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Sample Solutions 11

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1 Minimum Enclosing Circle

1. Loop over all subsets of three points in P, and check if the minimum enclosing circle of just those three points contains all points in P. There are $|P|^3$ subsets, and for each we test $|P|$ points, thus this takes $O(|P|^4)$ time total.

2. We give an algorithm that finds the minimum enclosing circle in time $O(n \log n)$ with probability at least $\frac{1}{2}$. As checking if a circle is the minimum enclosing circle can be done in linear time, repeating this algorithm until the minimum enclosing circle is found is sufficient.

   Let $t = O(\log n)$ and $r = O(1)$ be values we fix later. We initialize the weight of every point to $w^0(p) = 1$, and repeat the following for $i \in [t]$: sample a set R using Lemma 1 based on the weights w^{i-1}. We compute the minimum enclosing circle of R in constant time using the algorithm of part 1. Then, for any point p such that $p \notin C(R)$, we let $w^i(p) = w^{i-1}(p) \cdot e$. For other points, we let $w^i(p) = w^{i-1}(p)$. If every point is contained in the minimum enclosing circle of R, we return the minimum enclosing circle of R.

   For $i \in [t]$, fix w^{i-1}, and let R be a subset sampled using Lemma 1. We have

   $$\mathbb{E}\left[\sum_{p \in P} w^i(p)\right] = \sum_{p \in P} w(p)\left(1 + e \cdot \mathbb{P}(p \notin C(R))\right) \leq \left(\sum_{p \in P} w(p)\right) \cdot \left(1 + \frac{3e}{r+1}\right).$$

   Thus, using Markov, with probability at least $\frac{1}{2}$, we have

   $$\sum_{p \in P} w^t(p) \leq 2\mathbb{E}\left[\sum_{p \in P} w^t(p)\right] \leq 2n \cdot \left(1 + \frac{3e}{r+1}\right)^t \leq \exp\left(\frac{3e}{r+1} \cdot t + \ln(2n)\right).$$

   Assume now this holds, and suppose that the minimum enclosing circle was not found. Then, for each $i \in [t]$, one of the points p_1, p_2 or p_3 defining the minimum enclosed circle must not have been contained in the minimum enclosing circle of R. Thus, there exists a point $p' \in P$ such that p' was contained in at most $\frac{t}{3}$ minimum enclosing circles of sets R, thus in particular, we have $w^t(p') \geq e^{t/3}$. But we also have $w^t(p') < \sum_{p \in P} w^t(p)$. Thus, as long as

   $$\frac{1}{3} \cdot t \geq \frac{3e}{r+1} \cdot t + \ln(2n),$$

   we have a contradiction, thus we must have found the minimum enclosing circle. Selecting $t = 6\ln(2n) = O(\log n)$ and $r = \lceil 6 \cdot 3e - 1 \rceil = O(1)$ is sufficient.

3. Let $x_1, x_2, \ldots, x_{r+1}$ be $r+1$ points sampled independently from the same distribution. Let $i \in [r+1]$ be a random index. Then, the probability that $x_{r+1} \notin C(x_1, x_2, \ldots, x_r)$ equals the probability that $x_i \notin C(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+1})$.

   We can now fix any arbitrary $x_1, x_2, \ldots, x_{r+1}$. Let p_1, p_2, p_3 be any three points defining the minimum enclosing circle of the set. Then, the event that x_i is not in the minimum enclosing circle of the other points can only occur if $x_i \in \{p_1, p_2, p_3\}$, and the probability of the event is at most $\frac{3}{r+1}$.

Thus, letting R be a set of points sampled by sampling r points weighted by w with replacement from P, and x be a point sampled from P weighted by w, we have

$$\mathbb{P}(x \notin C(R)) \leq \frac{3}{r+1},$$

thus, we are done, as the expected total weight of points not contained in the minimum enclosing circle of R is

$$\begin{aligned}
\mathbb{E}_R\left[\sum_{p \in P \setminus C(R)} w(p)\right] &= \sum_{p \in P} w(p)\mathbb{P}(p \notin C(R)) \\
&= \left(\sum_{p \in P} w(p)\right) \cdot \mathbb{P}(x \notin C(R)) \\
&\leq \frac{3}{r+1} \cdot \sum_{p \in P} w(p).
\end{aligned}$$