Multistage Vertex Cover*

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Vertex Cover (VC) asks, given an undirected graph G and an integer $k \geq 0$, whether at most k vertices can be deleted from G such that the remaining graph contains no edge. VC is NP-hard and a formative problem of algorithmics and combinatorial optimization. We study a *time-dependent*, "*multistage*" version, namely a variant of VC on temporal graphs. A *temporal graph* $\mathcal{G}$ is a tuple $(V, \mathcal{E}, \tau)$ consisting of a set V of vertices, a discrete time-horizon τ, and a set of temporal edges $\mathcal{E} \subseteq \binom{V}{2} \times \{1, \ldots, \tau\}$. Equivalently, a temporal graph $\mathcal{G}$ can be seen as a vector $(G_1, \ldots, G_\tau)$ of static graphs (*layers*) over the same vertex set V. Then, our specific goal is to find a small vertex cover S_i for each layer G_i such that the sizes of the symmetric differences $S_i \triangle S_{i+1}$ between the vertex covers S_i and S_{i+1} of every two consecutive layers G_i and G_{i+1} are small. Formally, we thus introduce and study the following problem.

Multistage Vertex Cover (MSVC)
Input: A temporal graph $\mathcal{G} = (V, \mathcal{E}, \tau)$ and two integers $k \in \mathbb{N}, \ell \in \mathbb{N}_0$.
Question: Is there a sequence $\mathcal{S} = (S_1, \ldots, S_\tau)$ such that
(i) for all $i \in \{1, \ldots, \tau\}$, the set $S_i \subseteq V$ is a size-at-most-k vertex cover for G_i, and
(ii) for all $i \in \{1, \ldots, \tau - 1\}$, it holds that $|S_i \triangle S_{i+1}| \leq \ell$?

In our model, we follow the recently proposed *multistage* [4, 5, 9] view on classical optimization problems on temporal graphs. In general, the motivation behind a multistage variant of a classical problem such as Vertex cover is that the environment changes over time (here reflected by the changing edge sets in the temporal graph) and a corresponding adaptation of the current solution comes with a cost. In this spirit, the parameter ℓ in the definition of MSVC allows to model that only moderate changes concerning the solution vertex set may be wanted when moving from one layer to the subsequent one. Indeed, in this sense ℓ can be interpreted as a parameter measuring the degree of (non-)conservation [1, 10].

Related Work. The literature on vertex covering is extremely rich, even when focusing on parameterized complexity studies. Indeed, Vertex Cover (VC) can be seen as "drosophila" of parameterized algorithmics. Thus, we only consider VC studies closely related to our setting. First, we mention in passing that VC is studied in dynamic graphs [3, 12] and graph stream models [6].

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Table 1: Overview on our results. The column headings describe the restrictions on the input and each row corresponds to a parameter. p-NP-hard, PK, and NoPK, abbreviate para-NP-hard, polynomial problem kernel, and no problem kernel of polynomial size unless coNP ⊆ NP/ poly.

	general layers $1 \leq \ell < 2k$	general layers $\ell \geq 2k$	tree layers $1 \leq \ell < 2k$	one-edge layers $1 \leq \ell < 2$
	NP-hard		NP-hard	NP-hard
τ	p-NP-hard		p-NP-hard	FPT, PK
k	XP, W[1]-h.,	FPT, NoPK	XP, W[1]-h.	*open*, NoPK
$k + \tau$	FPT, PK		FPT, PK	FPT, PK

More importantly for us, Akrida et al. [2] studied a variant of VC on temporal graphs. Their model significantly differs from ours: They want an edge to be covered at least once over every time window of some given size Δ. Hence, they do not take into account the symmetric difference between two vertex covers for two consecutive layers.

A second related line of research, not directly referring to temporal graphs though, studies reconfiguration problems which arise when we wish to find a step-by-step transformation between two feasible solutions of a problem such that all intermediate results are feasible solutions as well[8, 11]. Mouawad et al. [14, 15] studied, among other reconfiguration problems, Vertex Cover Reconfiguration which takes as input a graph G, two vertex covers S and T of size at most k each, and an integer τ. The goal is to determine whether there is a sequence $(S = S_1, \ldots, S_\tau = T)$ such that each S_t is a vertex cover of size at most k. The essential difference to our model is that from one "sequence element" to the next only one vertex may be changed and that the input graph does not change over time. Indeed, there is an easy reduction of this model to ours while the opposite direction is unlikely to hold. This is substantiated by the fact that Mouawad et al. [14] showed that Vertex Cover Reconfiguration is fixed-parameter tractable when parameterized by vertex cover size k while we show W[1]-hardness for the corresponding case of MSVC.

Finally, there is also a close relation to the research on dynamic parameterized problems [1, 13]. Krithika et al. [13] studied Dynamic Vertex Cover where one is given two graphs on the same vertex set and a vertex cover for one of them together with the guarantee that the cardinality of the symmetric difference between the two edge sets is upper-bounded by a parameter d. The task then is to find a vertex cover for the second graph that is "close enough" (measured by a second parameter) to the vertex cover of the first graph. They show fixed-parameter tractability and a linear kernel with respect to d.

Our Contributions. Our results, focusing on the three perhaps most natural parameters, are summarized in Table 1. We highlight a few specific results. Multistage Vertex Cover remains NP-hard even if every layer consists of only one edge; clearly, the corresponding hardness reduction then exploits an unbounded number τ of time layers. If one only has two layers, however, one of them being a tree and the other being a path, then again Multistage Vertex Cover already becomes NP-hard. MSVC parameterized by solution size k is fixed-parameter tractable if $\ell \geq 2k$, but becomes W[1]-hard if $\ell < 2k$. Considering the tractability results for Dynamic Vertex Cover [13] and Vertex Cover Reconfiguration [14], this seems to be surprising and is our most technical result.

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