

Numerical Simulation of Dynamic Systems XXV

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Introduction

All of the solvers that we studied until now have the following property in common:

Given the time instant t_{k+1} , the solvers perform a polynomial extrapolation to calculate the values of all state variables at that time instant.

Now we shall study what happens when we reformulate the problem in a reverse fashion. We shall determine *when* a state variable reaches a predetermined value, or more precisely:

Given that the state variable x_i currently assumes the value $x_i(t_k)$, we would like to determine the shortest time distance h , such that $x_i(t_k + h) = x_i(t_k) \pm \Delta Q_i$.

where ΔQ_i is a predetermined *state quantum* associated with the state variable x_i .

As we are simulating a continuous system on a digital computer, we need to discretize something, as no digital computer can compute infinitely many state changes within a finite time interval.

Until now, we always *discretized the time axis*, while *keeping the state variables continuous*. In the sequel, we shall *discretize (quantize) the state variables*, while *keeping the time axis continuous*.

Introduction II

If we can construct a solver based on this principle, it follows that:

- ▶ The integration method will use a *variable step size*, and the step size in use will depend on the gradient of that state variable.
- ▶ The step size h will be different for each state variable x .
- ▶ We shall no longer be able to represent the discretized system by a set of *difference equations*, and we shall *lose the linearity* of the discretized system when approximating a linear continuous system.

$$\dot{x} = A \cdot x \not\Rightarrow x_{k+1} = F \cdot x_k$$

Space Discretization: A Simple Example

Let us start by considering the following first-order system:

$$\dot{x}_a(t) = -x_a(t) + 10 \cdot \varepsilon(t - 1.76)$$

with initial condition $x_a(t_0 = 0) = 10$.

Rather than simulating this model directly, we shall analyze the following related model:

$$\dot{x}(t) = -\text{floor}[x(t)] + 10 \cdot \varepsilon(t - 1.76)$$

or:

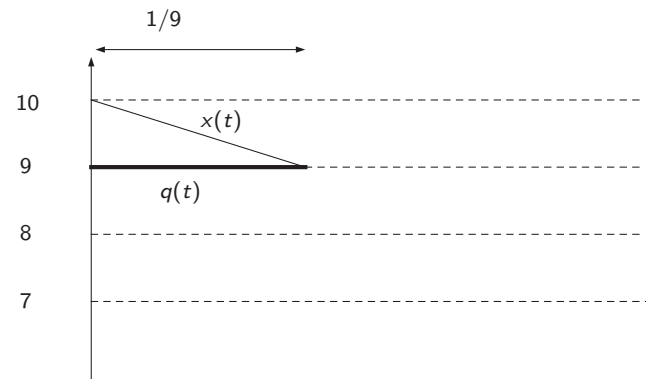
$$\dot{x}(t) = -q(t) + 10 \cdot \varepsilon(t - 1.76)$$

where $q(t) \triangleq \text{floor}[x(t)]$ is the integer part of the variable $x(t) > 0$.

The latter model can be simulated very easily.

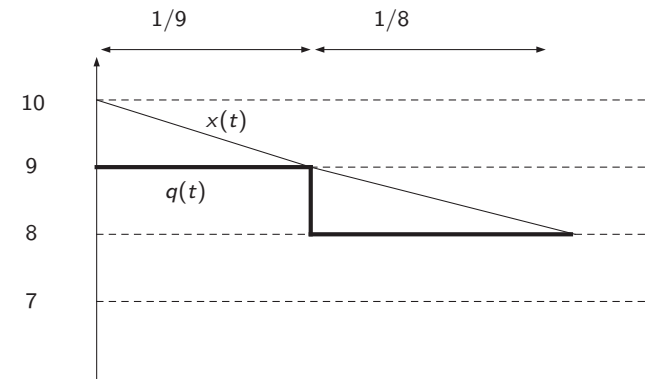
Space Discretization: A Simple Example II

$$\dot{x}(t) = -q(t) + 10 \cdot \varepsilon(t - 1.76) ; \quad q(t) = \text{floor}[x(t)] ; \quad x(0) = 10$$



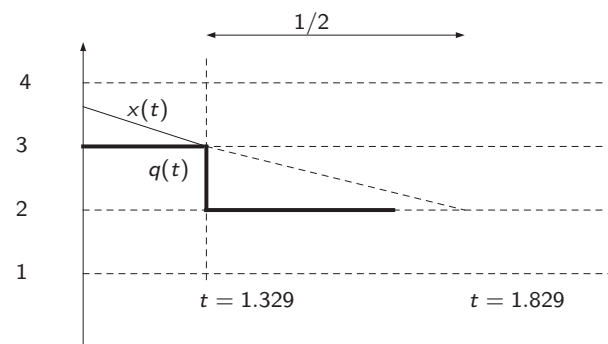
Space Discretization: A Simple Example II

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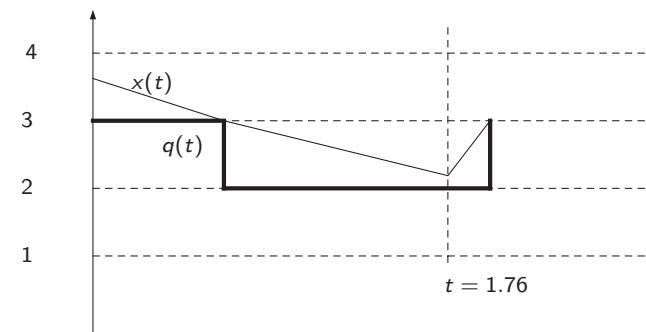
Space Discretization: A Simple Example II

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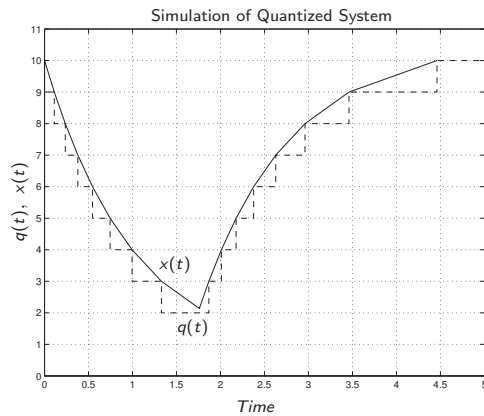


Space Discretization: A Simple Example II

$$\dot{x}(t) = -q(t) + 10 \cdot \varepsilon(t - 1.76) ; \quad q(t) = \text{floor}[x(t)] ; \quad x(0) = 10$$

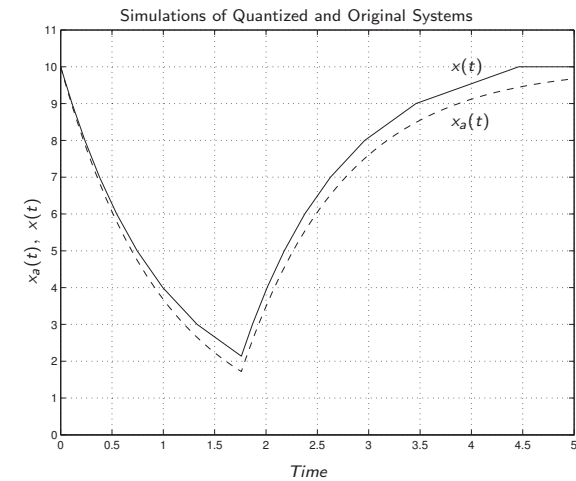


Space Discretization: A Simple Example III



- ▶ We were able to complete the simulation in **17 very simple steps**, thereby obtaining the **exact solution** of the quantized system.
- ▶ The solution of the quantized system is similar to that of the original continuous system.

Space Discretization: A Simple Example IV



Discrete Event Systems

- ▶ Clearly, the quantized system is a **discrete system**. However, it cannot be represented by a **set of difference equations**, i.e., it is not a **discrete-time system**.
- ▶ We recognize easily that it may be represented as a **discrete-event system**.
- ▶ The model can be encoded using the **DEVS formalism**.
- ▶ DEVS stands for **Discrete Event System specification**. The formalism was first introduced in the 1970s by **Bernard Zeigler**.
- ▶ All systems, the input/output behavior of which can be described by **sequences of discrete events**, can be represented using the DEVS formalism.

The Definition of DEVS

Atomic DEVS Models

A **DEVS model** processes a **sequence of input events** and, in reaction to those events and its own **initial discrete state**, generates a **sequence of output events**.



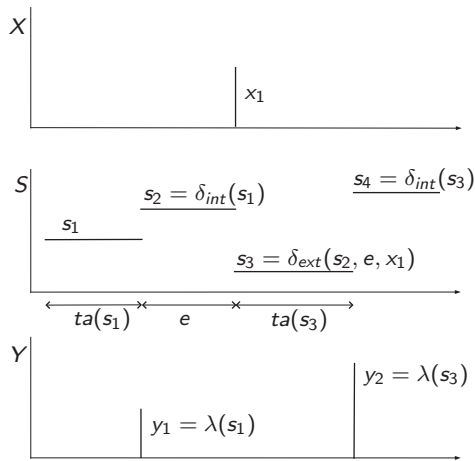
An **atomic DEVS model** is defined by the structure:

$$M = (X, Y, S, \delta_{int}, \delta_{ext}, \lambda, ta)$$

- ▶ X is the set of input values.
- ▶ Y is the set of output values.
- ▶ S is the set of state values.
- ▶ $\delta_{int}()$, $\delta_{ext}()$, $\lambda()$, and $ta()$ are functions defining the dynamics of the system.

The Definition of DEVS II

The Behavior of an Atomic DEVS Model



- $\delta_{int}(s)$ is the *internal transition function*.
- $\delta_{ext}(s, e, x)$ is the *external transition function*.
- $\lambda(s)$ is the *output function*.
- $ta(s)$ is the *time advance function*.

The Definition of DEVS III

The Specification of an Atomic DEVS Model

- Each possible state s ($s \in S$) has an associated *time advance* calculated by the *time advance function* $ta(s)$ ($ta(s) : S \rightarrow \mathbb{R}_0^+$). The time advance is a non-negative real number, determining how long the system remains in a given state in absence of input events.
- If the state adopts the value s_1 at time t_1 , after $ta(s_1)$ units of time (i.e., at time $t_1 + ta(s_1)$), the system performs an *internal transition*, taking it to a new state s_2 . The new state is calculated as $s_2 = \delta_{int}(s_1)$. Function $\delta_{int} : S \rightarrow S$ is called the *internal transition function*.
- When the state changes its value from s_1 to s_2 , an *output event* is produced with the value $y_1 = \lambda(s_1)$. Function $\lambda : S \rightarrow Y$ is called the *output function*. In this way, the functions ta , δ_{int} and λ define the *autonomous behavior of a DEVS model*.

The Definition of DEVS IV

The Specification of an Atomic DEVS Model

- When an input event arrives, the state changes instantaneously. The new state value depends not only on the value of the input event, but also on the previous state value and the elapsed time since the last transition. If the system assumes the state value s_2 at time t_2 , and subsequently, an input event arrives at time $t_2 + e < ta(s_2)$ with value x_1 , the new state is calculated as $s_3 = \delta_{ext}(s_2, e, x_1)$. In this case, we say that the system performs an *external transition*. Function $\delta_{ext} : S \times \mathbb{R}_0^+ \times X \rightarrow S$ is called the *external transition function*. No output event is produced during an external transition.

The Definition of DEVS V

The Specification of an Atomic DEVS Model

Let us consider the following simple example: A system receives positive numbers in an asynchronous way. After it received a number x , it generates an output event with the number $x/2$ after $3 \cdot x$ time units.

A DEVS model that correctly represents this behavior is the following:

$$M_F = (X, Y, S, \delta_{int}, \delta_{ext}, \lambda, ta), \text{ where}$$

$$X = Y = S = \mathbb{R}^+$$

$$\delta_{int}(s) = \infty$$

$$\delta_{ext}(s, e, x) = x$$

$$\lambda(s) = s/2$$

$$ta(s) = 3 \cdot s$$

Observe that the state can assume a time advance equal to ∞ . When this occurs, we say that the system is in a *passive state*, since it will no longer change its state, unless and until it receives an input event.

The Specification of an Atomic DEVS Model

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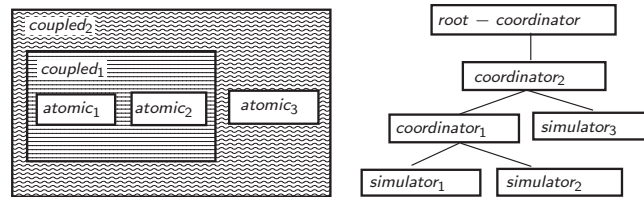
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Simulation of DEVS Models II

One of the simplest ways for implementing these steps is by writing a program with a hierarchical structure equivalent to the hierarchical structure of the model to be simulated.

A routine called *DEVS-simulator* is associated with each *atomic DEVS model*, and a different routine called *DEVS-coordinator* is related to each *coupled DEVS model*. At the top of the hierarchy, there is a routine called *DEVS-root-coordinator* that manages the global simulation time.



There exist several software tools that support directly the simulation of DEVS models. The one that we shall be using is called **PowerDEVS**. It was developed by *Ernesto Kofman* at the Universidad Nacional de Rosario (Argentina). It is the DEVS modeling and simulation environment that is most suitable for our purposes.



DEVS and Continuous System Simulation

In the example of the DEVS model of the static function, we represented piecewise constant trajectories as sequences of events. The same idea can also be used to *approximate continuous systems* using DEVS.

We can divide the quantized continuous system into:

a *dynamic system*:

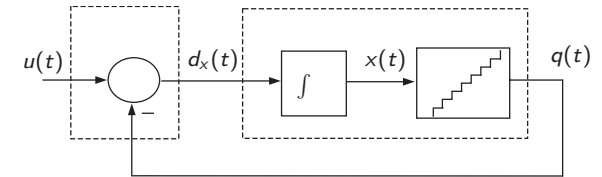
$$\begin{aligned}\dot{x}(t) &= d_x(t) \\ q(t) &= \text{floor}[x(t)]\end{aligned}$$

and a *static function*:

$$d_x(t) = -q(t) + u(t)$$

where $u(t) = 10 \cdot \varepsilon(t - 1.76)$.

The system can be represented using the following *block diagram*:



Each of the two subsystems has *input and output trajectories that are piecewise constant*. It is thus possible to represent them through DEVS models.



DEVS Models of Quantized Systems

The *static function* can be represented using the DEVS model M_F introduced earlier.

The *dynamic system* can be represented by the following DEVS model:

$$M_{QI} = (X, Y, S, \delta_{int}, \delta_{ext}, \lambda, ta), \text{ where}$$

$$X = Y = \mathbb{R} \times \mathbb{N}$$

$$S = \mathbb{R}^2 \times \mathbb{Z} \times \mathbb{R}_0^+$$

$$\delta_{int}(s) = \delta_{int}(x, d_x, q, \sigma) = (x + \sigma \cdot d_x, d_x, q + \text{sign}(d_x), \frac{1}{|d_x|})$$

$$\delta_{ext}(s, e, x) = \delta_{ext}(x, d_x, q, \sigma, e, x_v, p) = (x + e \cdot d_x, x_v, q, \tilde{\sigma})$$

$$\lambda(s) = \lambda(x, d_x, q, \sigma) = (q + \text{sign}(d_x), 0)$$

$$ta(s) = ta(x, d_x, q, \sigma) = \sigma$$

where:

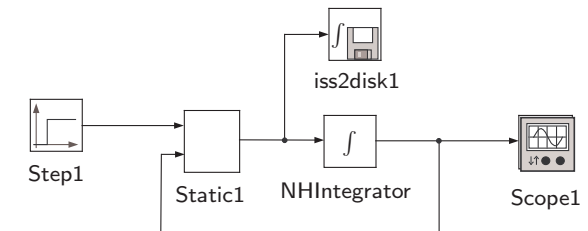
$$\tilde{\sigma} = \begin{cases} \frac{q+1-x}{x_v} & \text{if } x_v > 0 \\ \frac{q-x}{x_v} & \text{if } x_v < 0 \\ \infty & \text{otherwise} \end{cases}$$



PowerDEVS Model of a Quantized System

The DEVS models M_F (called *static function*) and M_{QI} (called *quantized integrator*) are graphically represented as *PowerDEVS blocks*.

The blocks can then be graphically coupled to each other:



and the system can be simulated easily.



Quantized Systems: Generalization

We can generalize this idea:

Given the *time-invariant continuous system* (state-space model):

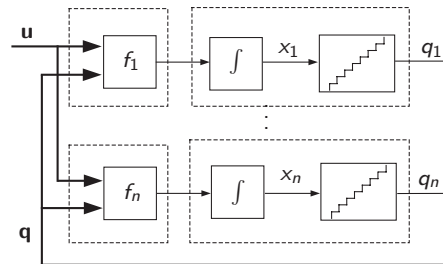
$$\begin{aligned}\dot{x}_{a_1} &= f_1(x_{a_1}, x_{a_2}, \dots, x_{a_n}, u_1, \dots, u_m) \\ &\vdots \\ \dot{x}_{a_n} &= f_n(x_{a_1}, x_{a_2}, \dots, x_{a_n}, u_1, \dots, u_m)\end{aligned}$$

The system can be approximated by the following *quantized system*:

$$\begin{aligned}\dot{x}_1 &= f_1(q_1, q_2, \dots, q_n, u_1, \dots, u_m) \\ &\vdots \\ \dot{x}_n &= f_n(q_1, q_2, \dots, q_n, u_1, \dots, u_m)\end{aligned}$$

We can model a generic time-invariant quantized system using DEVS models of the static function and quantized integrator types.

which can be represented by the following *block diagram*:



Quantized Systems: Illegitimacy

Unfortunately, there is a problem with the *legitimacy* of the resulting DEVS model.

A DEVS model is said to be *illegitimate* if it performs an infinite number of transitions in a finite interval of time.

Let us consider the quantized system:

$$\dot{x}(t) = -q(t) + 9.5 ; \quad q(t) = \text{floor}[x(t)]$$

with initial condition $x(0) = 10$:

- ▶ At $t = 0$, we have $q = 10$ and thus $\dot{x}(0) = -10 + 9.5 = -0.5$.
- ▶ Consequently, at $t = 0^+$, we have $x(t) = 9.999\dots$ and therefore $q(t) = 9$.
- ▶ This means that $\dot{x}(0) = -9 + 9.5 = +0.5$.
- ▶ As a consequence, we get immediately $x(t) = 10$ and thus return to the initial situation.

We notice that $q(t)$ *oscillates* between 10 and 9 with *infinite frequency*. For this reason, the DEVS model enters an *infinite loop*, and the simulation cannot advance.

Luckily, this problem can be solved easily by adding *hysteresis*.

Quantization Functions with Hysteresis

If we add *hysteresis* to the relationship between $x(t)$ and $q(t)$, the oscillations in $q(t)$ can only be produced by *large oscillations* in $x(t)$ that cannot occur instantaneously, as long as the magnitude of the state derivatives remains bounded.

Definition (Function of Quantization with Hysteresis)

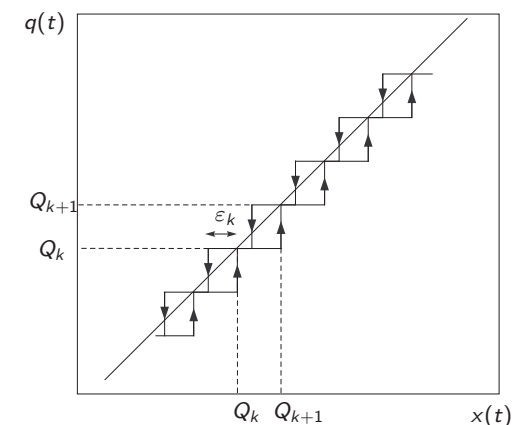
Given an ordered sequence of increasing real-valued numbers $(\dots, Q_{-1}, Q_0, Q_1, \dots)$, we say that $q(t)$ is related to $x(t)$ through a quantization function with hysteresis, if:

$$q(t) = \begin{cases} Q_m & \text{if } t = t_0 & \wedge & Q_m \leq x(t_0) < Q_{m+1} \\ Q_{k+1} & \text{if } x(t) = Q_{k+1} & \wedge & q(t^-) = Q_k \\ Q_{k-1} & \text{if } x(t) = Q_k - \varepsilon_k & \wedge & q(t^-) = Q_k \\ q(t^-) & \text{otherwise} \end{cases}$$

The discrete values Q_k are called *quantization levels*, and the distance $Q_{k+1} - Q_k$ is called *quantum*. The quantum is often chosen constant. ε_k is the *hysteresis width*.

Quantization Functions with Hysteresis II

The graph depicted below shows a *quantization function with hysteresis* with a *uniform quantum*.



QSS Method: Definition

Given the *time-invariant continuous system*:

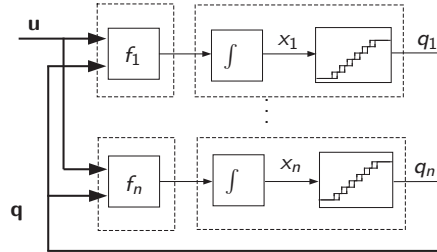
$$\begin{aligned}\dot{x}_{a_1} &= f_1(x_{a_1}, x_{a_2}, \dots, x_{a_n}, u_1, \dots, u_m) \\ &\vdots \\ \dot{x}_{a_n} &= f_n(x_{a_1}, x_{a_2}, \dots, x_{a_n}, u_1, \dots, u_m)\end{aligned}$$

approximated by the *quantized state system (QSS)*:

$$\begin{aligned}\dot{x}_1 &= f_1(q_1, q_2, \dots, q_n, u_1, \dots, u_m) \\ &\vdots \\ \dot{x}_n &= f_n(q_1, q_2, \dots, q_n, u_1, \dots, u_m)\end{aligned}$$

where each q_i is related to x_i by a *hysteretic quantization function*.

The QSS can be represented by the following *block diagram*:



As before, the QSS can be subdivided into *static functions* and *quantized integrators*.

DEVS Representation of a QSS

The DEVS models of the *static functions* are the same as before (M_F).

The DEVS model of the *quantized integrators* changes a bit due to the presence of *hysteresis*:

$M_{HQI} = (X, Y, S, \delta_{int}, \delta_{ext}, \lambda, ta)$, where

$$X = Y = \mathbb{R} \times \mathbb{N}; \quad S = \mathbb{R}^2 \times \mathbb{Z} \times \mathbb{R}_0^+$$

$$\delta_{int}(s) = \delta_{int}(x, d_x, k, \sigma) = (x + \sigma \cdot d_x, d_x, k + \text{sign}(d_x), \sigma_1)$$

$$\delta_{ext}(s, e, x_u) = \delta_{ext}(x, d_x, k, \sigma, e, x_v, p) = (x + e \cdot d_x, x_v, k, \sigma_2)$$

$$\lambda(s) = \lambda(x, d_x, k, \sigma) = (Q_{k+\text{sign}(d_x)}, 0)$$

$$ta(s) = ta(x, d_x, k, \sigma) = \sigma$$

with:

$$\sigma_1 = \begin{cases} \frac{Q_{k+2} - (x + \sigma \cdot d_x)}{d_x} & \text{if } d_x > 0 \\ \frac{(x + \sigma \cdot d_x) - (Q_{k-1} - \epsilon)}{|d_x|} & \text{if } d_x < 0 \\ \infty & \text{if } d_x = 0 \end{cases} \quad \sigma_2 = \begin{cases} \frac{Q_{k+1} - (x + e \cdot d_x)}{x_v} & \text{if } x_v > 0 \\ \frac{(x + e \cdot d_x) - (Q_k - \epsilon)}{|x_v|} & \text{if } x_v < 0 \\ \infty & \text{if } x_v = 0 \end{cases}$$

Simulation with QSS

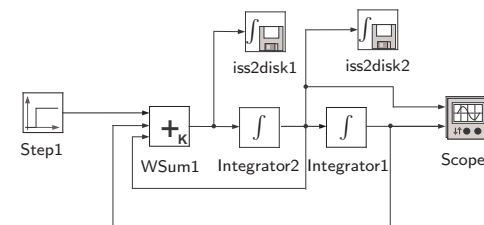
- In order to simulate a model using the QSS algorithm, we begin by choosing the *quantum* to be used by each state variable, i.e., by each *hysteretic quantized integrator*.
- We then would need to program the *static functions* and the *hysteretic quantized integrators*.
- However, **PowerDEVS** already comes with a library of pre-coded models of hysteretic quantized integrators (the user only needs to choose the quantum) and many different frequently used static functions (summers, limiters, etc.).
- It usually suffices to graphically construct the *block diagram* describing the system, choosing the quantum used by each of the state variables, and dragging the appropriate static functions from the graphical library and dropping them into the diagram window.
- It should be mentioned, however, that the QSS algorithm is independent of DEVS. We chose DEVS for the implementation of the QSS method, because DEVS simplified our work. However, we could have programmed the QSS method also independently of DEVS using any other event description formalism.

Simulation with QSS: An Illustrative Example

Let us consider the following second-order system and its QSS approximation:

$$\begin{aligned}\dot{x}_{a_1}(t) &= x_{a_2}(t) & \dot{x}_1(t) &= q_2(t) \\ \dot{x}_{a_2}(t) &= 1 - x_{a_1}(t) - x_{a_2}(t) & \dot{x}_2(t) &= 1 - q_1(t) - q_2(t)\end{aligned}$$

To simulate this system, we simply construct the *block diagram* using the hysteretic quantized integrator and the appropriate static functions of **PowerDEVS**:



- The *initial conditions* are parameters of the integrators (in our case: $x_1(0) = x_2(0) = 0$).
- The *quantum* and the *hysteresis* are parameters of each integrator (here: $Q_{k+1} - Q_k = \Delta Q = \epsilon_k = 0.05$).
- The QSS method intrinsically exploits *sparsity* (events are only propagated between directly connected blocks).

