a target-shooting algorithm which we will show to be an FPRAS faster than the DFN-COUNT algorithm as described in the lecture notes. To begin, let F be a disjunction of m clauses C_i, where each C_i is a conjunction of up to n literals. Let $f(F)$ be the number of satisfying assignments to F. By assumption, there exists an $i \in [m]$ such that $|C_i| = 10$. It follows that $f(F) \geq 2^{n-|C_i|} = 2^{n-10}$.

For our target-shooting algorithm, will sample k assignments α_j uniformly at random and count how many of these satisfy F. To show that this target-shooting algorithm is an FPRAS, we start by defining:

$$X_j = \begin{cases} 1 & F[\alpha_j] = 1, \\ 0 & \text{otherwise} \end{cases}.$$

Note that these are independent Bernoulli random variable with probability of success as follows:

$$\Pr[X_j = 1] = \frac{f(F)}{2^n} \geq \frac{2^{n-10}}{2^n} = \frac{1}{2^{10}},$$

by the previous observation of $f(F) \geq 2^{n-10}$. Define $X = \sum_{j=1}^{k} X_j$, then using linearity of expectations we can compute $\mathbb{E}[X]$ as follows:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{j=1}^{k} X_j\right] = \sum_{j=1}^{k} \mathbb{E}[X_j] = k\frac{f(F)}{2^n} \geq \frac{k}{2^{10}}. \tag{1}$$

As $f(F)$ is the quantity we would like to estimate, we can take the value of X and multiply it by $\frac{2^n}{k}$ so that in expectation it is $f(F)$. We therefore need to show that it is also ϵ-close with probability at least 3/4. To achieve this, let us first consider the probability that $f(F)$ is *not* ϵ-close to $\frac{2^n}{k}X$ in expectation:

$$\Pr\left[\left|\frac{2^n}{k}X - f(F)\right| \geq \epsilon f(F)\right] = \Pr\left[\left|X - k\frac{f(F)}{2^n}\right| \geq \epsilon k\frac{f(F)}{2^n}\right] = \Pr\left[|X - \mathbb{E}[X]| \geq \epsilon\mathbb{E}[X]\right].$$

As X is the sum of independent Bernoulli random variables, we can use the Chernoff bound to get an upper bound for the above probability:

$$\Pr\left[|X - \mathbb{E}[X]| \geq \epsilon\mathbb{E}[X]\right] \leq 2\exp\left(-\frac{\epsilon^2\mathbb{E}[X]}{3}\right) \leq 2\exp\left(-\frac{\epsilon^2 k}{3 \cdot 2^{10}}\right),$$

where we notably use the lower bound for $\mathbb{E}[X]$ in (1). In order for the proposed algorithm to be an FPRAS, we need to show that we can choose $k \in \text{poly}(n, 1/\epsilon)$ such that the upper bound above is less than or equal to 1/4. For this, observe the following:

$$2\exp\left(-\frac{\epsilon^2 k}{3 \cdot 2^{10}}\right) \leq \frac{1}{4} \iff k \geq 9 \cdot 2^{10} \ln 2 \cdot \frac{1}{\epsilon^2}.$$

Thus, by setting $k = O(\epsilon^{-2})$, we get:

$$\Pr\left[|X - \mathbb{E}[X]| \geq \epsilon\mathbb{E}[X]\right] \leq \frac{1}{4} \iff \Pr\left[\left|\frac{2^n}{k}X - f(F)\right| \leq \epsilon f(F)\right] \geq \frac{3}{4}.$$

What remains to be shown is to analyze the runtime of this algorithm. Observe that we can sample an assignment α_j in $\mathcal{O}(n)$ and check whether it satisfies F in $\mathcal{O}(m)$, which gives a complexity of $\mathcal{O}(nm)$ for each of the $k \in \text{poly}(n, 1/\epsilon)$ assignments. Overall, this gives $\mathcal{O}(nm/\epsilon^2)$, thus showing that this target-shooting algorithm is indeed an FPRAS. Moreover, comparing this with DNF-COUNT as given in the lecture notes, this algorithm is indeed faster, as desired.