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Colorings and Transversals of Graphs

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Abstract

The subject of this thesis is a relaxation of proper graph colorings - *bounded monochromatic component* colorings (*bmc* colorings). A vertex-coloring of a graph is called a bmc coloring if every color-class induces monochromatic components containing at most a certain bounded number of vertices. A proper coloring for instance is a bmc coloring in which every color-class induces monochromatic components of order one. We investigate three different aspects of bmc colorings.

We investigate extremal graph theoretic problems of bmc colorings. For certain families of graphs we determine bounds for the smallest monochromatic component order C, the critical component order, such that every graph contained in this family accommodates for a bmc coloring with respect to C. We determine bounds for the critical component order C for graphs with a bounded maximum degree: Every graph of maximum degree at most three admits a bmc 2-coloring with one color-class inducing monochromatic components of order one and the other color-class inducing monochromatic components of order at most 22; and every graph of maximum degree at most five admits a bmc 2-coloring inducing monochromatic components of order at most 1908 in each of the two color-classes. Additionally we restrict the graphs to being planar and show that every maximal planar graph (a triangulation) with maximum degree Δ and containing at most d vertices of odd degree admits a bmc 3-coloring inducing monochromatic components of order at most $2\Delta d$.

Secondly we study algorithmic aspects of bmc colorings. The proof of the existence of a bmc 2-coloring with one color-class inducing monochromatic components of order one and the other color-class of order at most 22 can be turned into an efficient algorithm that actually 2-colors these graphs. For some large constant C' we derive an efficient algorithm that 2-colors every graph with maximum degree at most five

with monochromatic components of order at most C' in each color-class.

As a third aspect we focus on complexity theoretic problems of bmc colorings. For a fixed monochromatic component order C'' we investigate the decision problem whether a graph from the family of graphs with bounded maximum degree admits a bmc coloring with respect to C''. We exhibit a sudden "hardness jump" in the complexity of this decision problem for graphs of maximum degree at most three at the critical component order C.

To almost all proofs related to bmc colorings there is a common denominator: *bounded component transversals* of multipartite graphs. Thus we devote the first part of this thesis to these transversals, proving both extremal and algorithmic results. We investigate the smallest number of vertices $n(\Delta')$ that still guarantees the existence of (an efficient algorithm for finding) a transversal inducing bounded components in every multipartite graph with partite sets of order at least $n(\Delta')$ and maximum degree at most Δ'. We further emphasize the importance of transversals inducing bounded components with an application to the Linear Arboricity Conjecture.

Zusammenfassung

Gegenstand der vorliegenden Arbeit ist eine Relaxierung gültiger Färbungen – sogenannte *bmc* Färbungen. Eine Knotenfärbung eines Graphen heisst bmc Färbung, falls jede monochromatische Komponente einer Färbungsklasse höchstens eine gewisse beschränkte Anzahl Knoten enthält. Zum Beispiel enstpricht jede gültige Färbung einer bmc Färbung mit monochromatische Komponenten der Grösse eins. Im folgenden betrachten wir drei Aspekte der bmc Färbungen.

Wir untersuchen Fragestellungen aus der Extremalen Graphentheorie bezüglich bmc Färbungen und beweisen Schranken für die kleinste monochromatische Komponentengrösse C, die kritische Komponentengrösse, sodass jeder Graph aus einer gewissen Graphenfamilie eine bmc Färbung mit monochromatischen Komponenten der Grösse höchstens C besitzt. Im Zusammenhang mit dieser Fragestellung bestimmen wir Schranken für die kritische Komponentengrösse für die Familie aller Graphen mit beschränktem Maximalgrad: Jeder Graph mit Maximalgrad höchstens drei besitzt eine bmc 2-Färbung, sodass die monochromatischen Komponenten der ersten Färbungsklasse einen, diejenigen der zweiten Färbungsklasse höchstens 22 Knoten enthalten; und jeder Graph mit Maximalgrad höchstens fünf besitzt eine 2-Färbung, wobei die monochromatischen Komponenten beider Färbungsklassen höchstens 1908 viele Knoten enthalten. Desweiteren schränken wir uns zusätzlich auf planare Graphen ein und beweisen, dass jeder maximale planare Graph (d.h., eine Triangulierung) mit Maximalgrad Δ und höchstens k vielen Knoten ungeraden Grades eine bmc 3-Färbung besitzt, wobei die monochromatischen Komponenten aller drei Färbungsklassen höchstens $2\Delta k$ viele Knoten enthalten.

Wir untersuchen algorithmische Aspekte der bmc Färbungen und entwerfen, für eine konstante Komponentengrösse C', einen effizienten Algorithmus, der jeden Graphen mit Maximalgrad höchstens fünf

mit zwei Farben einfärbt, sodass alle monochromatischen Komponenten höchstens C' viele Knoten enthalten. Auch kann die zuvor erwähnte bmc 2-Färbung der Graphen mit Maximalgrad höchstens drei, wobei eine der Färbungsklassen monochromatische Komponenten der Grösse eins, die andere der Grösse höchstens 22, enthält, effizient gefunden werden.

An dritter Stelle behandeln wir komplexitätstheoretische Aspekte der bmc Färbungen. Für konstante monochromatische Komponentengrösse C'' betrachten wir das Entscheidungsproblem, ob ein Graph mit beschränktem Maximalgrad eine bmc Färbung besitzt. Wir stellen einen Sprung der Komplexität des Entscheidungsproblems für Graphen mit Maximalgrad drei an der kritischen Komponentengrösse fest.

Ein grossteil der Beweise bezüglich bmc Färbungen benutzen ein gemeinsames Prinzip: *Transversalen* multipartiter Graphen mit beschränkten Komponenten – sogenannte *bc Tranversalen.* Deshalb beschäftigen wir uns im ersten Teil dieser Arbeit mit bc Tranversalen und beweisen für diese extremale – sowie algorithmische Resultate. Wir betrachten die kleinstmögliche Anzahl Knoten $n(\Delta)$, sodass jeder multipartite Graph mit Maximalgrad höchstens Δ und Knotenmengen der Grösse mindestens $n(\Delta)$ eine bc Transversale enthält und entwerfen effiziente Algorithmen zum Auffinden solcher bc Transversalen. Desweiteren belegen wir die Bedeutung der bc Tranversalen mittels einer Anwendung für die Lineare Arborizität eines Graphen.

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Contents

It's a Bit Complicated!

Eddie Argos

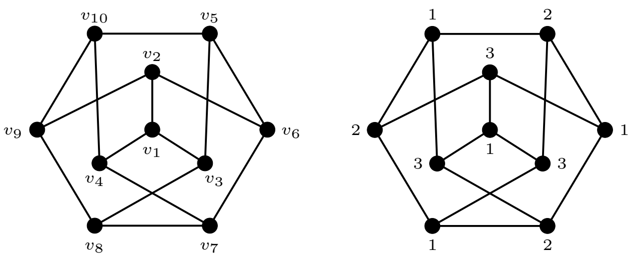
Chapter 0

Introduction

We start this thesis with a short introduction to graph theory and in particular to graph colorings.

0.1 Graph Colorings

A graph G is a set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, each connecting a pair of vertices, its *endpoints*. We say that the two endpoints of an edge are *adjacent*. A drawing that shows the well known Petersen Graph can be found in Figure 1(a).



(a) Petersen Graph (b) proper 3-coloring

Figure 1: The Petersen Graph and a proper 3-coloring of its vertices.

A *proper k-coloring* of the vertices of a graph G is an assignment of

k colors (often the integers $1, \ldots, k$) so that no two adjacent vertices get the same color, see for instance Figure 1(b) for a proper 3-coloring of the Petersen Graph. The set of vertices receiving color j is a *color-class* and induces a graph with no edges, i.e., it is an *independent set* of G. So, a proper k-coloring of the vertices of G is simply a partition of $V(G)$ into k independent sets (compare Figure 1(b) with Figure 2).

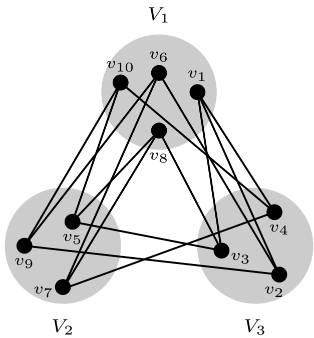


Figure 2: A partition of the Petersen Graph into three independent sets.

The chromatic number $\chi(G)$ of a graph G is the minimum k for which there is a k-coloring of G. One straightforward reason for a graph to have large chromatic number is the containment of a large clique (i.e., a large complete subgraph). Obviously if a graph G contains a clique of order k, then $\chi(G) \geq k$. Similarly for every subgraph $H \subseteq G$ it holds that $\chi(H) \leq \chi(G)$. Since the Petersen Graph contains odd cycles, and an odd cycle requires at least three colors in any proper coloring, the chromatic number of the Petersen Graph is at least three.

The following proposition states a relation between the maximum degree $\Delta(G)$ of a graph G and its chromatic number.

Proposition 0.1 (Greedy Coloring). *For every graph G, $\chi(G) \leq \Delta(G) + 1$.*

This can be easily seen as follows. Fix an arbitrary ordering, $v_1, \ldots, v_n$, of the vertices of G. For each v_i in turn, color v_i with the smallest color not appearing on any neighbor earlier in this ordering. Thus every vertex receives a color between 1 and $\Delta(G) + 1$.

Numerous problems of pure mathematics and theoretical computer science require the study of proper colorings and even more real-life problems require the calculation or at least an estimation of the chromatic number. Nevertheless, there is the discouraging fact that the calculation of the chromatic number of a graph or the task of finding an optimal proper coloring are both intractable problems. The following theorem by Karp [45] and its specialization to planar graphs by Stockmeyer, [59] formalize what we have just mentioned.

Theorem 0.1 ([45, 59]). *Deciding whether a graph is* 3*-colorable is* NP*-complete, even if the input graphs are restricted to being planar and having maximum degree at most four.*

Even fast approximation of the chromatic number is probably not possible according to a theorem by Lund and Yannakakis in [24].

Theorem 0.2 ([24]). *There is no polynomial-time algorithm that approximates the chromatic number of an n-vertex graph within a factor $n^{1-\epsilon}$ for some particular small $\epsilon > 0$, unless* $\mathrm{P} = \mathrm{NP}$.

Also the extremal graph theoretic question of finding upper bounds for the chromatic number for some restricted classes of graphs turned out to be very difficult and inspired researchers for more than a hundred years. Arguably the genesis of all graph coloring problems is the question whether every planar graph admits a proper 4-coloring. This has been answered in the affirmative by Appel, Haken, and Koch in [13].

Theorem 0.3 (Four Color Theorem [13]). *For every planar graph G, $\chi(G) \leq 4$.*

In some sense this proof of the Four Color Theorem is not satisfactory, due to the fact that its verification forces the verifier to go through a lengthy and computer aided proof. A shorter proof by Robertson Sanders Seymour, and Thomas appeared in [55] – still it is lengthy, uses computers and is hard to verify.

Many variants of proper colorings have been studied. A weaker result for planar graphs for a relaxation of proper colorings by Cowen, Cowen, and Woodall [25] states the following:

Theorem 0.4 ([25]). *The vertex set of every planar graph G can be partitioned into four sets $V_1, \ldots, V_4$ such that each set V_i induces a graph with isolated vertices and isolated edges only.*

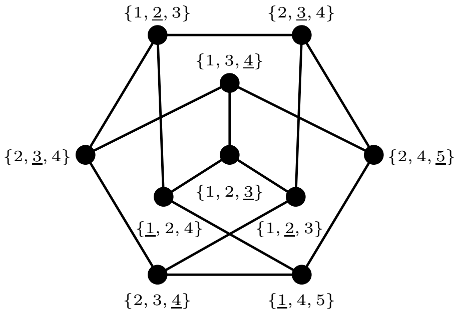


Figure 3: Lists of colors assigned to the vertices of the Petersen graph.

The proof of this statement does not assume the truth of the Four Color Theorem, is much shorter and can be verified without using a computer. We introduce the notion of *bounded monochromatic component colorings*, bmc colorings for short. A k-coloring is said to be a $(C_1, \ldots, C_k)$-bmc k-coloring if the graph induced by the ith color-class V_i contains no component larger than C_i. In particular, Theorem 0.4 shows that every planar graph admits a $(2, 2, 2, 2)$-bmc 4-coloring. The main focus of this thesis, Part II, is the investigation of bounded monochromatic component colorings.

A generalization of colorings with many applications, also in practice, are *list-colorings*. Instead of choosing the colors from the set $\{1, \ldots, k\}$, every vertex v has a list $L_v = \{c_1, \ldots, c_{k'}\}$ of available colors assigned to it. A coloring is called a list-coloring if for every vertex v in G the color of vertex v is contained in L_v. We call a graph G properly *k-choosable* if for every assignment of lists of length at least k to the vertices of G, the graph G has a proper list-coloring, see also Figure 3. Let us define the list-chromatic number $\chi_l(G)$ of a graph G to be the smallest list-length k such that G is k-choosable. Somewhat unexpectedly, the list-chromatic number can be much larger than the chromatic number. For instance the complete bipartite graph $K_{n,n}$, with $n = \binom{2k-1}{k}$, is not k-choosable (but $\chi(K_{n,n}) = 2$). Nevertheless, we can again by a greedy algorithm show the following upper-bound:

Proposition 0.2 (Greedy list-coloring). *For every graph G, $\chi_l(G) \leq \Delta(G) + 1$.*

For planar graphs the following statement has been proved by Thomassen in [61].

Theorem 0.5 ([61]). *For every planar graph G, $\chi_l(G) \leq 5$.*

The proof of this theorem applies an ingenious inductive argument. In [64] Voigt exhibits planar graphs and lists of length exactly four, such that there is no proper coloring of the graph consistent with the lists.

Another concept that is closely related to colorings and that can be seen as a generalization of list-colorings are *bounded component transversals* (bc transversals) of multipartite graphs G with $V(G) = V_1 \cup \cdots \cup V_m$. A transversal T of G is a subset of its vertices containing exactly one vertex from each partite set V_i. Moreover if the graph induced by the vertices of T, the *transversal graph*, contains no component larger than f, then T is called an *f-bc transversal*.

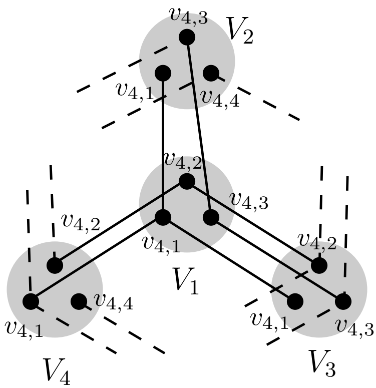


Figure 4: Independent transversals generalize list-colorings.

Why are list-colorings a special-case of transversals? For a graph G with assigned lists of length exactly k we define a new multipartite graph H as follows. Take k copies of each vertex v_i and for each color $c \in L_{v_i}$ denote one copy of v_i by $v_{i,c}$. Let two vertices $v_{i,c}$ and $v_{j,c'}$ be adjacent in H if $i \neq j$, $\{v_i, v_j\} \in E(G)$ and $c = c'$. The partite set V_i of H contains the k vertices $v_{i,c}, c \in L_{v_i}$. For an example we show such a transformation in Figure 4 of the four inner vertices $v_1, \ldots, v_4$ of the Petersen Graph and corresponding lists as in Figure 3. Hence choosing a proper list-coloring of G corresponds to choosing an *independent transversal* of H, i.e., a transversal inducing an independent set in G.

It turns out that bc transversals are omnipresent in proofs for bmc colorings. Therefore we devote Part I of this thesis to bc transversals.

0.2 Short Outline of the Thesis

In this section we attempt to give an informal and compact overview of the thesis. Nevertheless we insist on setting our problems into their corresponding context by stating existing results and also mentioning related problems. Also, we take the opportunity to mention the corresponding collaborators for each of the topics presented.

The thesis is organized into two parts, in one part results on bounded component transversals are presented, in the other part results on bounded monochromatic component colorings are shown. As mentioned earlier, bounded monochromatic component colorings go hand in hand with bounded component transversals – many proofs for bounded monochromatic component colorings apply results on bounded component transversals. Thus we first present the part on bounded component transversals before we come to bounded monochromatic component colorings.

Part I: Bounded Component Transversals

Let us recall that a *transversal* of a multipartite graph is a subset of its vertices containing exactly one vertex from each partite set. The *transversal graph* is the graph induced by the vertices of the transversal. For an integer $f \geq 1$ we call a transversal an *f-bc transversal* if the largest component of the transversal graph contains at most f vertices.

The main goal of this part of the thesis is to guarantee the existence of bounded component transversals for multipartite graphs that have large partite sets, with respect to their maximum degree. Moreover we want to elaborate on algorithms that even find such bounded component transversals. It seems natural that for some fixed maximum degree, the more vertices a multipartite graph contains in each of its partite sets, the more freedom is given to choose a transversal, possibly inducing only small components in the transversal graph. We have seen in Proposition 0.2 that the (list-)chromatic number of a graph can easily be upper bounded by its maximum degree, independent of the number of vertices contained in the graph. Let us for a moment focus on independent transversals, i.e., transversals inducing an independent set.

Let G be a multipartite graph with maximum degree Δ for which the number of vertices in each of its partite sets is lower-bounded by some value $p(\Delta)$ depending only on Δ. In contrast to the greedy algorithm for proper (list-)colorings, one can observe that a greedy algorithm for choosing an independent transversal of G – choose a vertex t_i from V_i that is not adjacent to any of the earlier chosen vertices t_j in G, $j < i$ – is likely to fail. This algorithm can get stuck in a partial transversal $t_1, \ldots, t_k$ such that every vertex of V_{k+1} is adjacent to a vertex of $t_1, \ldots, t_k$, with $k < m$.

Nevertheless using the well known Lovász Local Lemma one can prove that every multipartite graph G with partite sets containing at least $2e\Delta(G)$ many vertices contains an independent transversal. Moreover Haxell [35] found a proof for the fact that every multipartite graph G with partite sets containing at least $2\Delta(G)$ many vertices contains an independent transversal. A construction by Bollobás, Erdős, and Szemerédi [22] proves this result to be best possible. The result has then been extended to f-bc transversals, $f \geq 1$ by Haxell, Szabó, and Tardos [39] as follows. Every multipartite graph with partite sets containing at least $\left\lfloor \left(\frac{f+1}{f}\right)\Delta(G) \right\rfloor$ many vertices contains an f-bc transversal. Subsequently we refer to this result as the "Transversal Theorem".

Another closely related extremal problem that has gained lots of attention is the minimum order of partite sets in an m-partite graph with a certain maximum degree, such that the graph is guaranteed to contain an f-bc transversal, for some fixed number of partite sets m and fixed component order f (note that before we did not restrict on the number of partite sets in the multipartite graph).

Bounded component transversals have proved to be very applicable in many areas of graph theory. In Chapter 3 we show an application of bounded component transversals in order to get one tiny step closer to a proof of the Linear Arboricity Conjecture. Most importantly for us we will see many applications of bounded component transversals for bounded monochromatic component colorings in Part II of this thesis. applications, we mention some of the

Chapter 1: About f-Bc Transversal

We start this chapter with a new short proof of the Transversal Theorem by proving a stronger statement.

Independent Transversal Hereupon we shortly turn our focus to independent transversals. Again due to the Transversal Theorem applied with $f = 1$, every multipartite graph G with partite sets containing at least $2\Delta(G)$ many vertices has an independent transversal. On the other hand it has been shown by Bollobás, Erdős, and Szemerédi [22] that there are multipartite graphs with partite sets containing $2\Delta(G)-1$ many vertices with no independent transversal. Here we give a simplified construction for the same fact.

For some fixed numbers m and Δ, the minimum partite set order such that every m-partite graph G with maximum degree at most Δ is guaranteed to have an independent transversal has been thoroughly studied. A line of research including work from Aharoni, Alon, Bollobás, Erdős, Haxell, Jin, Szabó, Szemerédi, Tardos, and Yuster ([22, 6, 8, 44, 35, 36, 67, 2, 9, 60]) culminated in the work of Haxell and Szabó [38] that completely determines this smallest number of vertices in each partite set of G with respect to the maximum degree of G such that G contains an independent transversal.

Matching Transversals For the remainder of this chapter we focus on 2-bc transversals, subsequently referred to as *matching* transversals. Matching transversals have been introduced by Haxell, Szabó, and Tardos in [39] and successfully applied for the problem of bmc 2-coloring of 4-regular graphs. Also in Chapter 4 transversals are an invaluable tool in handling bmc 2-coloring of graphs of maximum degree three. The result used in the two proofs states that every multipartite graph of maximum degree at most two such that each part contains at least two vertices contains a matching transversal (see [39]). Note that this is a strengthening of the Transversal Theorem (which would only yield that the partite sets should contain at least three vertices provided that the maximum degree of the graph remains two). Our knowledge about matching transversals is much sparser than about independent transversals. Already the question whether every multipartite graph of maximum degree at most three and partite sets containing at least three vertices has a matching transversal remains open. If every partite set of the multipartite graph contains at least four vertices, then the answer is "YES" and if they contain only two vertices, then it can be easily verified that the answer is "NO".

From the Transversal Theorem we can conclude that every multipartite graph with partite sets containing at least $\lfloor 3\Delta(G)/2 \rfloor$ many vertices contains a matching transversal. On the other hand there are multipartite graphs H with partite sets containing at most $\Delta(H) - 1$ many vertices such that H does not contain a matching transversal. It is a great challenge to close this gap between the upper and lower bounds. We conjecture that every multipartite graph G with partite sets of order at least $\Delta(G)$ contains a matching transversal.

In order to get closer to this conjecture we considered the problem restricted to m-partite graphs, for some constant m. We first determine the minimal maximum degree of 3-partite graphs with partite sets of order exactly n with *no* matching transversal to be $\lceil 3n/2 \rceil$. Secondly restricted to 4-parite graphs we obtain the lower bound $4n/3$ and the upper bound $\lfloor 10n/7 \rfloor + 12$.

The results presented in this chapter are joint work with Penny Haxell and Tibor Szabó [16].

Chapter 2: Algorithmic Aspects

The lack of a greedy algorithm for (independent) transversals also leaves us with no algorithm for finding such a transversal. The proof of the Transversal Theorem is purely existential, and also its original proof by Haxell, Szabó, and Tardos in [39] and its topological proof by Aharoni, Chudnovsky and Kotlov [1] (and its stengthening by Szabó and Tardos in [60]) seem to resist any attempt to turn them into an efficient algorithm.

In this chapter we show the existence of a simple, deterministic, and polynomial-time (in the number of parts m) algorithm for finding an independent transversal for every m-partite graph of maximum degree at most Δ and parts of size $\Omega(\Delta^3)$, provided that Δ is constant.

Let us remark that using proof ideas by Alon (see [8]) one can improve the bound on the partite set order to be linear in $\Delta(G)$. Moreover – as an outlook – this technique can then be combined with our result above yielding the currently smallest factor C such that there is such a polynomial-time algorithm for finding an independent transversal of a multipartite graph with partite sets containing at least $C\Delta(G)$ many vertices.

Let us also mention that several proofs for results for the existence of certain bounded monochromatic component colorings can be turned into efficient algorithms using efficient algorithms on bounded component transversals, see for instance Section 4.2.

The results shown in this chapter can be found in [15].

Chapter 3: Application – Linear Arboricity

We promised several times that bounded component transversals find many applications in graph theory. In this chapter we want to highlight one such application that is not related to bounded monochromatic component colorings. The Linear Arboricity Conjecture, raised by Akiyama, Exoo, and Harary [4] states that every r-regular graph can be edge-colored with $\lceil \frac{r+1}{2} \rceil$ many colors such that each monochromatic component forms a path. Let us observe here that this bound on the number of colors is just the obviously necessary one. Every color-class contains only paths and hence at most $n-1$ many edges. Since an r-regular graph on n vertices contains $rn/2$ many edges, any such edge-coloring requires at least $\frac{rn}{2(n-1)} > r/2$ many colors. The conjecture has been answered in the affirmative for $r \in \{3, 4\}$ by Akiyama, Exoo, and Harary in [4, 5], for $r \in \{5, 6, 8\}$ by Enomoto and Péroche in [28], and for $r = 10$ by Guldan in [31]. Moreover the conjecture has been verified by Alon in [6] for r-regular graphs G if r is even and G has high girth, and for r-regular graphs if r is odd, G has a perfect matching and G has high girth.

We show that the above condition on the graph G having a perfect matching for r being odd can be relaxed to G having a 3-factor using the Transversal Theorem. Also we prove that every 7-regular graph with high girth fulfills the conjecture.

The results presented in this chapter are joint work with Thomas Rauber [17].

Part II: Bounded Monochromatic Component Colorings

Sometimes the number of colors available to color a graph is less than its chromatic number. Therefore one is forced to relax the properness condition and to find a good approximation of its properness. Another good reason to introduce relaxations of proper colorings is that in some theoretical or practical situations a small deviation from proper is still acceptable, while the problem could become tractable, or in certain problems the use of the full strength of proper coloring is an “overkill”. Often a weaker concept suffices and provides better overall results.

The variant of relaxations of proper colorings we study in this part of the thesis allows the presence of some small level of conflicts in the color assignment. Namely, we will allow vertices of one or more color-class(es) to participate in one conflict or, more generally, let each conflicting component have a bounded number of vertices. Since we impose bounds on the number of vertices in monochromatic components of a coloring, this relaxation is referred to as *bounded monochromatic component colorings* or short bmc colorings. Recall that a k-coloring is $(C_1, C_2, \ldots, C_k)$-bmc if every monochromatic component in the graph induced by the ith color-class contains at most C_i many vertices, for $1 \leq i \leq k$. Note that a $(1, \ldots, 1)$-bmc k-coloring corresponds to a proper k-coloring. We will mostly be concerned with bmc 2-colorings and especially the two most natural cases of it: *symmetric* bmc 2-colorings (when $C_1 = C_2$), and *asymmetric* bmc 2-colorings (when $C_1 = 1$).

Bmc colorings have been introduced by Kleinberg, Motwani, Raghavan, and Venkatasubramanianby in [46] motivated by a problem in computer science concerning storage management problems for evolving databases. Symmetric bmc 2-colorings were first studied by Alon, Ding, Oporowski, and Vertigan [10]. Asymmetric bmc 2-colorings were introduced in a joint paper with Tibor Szabó [19].

Chapter 4: Bmc 2-Colorings

This chapter can be considered as the main chapter of the thesis. We deal with bmc 2-colorings – the asymmetric case in the first section – the symmetric case in the second section.

Asymmetric Bmc 2-Colorings Any graph of maximum degree two – a graph consisting of disjoint paths and cycles only – easily admits a $(1, 2)$-bmc 2-coloring. On the other hand, we show a construction of 4-regular graphs on n-vertices, where the removal of any independent set leaves a graph with components of order $2n/3$. Thus such graphs do not admit a $(1, 2n/3 - 1)$-bmc 2-coloring.

Henceforth we subsequently focus on graphs with maximum degree at most three and investigate whether every such graph G contains an independent set such that its removal from G leaves only component of constant order (in other words, whether G admits a $(1, C)$-bmc 2-

coloring, for some universal constant C). Indeed we prove that every graph with maximum degree three admits a $(1,22)$-bmc 2-coloring. Let us call the smallest component order C for which there exists a $(1,C)$-bmc 2-coloring for every graph with maximum degree three the *critical component order* C^*. At the moment we can only construct graphs that are not $(1,5)$-bmc 2-colorable, hence $6 \leq C^* \leq 22$. The proof of the upper bound is constructive and implies an algorithm that actually finds such a $(1,22)$-bmc 2-coloring in quasilinear-time in the order of the graph. The algorithm and also its analysis is quite lengthy, therefore we introduce the main concepts of the algorithm on the more restrictive class of triangle-full graphs with maximum degree at most three. There we show that every triangle-full graph of maximum degree at most three admits a $(1,6)$-bmc 2-coloring. Moreover we can construct triangle-full graphs with maximum degree three showing that this results is best possible.

A similar statement – using a completely different approach for its proof – is true for triangle-free graphs of maximum degree at most three. Namely we show that every such graph admits a $(1,6)$-bmc 2-coloring. This result turns out to be crucial later when it comes to the determination of the complexity of asymmetric bmc 2-colorability problems.

We study bmc 2-colorings not only from the point of view of extremal graph theory, but also of complexity theory, and find that these aspects eventually meet for asymmetric bmc 2-colorings. We show that for every component order $2 \leq C < C^*$, the decision problem whether a graph with maximum degree at most three is $(1,C)$-bmc 2-colorable is NP-complete. Let us emphasize here that this result implies a "sudden jump" in the hardness of the decision problem, since for $C \geq C^*$, by definition, every graph with maximum degree at most three admits a $(1,C)$-bmc 2-coloring. Moreover, as mentioned earlier, we can construct graphs for any $\Delta \geq 4$ and positive C, that are not $(1,C)$-bmc 2-colorable and have maximum degree Δ. These graphs can then be used to show that the decision problem whether a graph with maximum degree at most Δ, with $\Delta \geq 4$, is $(1,C)$-bmc 2-colorable is NP-complete, for every $C \geq 2$.

In Figure 5 we overview the results about the hardness of deciding whether a graph of maximum degree Δ admits a $(1,C)$-bmc 2-coloring. We divide the results into three classes, depending on whether the problem is trivial (**T**), polynomial-time decidable (**P**) or NP-complete

(**NPc**). Similar hardness jumps have been shown for instance for the

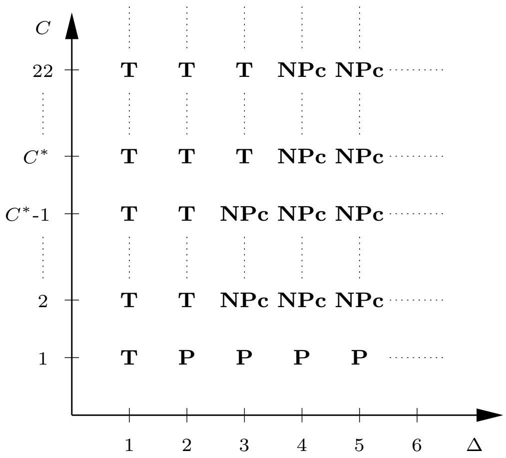


Figure 5: Hardness of the decision problem whether a graph of maximum degree Δ admits a $(1, C)$-bmc 2-coloring.

k-SAT problem with limited occurrences of each variable by Tovey [63] for $k = 3$ and Kratochvíl, Savický, and Tuza [48] for arbitrary k.

The results presented in this chapter are joint work with Tibor Szabó [19, 18].

Symmetric Bmc 2-Colorings In the second section we investigate symmetric bmc 2-colorings of graphs with bounded maximum degree.

Note first that the two vertex sets of a maximum edge-cut of a graph G with $\Delta(G) \leq 3$ immediately impose a 2-coloring of G such that every monochromatic component contains at most two vertices. In [10] Alon, Ding, Oporowski, and Vertigan showed that any graph of maximum degree four has a $(57, 57)$-bmc 2-coloring. This was improved by Haxell, Szabó, and Tardos in [39], who showed that even a $(6, 6)$-bmc 2-coloring is possible, and such a $(6, 6)$-bmc 2-coloring can be constructed in polynomial time. They also proved that the family of graphs of maximum degree at most five is $(17617, 17617)$-bmc 2-colorable, but the proof does not directly imply an efficient algorithm for finding such a coloring.

In this section we improve on this last result, by making use of independent transversals, showing that every graph of maximum degree at most five is $(1908, 1908)$-bmc 2-colorable. This new proof can actually be turned into an efficient algorithm that (C, C)-bmc 2-colors every graph of maximum degree at most five, with $C = 94371840$.

For the sake of completeness let us note here that a similar statement is not possible for graphs of maximum degree six. In [10] Alon, Ding, Oporowski, and Vertigan constructed for every component order C a graph H_C of maximum degree six such that in every 2-coloring of H_C there is a monochromatic component of order larger than C. The graph H_C actually implies a much stronger statement. In every 2-coloring of H_C there is a monochromatic component of order $\Omega(\sqrt{n})$, with n being the number of vertices of H_C.

For the complexity theoretic aspect of sbmc 2-colorings we show that deciding whether a graph of maximum degree at most six is (C, C)-sbmc 2-colorable is NP-complete, for $C \geq 2$. Moreover we conjecture that there is a sudden jump in the hardness of the symmetric case as well, for both maximum degree four and maximum degree five, similar to the asymmetric case. Such a result would particularly be interesting for graphs of maximum degree at most four, since here the determination of the critical component order is even more within reach (between 4 and 6). So far we can show that deciding whether a graph of maximum degree at most four is (C, C)-sbmc 2-colorable is NP-complete, for $C \in \{2, 3\}$. The similar problem is wide open for graphs with maximum degree 5: Currently the critical component order lies between 6 and 1908.

A summary of our knowledge about symmetric sbmc 2-colorings is shown in Figure 6. The results appearing in this chapter can be found in [15] and in a joint paper with Tibor Szabó [18].

Chapter 5: Bmc k-Colorings, $k > 2$

The last chapter is devoted to the study of bmc k-colorings with $k > 2$.

Asymmetric bmc (k, l)-Colorings In a first section we investigate a generalization of asymmetric bmc 2-colorings. That is, several color-classes form an independent set and some other color-classes induce monochromatic components of order at most C, for some fixed parameter C. We say that a class of graphs is abmc (k, l)-colorable if there is a constant C' depending only on k and l such that every graph in the

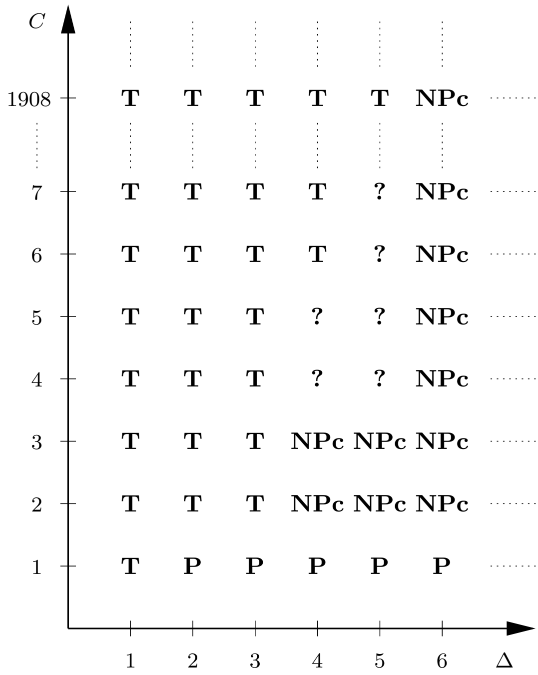


Figure 6: Hardness of the decision problem whether a graph of max. degree Δ admits a (C, C)-bmc 2-coloring.

class admits a $(C_1, \ldots, C_k, C_{k+1}, \ldots, C_{k+l})$-bmc $(k+l)$-coloring with $C_i = 1$, for $i \in [k]$ and $C_j = C'$ for $j \in \{k+1, \ldots, k+l\}$. For every two integers k, l we give lower and upper bounds for the smallest maximum degree Δ such that there is a graph of maximum degree at most Δ that is *not* abmc (k, l)-colorable.

The results appearing in this first section are joint work with Tibor Szabó [19].

Symmetric Bmc Colorings of Planar Graphs In the second section of this chapter we focus on planar graphs. Recall that every planar graph is properly 4-colorable due to the Four Color Theorem, see Theorem 0.3. On the other hand there are planar graphs that are not properly 3-colorable. Let us consider symmetric k-colorings of planar graphs with $k \in \{2, 3\}$.

Considering symmetric bmc 2-colorings, we mentioned earlier that in [10] the authors construct planar n-vertex graphs H such that in every 2-coloring of H there is a monochromatic component of order $\Omega(\sqrt{n})$. Note that these graphs have maximum degree six. On the other hand if we restrict to outerplanar graphs G, then we can show that G admits a (C, C)-bmc 2-coloring with $C = 2\Delta(G) - 1$. This result is tight up to a factor two, there are outerplanar graphs H' for which in every 2-coloring of H' there is a monochromatic component containing at least $\Delta(H') - 3$ many vertices.

Kleinberg, Motwani, Raghavan, and Venkatasubramanianby in [46] and also Linial, Matoušek, Sheffet, and Tardos [49] construct planar graphs H'' such that in every 3-coloring of H'' there is a monochromatic component of order $\Omega(n^{1/3})$, $n = |V(H'')|$. Unlike the graphs H for the symmetric bmc 2-colorings, these graphs H'' contain vertices of large degree, i.e., linear in the number of vertices of H''. Motivated by this fact, Kleinberg et al [46] ask the following natural question which we will formulate as a conjecture. Is there a constant $f(\Delta)$ depending only on Δ such that every planar graph with maximum degree at most Δ admits a $(f(\Delta), f(\Delta), f(\Delta))$-bmc 3-coloring? We solve a weaker but similar problem. Every triangulation of maximum degree at most Δ with at most d many vertices of odd degree admits a $(2d\Delta^3, 2d\Delta^3, 2d\Delta^3)$-bmc 3-coloring.

The results presented in this section are joint work with Gábor Tardos [20].

0.3 Preliminaries and Notation

We denote the set of integers $\{1, \dots, k\}$ by $[k]$.

Let G be a graph, then $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G. The *order* of a graph G is the cardinality of $V(G)$ (often simply denoted by n), the *size* of G is the cardinality of $E(G)$. The edge joining two vertices u and v is denoted by $\{u, v\}$ (when an edge is directed from u to v, then we denote this *arc* by the ordered pair (u, v) instead). Sometimes we want to allow multiple edges between the same pair of vertices, such graphs are denoted as *multigraphs*. For a vertex v in $V(G)$, $N_G(v)$ denotes the *set of neighbors* of v, and $d_G(v) = |N_G(v)|$ is the *degree* of v in G. If the graph G is obvious from the context, we write

$d(v)$ and $N(v)$ instead of $d_G(v)$ and $N_G(v)$. The maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. We call G an *r-regular graph*, if $d_G(v) = r$ for all v in $V(G)$. A graph where all vertex degrees are even is also called an *Eulerian* graph.

Similarly we define for a *directed* graph G, i.e., a graph with only directed edges, the *in-neighborhood* $N_G^-(v) = \{(u,v) \in E(G)\}$, the *out-neighborhood* $N_G^+(v) = \{(v,u) \in E(G)\}$ and correspondingly the *indegree* $d_G^-(v) = |N_G^-(v)|$ and the *outdegree* $d_G^+(v) = |N_G^+(v)|$. Moreover we define $\delta^-(G), \Delta^-(G), \delta^+(G)$, and $\Delta^+(G)$ to be the minimum indegree, maximum indegree, minimum outdegree, and maximum outdegree of G.

A *path* P is a connected graph with $\Delta(P) \leq 2$ consisting either of only one single vertex or of two distinct vertices v_s and v_e of P with $d(v_S) = d(v_e) = 1$ and other vertices v with $d(v) = 2$. The *length* of a path P is the number of edges in P. The length of the shortest path between two vertices u and v in G is denoted by $dist_G(u,v)$. The *girth* of G, denoted by $g(G)$, is the minimum length of a shortest cycle in G. The *diameter* $dia(G)$ of a graph G is the length of a longest shortest path, i.e., $dia(G) = \max_{u,v \in V(G)} dist(u,v)$. A *tree* T is a connected graph with containing no cycles. The *depth* of a tree is the minimum distance from the root to a leave vertex.

A *component* C of a graph G is a containment-maximal connected subgraph of G. Sometimes it is more appropriate to think of a component as just the vertices of C. We will switch between both viewpoints.

An *edge-cut* of a graph G is a vertex-partition $\mathcal{U} = (U_1, U_2)$. We say that an edge of G with one endpoint in U_1 and the other endpoint in U_2 is *contained* in $\mathcal{U}$. We define $|\mathcal{U}|$ (the size of $\mathcal{U}$) to be the number of edges contained in $\mathcal{U}$. Also we say that we *switch the sides* of a vertex $v \in U_i$, if in fact we move v from U_i to U_{3-i}, $i \in \{1,2\}$. Similarly a subset $U' \subseteq V(G)$ of G is called a *vertex-cut* if $G \setminus U'$ consists of more than one connected component. A graph G is called *k-edge-connected* (*k-vertex-conntected*) if there is no edge-cut (vertex-cut, resp.) (a subset of the edges (vertices, resp.) of G that disconnects G) of size at most $k-1$. A *block* is a maximal (w.r.t. vertex inclusion) 2-vertex-connected component of a graph G. Similarly an *edge-block* is a maximal (w.r.t. edge inclusion) subgraph of G not containing any cut-edge.

A graph G together with a partition of its vertex set $V(G)$ into independent sets $V_1, \ldots, V_m$, is called an *m-partite* graph (or a *multi-partite* graph). Clearly, every m-partite graph is properly m-colorable.

A graph G is called *bipartite* if there is *some* partition of $V(G)$such that each partite set forms an independent set (note that the partition of $V(G)$ is not fixed for bipartite graphs). The complete bipartite graph with $|V_1| = n$ and $|V_2| = m$ is denoted by $K_{n,m}$.

We denote the complete graph on n vertices by K_n. The graph K_3 is also called a *triangle*. We say that a graph G is H*-free* if G does not contain H as a subgraph. Also we call a graph G H*-full* if every vertex in G is contained in a subgraph isomorphic to H. Another graph that we will meet in the course of this thesis is the *diamond* D, that is two triangles sharing exactly one edge.

A graph that can be embedded into the plane without crossing edges is referred to as a *planar graph*. A *plane graph* is a planar graph together with a particular planar embedding of it. A *face* of a plane graph is, informally speaking, a maximal region of the plane containing no point of the embedding. Let us denote the set of faces of a plane graph G by $F(G)$. A finite plane graph has one unbounded face - the *outerface*. A maximal planar graph (with respect to edge addition) is called a *triangulation* - all faces form a triangle. If a graph admits a planar embedding such that all vertices lie on the outerface, then we call the graph *outerplanar*. The *dual* G^* of a plane graph G is define to be the multigraph with $V(G^*) = F(G)$ and $E(G^*) = (F_e \mid e \in E(G))$, where F_e denotes the two (not necessarely distinct) faces in G incident to the edge e.

Similarly to proper vertex colorings (as defined in Section 0.1), a *proper edge k-coloring* of a graph G is an assignment of the integers $1, \ldots, k$ to the edges of G such that no two incident edges get the same color. The *line-graph* $L(G)$ of a graph G is defined as follows: $V(L(G)) = E(G)$ and $E(L(G)) = \{\{e_1, e_2\} \mid e_1, e_2 \in E(G), e_1$ is incident to e_2 in $G\}$. It is not hard to see that a proper edge-coloring of a graph G corresponds to a proper vertex-coloring of $L(G)$ and vice versa.

The subgraph of a graph G induced by a vertex set $U \subseteq V(G)$ is denoted throughout by $G[U]$. Vertices and edges in $G[U]$ are refered to as U*-vertices* and U*-edges*, respectively. Neighbors of a vertex $v \in V(G)$ in the induced subgraph $G[U]$ are called U*-neighbors* of v and connected components in an induced subgraph $G[U]$ are called U*-components*. Similarly, $G - U$ denotes the induced subgraph $G[V(G) \setminus U]$. And for a subset $T \subseteq E(G)$, $G - T$ denotes the graph obtained from G by deleting the edges of T.

A *spanning* subgraph H of a graph G is a graph $H \subseteq G$ with $V(H) =$

$V(G)$. An *f-factor* of G is a spanning subgraph F of G such that for every vertex v in F, $d_F(v) \in f$. We say that a graph G can be *factored* into the factors $F_1, \ldots, F_l$ if there is a partition $E_1, E_2, \ldots, E_j$ of $E(G)$ such that G restricted to the edges of E_i forms a factor from the family $F_1, \ldots, F_l$. For a factor F of a graph G we define $G - F$ to be the graph G with all edges of F removed.

Part I

Bounded Component Transversals

...Weisste Bescheid...

Horst Schlämmer

Chapter 1

About f-Bc Transversals

Let G be a multipartite graph with $V(G) = V_1 \cup \cdots \cup V_m$. A *transversal* T of G is a subset of the vertices in G containing exactly one vertex from each partite set V_i. An *f-bc transversal* of a graph with a vertex partition is a transversal T in which each connected component of the subgraph induced by T (the transversal graph) has at most f vertices. Let $\Delta_f(m, n)$ denote the smallest integer Δ such that there exists an m-partite graph G with maximum degree Δ and parts of size n and with no f-bc transversal. We define $\Delta_f(n) = \min_{m \in \mathbb{N}} \Delta_f(m, n)$ in case we do not want to restrict the number of parts in the graphs under consideration and let Δ_f denote $\lim_{n\to\infty} \Delta_f(n)/n$. It is not hard to check that this limit always exists.

A formal statement of the Transversal Theorem follows.

Theorem 1.1 (Transversal Theorem). *Let $f \geq 1$ be fixed. Let G be a multipartite graph with each partite set containing exactly n vertices and $\Delta(G) \leq \lfloor (\frac{f}{f+1})n \rfloor$. Then G has an f-bc transversal, and therefore $\Delta_f(n) > (\frac{f}{f+1})n$.*

For a graph G and a set $U \subseteq V(G)$ of vertices, we say that U *dominates* G if for every $v \in V(G)$ there exists $u \in U$ such that $\{u, v\} \in E(G)$. Note that even the vertices $v \in U$ have to be dominated. In order to prove Theorem 1.1 we first show the following stronger theorem.

Theorem 1.2. *Let G be a graph, and suppose $V_1 \cup \ldots \cup V_m$ is a partition of $V(G)$ into m independent vertex classes. Suppose G has no f-bc transversal, but the graph $G_1 = G[V_2 \cup \ldots \cup V_m]$ has an f-bc transversal.*

Then there exists a subset $S \subseteq \{V_1, \ldots, V_m\}$ and a subset $Z \subseteq \bigcup_{V_i \in S} V_i$ such that

(*i*) $V_1 \in S$,

(*ii*) *Z dominates $G_S = G[\bigcup_{V_i \in S} V_i]$,*

(*iii*) $|Z| \leq \lfloor (\frac{f+1}{f})(|S|-1) \rfloor$, *and*

(*iv*) *all components of $G[Z]$ have at least $f+1$ vertices.*

To see that Theorem 1.2 immediately implies Theorem 1.1, we assume on contrary that the graph G as in Theorem 1.1 does not contain an f-bc transversal. Suppose first that G_1 does contain an f-bc transversal. Therefore according to Theorem 1.2 there exists the subset S and the subset Z such that conclusions (i)-(iv) hold. Observe here that the number of vertices in a graph of maximum degree Δ that can be dominated by a set of size at most $\lfloor (\frac{f+1}{f})(|S|-1) \rfloor$, is at most $\lfloor (\frac{f+1}{f})(|S|-1) \rfloor \Delta$. Since G_S contains $n|S| \geq (\frac{f+1}{f})\Delta|S| > \lfloor (\frac{f+1}{f})(|S|-1) \rfloor \Delta$ vertices, not both, conclusions (ii) and (iii) of Theorem 1.1 can hold in G. Hence in this case G must have an f-bc transversal - a contradiction. On the other hand if G_1 does not contain an f-bc transversal, then we repeat the above argument with G replaced by G_1.

Proof of Theorem 1.2. We prove Theorem 1.2 by induction on m. Let G be as in the statement of the theorem. The assertion of the theorem is trivially true when $m = f$, so assume $m \geq f+1$ and that the statement is true for smaller values of m.

Choose an f-bc transversal T of G_1. Then every vertex v of V_1 has the property that the component C_v of $T \cup \{v\}$ containing v has at least $f+1$ vertices. Amongst all choices of v and T, we take those which minimize the order of the component C_v. We form a new graph H by

- removing the vertex set $W = N_G(C_v)$ from G (note $C_v \subset W$),
- unifying the remaining vertices in $\cup_{V_j \cap V(C_v) \neq \emptyset} V_j$ into one new vertex class Y^* (and removing any edges inside Y^*).

Each other class V_i just becomes $Y_i = V_i \setminus W$ in H. Note that each class apart from possibly Y^* is nonempty because it still contains an element of T, and indeed the remainder of T forms an f-bc transversal of all classes of H except Y^*.

Case 1: $Y^* = \emptyset$.

In this case set $S = \cup_{V_j \cap C_v \neq \emptyset} V_j$ and $Z = C_v$. Then, by definition of C_v with respect to T and v, Z dominates all of G_S as required. Moreover since C_v contains exactly one vertex from each class in S, and C_v has at least $f+1$ vertices, we have $|Z| = |S| \leq \lfloor (\frac{f+1}{f})(|S|-1) \rfloor$.

Case 2: $Y^* \neq \emptyset$.

First we verify that H does not have an f-bc transversal. Suppose on the contrary that T' is an f-bc transversal for H. Let z be the vertex of T' in Y^*. Then by definition of Y^*, in G we have $z \in V_j$ for some V_j with $V_j \cap C_v \neq \emptyset$. By definition of H, there are no edges joining any vertex of T' (including z) to any vertex of C_v. Thus if $z \in V_1$ we find that $T' \cup (C_v \setminus \{v\})$ is an f-bc transversal of G, which is a contradiction. If $z \in V_j$ for some $j \neq 1$, let w be the vertex of C_v in V_j. Then $T^* = T' \cup (C_v \setminus \{v, w\})$ is an f-bc transversal of G_1 with the property that the component of $T^* \cup \{v\}$ containing v is smaller than C_v. This contradicts our choice of T. We conclude that H has no f-bc transversal.

Let t denote the number of vertex classes that intersect C_v in G. Then $t \geq f+1$. Since H has $m-t+1 < m$ vertex classes, by the induction hypothesis applied to H and the class Y^*, there exists a set S' of vertex classes containing Y^* together with a set of vertices Z' of $H_{S'}$ that satisfies the conclusions (i)–(iv). We set $S = S' \setminus \{Y^*\} \cup \{V_j : V_j \cap C_v \neq \emptyset\}$ and $Z = Z' \cup C_v$. Then (i) holds. To check (ii), note that every vertex of G_S that was in H is dominated by a vertex of Z', and all the remaining vertices of G_S are dominated by C_v. For (iii) we have $|S| = |S'|-1+t$ and $|Z| = |Z'|+t \leq \lfloor (\frac{f+1}{f})(|S'|-1) \rfloor + t \leq \lfloor (\frac{f+1}{f})(|S'|-2+t) \rfloor$ since $t \leq \lfloor (\frac{f+1}{f})(t-1) \rfloor$. Therefore $|Z| \leq \lfloor (\frac{f+1}{f})(|S|-1) \rfloor$ as required. Finally for (iv) note that since C_v has at least $f+1$ vertices, each component of $G[Z]$ has at least $f+1$ vertices. □

Szabó and Tardos ([60]) construct graphs showing the following upper bound on $\Delta_f(n)$.

Proposition 1.1 ([60]). *For every three integers* $n \geq 1, f \geq 2$ *and* $m \geq \sum_{i=0}^{f} n^i$, $\Delta_f(m, n) \leq n+1$.

Proof. Let G be the disjoint union of n copies of an n-ary tree of depth f together with a partition of $V(G)$ as follows. For a vertex $v \in V(G)$

which is not a leaf of G, define the partite set V_v to contain every child of v. For the root vertices we define a part V_{root} including all n root vertices.

Obviously $\Delta(G) = n + 1$ and $|V_v| = n$ for every vertex v that is not a leaf of G.

It is not hard to see that in every transversal of the multipartite graph G with partite sets V_v, there is a path from one of the n root vertices to a leaf of the same tree. Hence G does not contain an f-bc transversal. □

We conclude according to Theorem 1.1 and Proposition 1.1 that

$$\Big(\frac{f}{f+1}\Big)n < \Delta_f(n) \leq n + 1,$$

for every two integers $f, n \geq 1$.

1.1 Independent Transversals

Historically the investigation of $\Delta_f(m,n)$ started with 1-bc transversals, subsequently called independent transversals. Independent transversals and in particular the determination of the number $\Delta_1(m,n)$, for $m \geq 2$ and $n \geq 1$ received a lot of attention. In a series of works ([22, 6, 8, 44, 35, 36, 67, 2, 9, 60, 38]) $\Delta_1(m,n)$ has been completely determined.

Theorem 1.3. *For integers $m \geq 2$ and $n \geq 1$ the following holds,*

$$\Delta_1(m,n) = \begin{cases} \lceil \frac{(m-1)n}{2(m-2)} \rceil, & \text{if } m \text{ is odd, and} \\ \lceil \frac{mn}{2(m-1)} \rceil, & \text{if } m \text{ is even.} \end{cases}$$

The upper bound for all m and n has been proved by Szabó and Tardos for even m in [60] and in [38] is has been shown that this construction is also optimal for odd m (adding one more partite set). We give here a simplified construction showing $n/2 < \Delta_1(m,n) \leq \lceil n/2 \rceil + 1$ for $m - 2 \geq N$ only. This yields a tight bound if n is even. Let G be the graph in Figure 1.1 (a vertex represents a whole class of vertices). That is, G is an r-partite graph with partite sets of size n, $m > n$ and each partite set $V_i, i \in [m-2]$ is partitioned into two almost equally sized classes $V_{i,1}$ and $V_{i,2}$ (i.e., $|V_{i,1}| = \lfloor n/2 \rfloor$ and $|V_{i,2}| = \lceil n/2 \rceil$). Let every vertex in part $V_{i,2}$ be adjacent to every vertex in $V_{i+1,1}$, with

$i \in [m-3]$ and let every vertex in $V_{m-2,2}$ be adjacent to every vertex in $V_{1,1}$. We assume that the two parts V_{m-1} and V_m each contain $n \leq m-2$ vertices $v_{m-1,1}, v_{m-1,2}, \ldots, v_{m-1,n}$ and $v_{m,1}, v_{m,2}, \ldots, v_{m,n}$, respectively. Further let every vertex of $V_{i,j}$ be adjacent to $v_{m+1-j,i}$, $i \in [n]$ and $j \in \{1,2\}$.

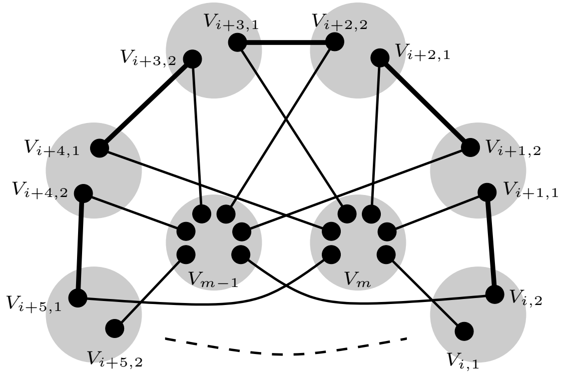


Figure 1.1: An m-partite graph G without an independent transversal.

Proposition 1.2. *For the graph G in Figure 1.1 it holds that $\Delta(G) = \lceil n/2 \rceil + 1$ and G does not contain an independent transversal.*

Proof. It is not hard to see that for every vertex $v \in V_{m-1} \cup V_r$, $d(v) \leq \lceil n/2 \rceil$. For a vertex $v \in \bigcup_{i=1}^{m-2} V_i$, $d(v) \leq \lceil n/2 \rceil + 1$.

Suppose for a moment that G contains an independent transversal T. Hence either $T \cap (\bigcup_{i=1}^{m-2} V_{i,1}) = \emptyset$ or $T \cap (\bigcup_{i=1}^{m-2} V_{i,2}) = \emptyset$. Without loss of generality assume the latter case. Thus there is an $j \in [n]$ such that the vertex $T \cap V_m$ is adjacent to the vertex $T \cap V_{j,1}$. □

1.2 Matching Transversals

Let us finally restrict to 2-bc transversals which we subsequently want to call a matching transversals. According to Theorem 1.1 every multipartite graph G with partite sets of order n and $\Delta(G) \leq \lfloor 2n/3 \rfloor$ contains a matching transversal. On the other hand according to Proposition 1.1 there are multipartite graphs with partite sets of order n and

$\Delta(G) = n + 1$ without a matching transversal. We think that these graphs are in some sense optimal.

Conjecture 1.1. *Let G be a multipartite graph with partite sets containing at least n vertices. If $\Delta(G) \leq n$, then G contains a matching transversal.*

The following theorem by Haxell, Szabó, and Tardos [39] turns out to be very useful later on.

Theorem 1.4 ([39]). *Every multipartite graph G with partite sets of order at least two and with maximum degree $\Delta(G) \leq 2$ contains a matching transversal T, i.e., $\Delta_2(2) = 2$. Moreover there is a linear-time (in the order of G) algorithm that finds T.*

We will investigate $\Delta_f(r, n)$ for matching transversals ($f = 2$). We can construct graphs such that the following holds.

Proposition 1.3. *For any two integers $m \geq 3, n \geq 1$,*

$$\Delta_2(m,n) \leq \begin{cases} n + \lceil \frac{n}{m-1} \rceil, & \text{if } m \text{ is odd,} \\ n + \lceil \frac{n}{m-2} \rceil, & \text{if } m \text{ is even.} \end{cases}$$

Proof. Assume first that m is odd. Let G be an m-partite graph such that $G[V_{2i} \cup V_{2i+1}]$ is isomorphic to $K_{n,n}$, for $1 \leq i \leq (m-1)/2$. Partition the part V_m into $m-1$ almost equally sized parts $V_{m,i}$, with $\lfloor n/(m-1) \rfloor \leq |V_{m,i}| \leq \lceil n/(m-1) \rceil$, for $i \in [m-1]$ and connect every vertex in V_j with every vertex in $V_{m,j}$, see Figure 1.2. Hence $\Delta(G) = n + \lceil \frac{n}{m-1} \rceil$. In case m is even, then simply add another part to G with n isolated vertices.

Obviously in every transversal T, the two vertices in $T \cap V_{2i}$ and $T \cap V_{2i+1}$ form an edge in G. Hence every vertex of V_m has a neighbor in an edge of $G[T]$ and thus G does not contain a matching transversal. □

Let us note here that Proposition 1.3 improves Proposition 1.1 for matching transversals in the following sense. For integers $n > 0$ and $m > n$ Proposition 1.3 shows the existence of m-partite graphs with parts of size n and maximum degree $\Delta(G) = n + 1$ without a matching transversal, while Proposition 1.1 requires $m \geq \sum_{i=0}^{2} n^i$.

Improving on the lower bound $\Delta_2(n) > 2n/3$ seems to be hard in general. In the remainder of this chapter we investigate 3-partite and 4-partite graphs.

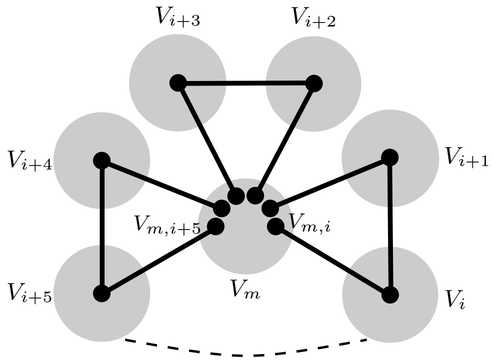


Figure 1.2: An m-partite graph with no matching transversal.

Lemma 1.1. $\Delta_2(3, n) = \lceil 3n/2 \rceil$.

Proof. Proposition 1.3 with $m = 3$ yields $\Delta_2(3, n) \leq \lceil 3n/2 \rceil$. For the lower bound we first observe that for every vertex $v \in V_i, i \in [3]$ there is a $j \in [3] \setminus \{i\}$, such that $d_{V_i}(v) = n$. For that suppose there is a vertex $v \in V_i$ with $d_{V_j}(v) < n$ and $d_{V_k}(v) < n$, where $\{i, j, k\} = [3]$. Hence there is a non-neighbor v_j of v in V_j and a non-neighbor v_k of v in V_k. Then the three vertices v, v_i and v_j form a matching transversal.

Hence we can define the following refined partition of G, see also Figure 1.3. For $\{i, m\} \subset [3]$ let

$$V_{i,m} = \{v \in V_i \mid d_{V_m}(v) = n\}.$$

Since no vertex has degree n into two other classes, this is really a partition. For a vertex $v \in V_{i,j}$ it holds that $N(v) \supseteq V_j \cup V_{k,i}$. Obviously there is a part V_k with $V_{k,i} \neq \emptyset \neq V_{k,j}$. Hence without loss of generality $|V_{k,i}| \geq \lceil n/2 \rceil$, and the proof is concluded. □

Now we turn our attention to $\Delta_2(4, n)$. We define $\mathcal{G}_4(n)$ to be the family of 4-partite graphs with partite sets containing exactly n vertices. We can subsequently assume for graphs $G \in \mathcal{G}_4(n)$ that $\Delta(G) \leq \Delta_2(3, n) = \lceil 3n/2 \rceil$.

Theorem 1.5. *Every graph $G \in \mathcal{G}_4(n)$ with $\Delta(G) < \frac{4n}{3}$ has a matching transversal. On the other hand for every n divisible by 7 there are graphs*

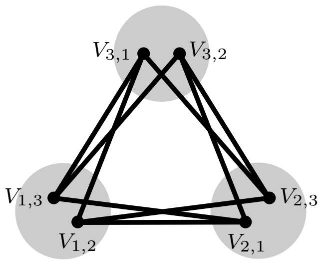


Figure 1.3: 3-partite graph G with refined partite sets $V_{i,j} \subseteq V_i$, $i \in [3], j \in [2]$

$G \in \mathcal{G}_4(n)$ with $\Delta(G) = \frac{10n}{7}$ and no matching transversal. Therefore $\lceil 4n/3 \rceil \le \Delta_2(4,n) \le \lfloor 10n/7 \rfloor + 9$.

Proof. For a given graph G, we say that a pair of vertices u, v *dominates* a vertex set U if and only if every vertex of U is adjacent to u or v.

Lemma 1.2. *A graph $G \in \mathcal{G}_4(n)$ contains* no *matching transversal if and only if every pair of vertices v_i, v_j from distinct partite sets V_i and V_j of G dominates at least one of the other partite sets V_k with $k \in [4] \setminus \{i, j\}$.*

Proof. Suppose there is a pair $v_1 \in V_1$, $v_2 \in V_2$ that dominates neither V_3 nor V_4. Hence there is a vertex $v_3 \in V_3$ and a vertex $v_4 \in V_4$, both not adjacent to v_1 and v_2. We immediately conclude that v_1, v_2, v_3 and v_4 form a matching transversal in G.

On the other hand suppose that G contains a matching transversal $T = \{v_1', v_2', v_3', v_4'\}$. Obviously the complement of $G[T]$ contains a 4-cycle as a subgraph, say v_1', v_3', v_2', v_4'. then the pair v_1', v_2' dominates neither V_3 nor V_4 (and the pair v_3', v_4' dominates neither V_1 nor V_2). □

The *crossing degree sum* $d^\times(v_i, v_j)$ of two vertices v_i, v_j from distinct parts V_i and V_j is defined as $d_{V_i}(v_j) + d_{V_j}(v_i)$.

Corollary 1.1. *If there are vertices v_i and v_j from distinct parts of a graph $G \in \mathcal{G}_4(n)$ with $\Delta(G) < 4n/3$ such that*

(i) $d^\times(v_i, v_j) > 2\Delta(G) - n$, or

(ii) $d^{\times}(v_i, v_j) = 2n$,
then G has a matching transversal.

Proof. (i) The two vertices v_i and v_j have together less than n neighboring vertices outside of $V_i \cup V_j$. Hence they do not dominate any other part of G. According to Lemma 1.2, G contains a matching transversal. (ii) follows immediately from (i) and $\Delta(G) < 4n/3$. □

Let $G \in \mathcal{G}_4(n)$ with no matching transversal. We define a directed graph $K(G)$ on the vertex set $V(K(G)) = \{V_1, V_2, V_3, V_4\}$ by putting an arc (V_i, V_j) in $K(G)$ if there exists a vertex $v_i \in V_i$ with $d_{V_j}(v_i) = n$. From Corollary 1.1(ii) we conclude that for every two parts V_i and V_j with $\{i, j\} \subseteq [4]$ not both arcs (V_i, V_j) and (V_j, V_i) exist.

Lemma 1.3. *Let $G \in \mathcal{G}_4(n)$ be a graph with no matching transversal, and let α be a constant such that $\Delta(G) < (1 + \alpha)n$. Suppose V_i is a vertex of outdegree at most 1 in $K(G)$. Let V_j be the outneighbor of V_i if it exists, otherwise let V_j be an arbitrary class different from V_i. Let V_k and V_l be the other two class es different from V_i and V_j. Then one of the following holds.*

(i) V_j contains a vertex y with $d_{V_i}(y) \geq \min\{n, (2 - 3\alpha)n\}$,

(ii) (V_l, V_k) is an arc of $K(G)$,

(iii) (V_k, V_j) is an arc of $K(G)$,

(iv) (V_k, V_l) is an arc of $K(G)$,

(v) (V_l, V_j) is an arc of $K(G)$.

Moreover, if none of (i), (ii) and (iii) hold then $\Delta(G) \geq \lceil 4n/3 \rceil$, and if none of (i), (iv) and (v) hold then $\Delta(G) \geq \lceil 4n/3 \rceil$.

Proof. Let $v \in V_j$ be a vertex with largest degree into V_i. If $d_{V_i}(v) = n$ then (i) holds. Thus we may assume that v has a non-neighbor x in V_i. Since there is no arc from V_i to V_k or V_l, we can find vertices $w \in V_k$ and $z \in V_l$ that are not adjacent to x. Then by Lemma 1.2, we see that v and w dominate V_l, w and z dominate V_j, and v and z dominate V_k. Thus

$$d(v) + d(w) + d(z) \geq 3n + d_{V_i}(v) + d_{V_i}(w) + d_{V_i}(z).$$

Let us first look at the vertex z. If $d_{V_k}(z) = n$ then (ii) holds. Thus we may assume z has a non-neighbor $u \in V_k$. If $d_{V_j}(u) = n$ then (iii) holds, so we may assume u has a non-neighbor $t \in V_j$. Then z and t

have a common non-neighbor in V_k, so by Lemma 1.2 we must have that z and t dominate V_i. By choice of v we conclude that $d_{V_i}(z) \geq n - d_{V_i}(t) \geq n - d_{V_i}(v)$. Thus in particular if none of (i), (ii) and (iii) hold then

$$3\Delta(G) \geq d(v) + d(w) + d(z) \geq 3n + n + d_{V_i}(w).$$

This implies $3\Delta(G) \geq 4n$ and therefore $\Delta(G) \geq \lceil 4n/3 \rceil$.

The same argument applied to w instead of z tells us that if (iv) and (v) fail to hold then $d_{V_i}(w) \geq n - d_{V_i}(v)$, so if none of (i), (iv) and (v) hold then $\Delta(G) \geq \lceil 4n/3 \rceil$. In addition if all of (ii), (iii), (iv), and (v) fail we have

$$3(1+\alpha)n > 3\Delta(G) \geq d(v) + d(w) + d(z) \geq 5n - d_{V_i}(v).$$

This implies $d_{V_i}(v) > (2 - 3\alpha)n$, so (i) holds. □

Lemma 1.4. *Let $G \in \mathcal{G}_4(n)$ with $\Delta(G) < (1+\alpha)n$, and suppose $K(G)$ has at most one arc. If G has no matching transversal then $\alpha > 3/8$.*

Proof. First suppose $K(G)$ has no arcs. By Lemma 1.3 applied with $V_i = V_2$ and $V_j = V_1$, we find that V_1 contains a vertex y with $d_{V_2}(y) \geq (2 - 3\alpha)n$. By Lemma 1.3 applied with $V_j = V_2$ and $V_i = V_1$, we find that V_2 contains a vertex z with $d_{V_1}(z) \geq (2-3\alpha)n$. Thus we have that the crossing degree sum $d^{\times}(y,z) \geq (4-6\alpha)n$. Corollary 1.1 then implies $(4 - 6\alpha)n \leq 2\Delta - n < 2(1+\alpha)n - n$ which implies $4 - 6\alpha < 1 + 2\alpha$, in other words $\alpha > 3/8$.

If (V_1, V_2) is the unique arc of $K(G)$ then Lemma 1.3 applied with $V_i = V_1$ and $V_j = V_2$ gives us a vertex $z \in V_2$ with $d_{V_1}(z) \geq (2 - 3\alpha)n$. Together with a vertex $y \in V_1$ with $d_{V_2}(y) = n$ we find a pair with crossing degree sum $d(y,z) \geq (3-3\alpha)n$. Thus $(3-3\alpha)n \leq 2(1+\alpha)n - n$ which tells us that $\alpha > 2/5 > 3/8$. □

Theorem 1.5 follows from three lemmas.

Lemma 1.5. *Let $G \in \mathcal{G}_4(n)$ be a graph with no matching transversal, and $\Delta(G) < 4n/3$. Then it holds true that*

(i) $\Delta^+(K(G)) = 3$, or

(ii) $\Delta^-(K(G)) \geq 2$.

Proof. Suppose on the contrary that there is a graph $G \in \mathcal{G}_4$, $\Delta(G) < 4n/3$, with no matching transversal, $\Delta^+(K(G)) \leq 2$ and $\Delta^-(K(G)) \leq 1$. We choose G from all graphs in $\mathcal{G}_4(n)$ subject to these conditions such that the number of edges in $K(G)$ is maximal. According to Lemma 1.4, $K(G)$ contains at least one arc. Without loss of generality let V_1 denote a vertex of $K(G)$ with maximum outdegree.

If $d^+(V_1) = 1$, then let V_2 be the vertex adjacent to V_1. Due to Lemma 1.3 with $V_i = V_1, V_j = V_2$, and the fact that none of the cases (i), (iii) and (v) applies, the edge (V_3, V_4) and the edge (V_4, V_3) exist in $K(G)$. A contradiction according to Corollary 1.1(ii).

If $d^+(V_1) = 2$, then let V_2 be the vertex in $K(G)$ not adjacent to V_1. Note first that the arcs (V_2, V_3) and (V_2, V_4) are not contained in $K(G)$, thus we apply Lemma 1.3 with $V_i = V_2$ and $V_j = V_1$. If either of the cases $(ii) - (v)$ occurs, then $\Delta^-(K(G)) \geq 2$. Hence case (i) occurs. We conclude that there is an arc from V_1 to V_2, again a contradiction. □

Lemma 1.6. *Let $G \in \mathcal{G}_4(n)$ be a graph with no matching transversal and $\Delta(G) < 4n/3$ such that there is a vertex $V_{j'}$ in $K(G)$ with $d^+_{K(G)}(V_{j'}) = 3$. Then $\Delta^-(K(G)) \geq 2$.*

Proof. Let us assume without loss of generality that $V_{j'} = V_4$ and choose three vertices $v_i \in V_4$ with $d_{V_i}(v_i) = n$, $i \in [3]$. First we observe that $K(G)$ contains no other arcs than (V_4, V_i), $i \in [3]$.

Claim 1.1. *For every $v \in V_i$ $(i \in [3])$ there is a $j \in [3] \setminus \{i\}$ such that $d_{V_j}(v) > 2n/3$.*

Proof. According to Lemma 1.2 the pair v, v_i dominates another part $V_j, j \in [3] \setminus \{i\}$. Due to the fact that $d_{V_i}(v_i) = n$, and thus $d_{V_j}(v) < 4n/3 - n = n/3$, it holds for v that $d_{V_j}(v) > 2n/3$. □

Since we assume $\Delta(G) < 4n/3$, we can classify every vertex $v \in V_i$ according to whether its degree to V_j or to V_k is larger than $2n/3$, for $\{i, j, k\} = [3]$. Hence we obtain a partition of V_i into classes $V_{i,j}$ and $V_{i,k}$ as follows.

$$v \in V_{i,m} (\subseteq V_i) \iff d_{V_m}(v) > 2n/3, \text{ with } m \in [3] \setminus \{i\}.$$

Claim 1.2. *For every vertex $v \in V_1 \cup V_2 \cup V_3$, $n/3 < d_{V_4}(v) < 2n/3$.*

Proof. $d_{V_4}(v) < 2n/3$ immediately follows from Claim 1.1. Since there is no arc in $K(G)$ other than the arcs leaving V_4, for every vertex $u_i \in V_i$ we find a second vertex in $u_j \in V_j$ such that they have a common non-neighbor in V_k. Thus u_i and u_j dominate V_4. According to Claim 1.1, $d_{V_l}(u_j) > 2n/3$, for a $l \in [3] \setminus \{j\}$ and hence $d_{V_4}(u_j) < 2n/3$. Moreover $d_{V_4}(u_i) > n/3$. □

We are now ready to prove the proposition. We observe that $|V_{i,k}| + |V_{j,k}| < n$, for $\{i, j, k\} = [3]$, since otherwise there is a vertex w in the common non-neighborhood of v_i and v_j in V_k with $d_{V_i \cup V_j}(w) \geq n$ (note here that w is adjacent to every vertex of $V_{i,k}$ and of $V_{j,k}$). This fact combined with $d_{V_4}(w) > n/3$ (Claim 1.2) yields to a contradiction. Therefore there is a choice of $\{i^*, j^*, k^*\} = [3]$ such that $|V_{i^*,j^*}| \geq n/2, |V_{j^*,k^*}| \geq n/2$ and $|V_{k^*,i^*}| \geq n/2$. Let us choose three vertices $v_{i^*,j^*} \in V_{i^*,j^*}, v_{j^*,k^*} \in V_{j^*,k^*}$ and $v_{k^*,i^*} \in V_{k^*,i^*}$. Suppose for a moment that any of the three vertices $v_{a,b}$ together with any vertex $w \in V_b$ dominate V_4. For every vertex $w \in V_b$, $N_{V_4}(w) \supseteq V_4 \setminus N_{V_4}(v_{a,b}) \neq \emptyset$ (since $d_{V_4}(v_{a,b}) < 2n/3 < n$). Therefore $d_{V_4}(w) > n/3$. Hence there is a vertex in V_4 that is complete to at least two of the three parts V_1, V_2 and V_3, a contradiction. Thus we can choose $v_{a,b}$ such that there is a vertex $w \in V_b$ for which the pair $v_{a,b}, w$ does not dominate V_4.

$$\begin{aligned} d(v_{a,b}) + d(w) &\geq d_{V_b}(v_{a,b}) + d_{V_4}(v_{a,b}) + d_{V_a}(w) + d_{V_4}(w) + |V_c| \\ &> 2n/3 + n/3 + n/2 + n/3 + n = 17n/6 > 2(4n/3), \end{aligned}$$

with $\{a, b, c\} = [3]$. This contradicts the fact that $\Delta(G) < 4n/3$. □

Lemma 1.7. *Let $G \in \mathcal{G}_4(n)$ be a graph with no matching transversal such that there is a vertex V_i in $K(G)$ with $d^-_{K(G)}(V_i) \geq 2$. Then $\Delta(G) \geq \lceil 4n/3 \rceil$.*

Proof. Assume that $\Delta(G) < 4n/3$ and let without loss of generality $\{(V_1, V_3), (V_2, V_3)\} \in E(K(G))$. Further let $v_i \in V_i$ be a vertex with $d_{V_3}(v_i) = n$, $i \in \{1, 2\}$. We first observe the following:

Claim 1.3. *No pair v_1, w with $w \in V_3$ dominates V_2.*

Proof. Suppose v_1, w dominate V_2. Since $d_{V_3}(v_1) = n$, and therefore $d_{V_2}(v_1) < n/3$ we have that $d_{V_2}(w) > 2n/3$. Hence $d^\times(w, v_2) > 5n/3$, a contradiction to Corollary 1.1 and the fact $\Delta(G) < 4n/3$. □

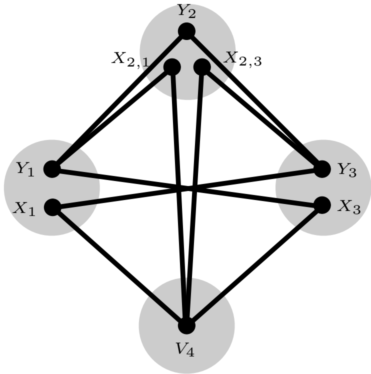


Figure 1.4: The graph G with $\Delta(G) = 10n/7$ that contains no matching transversal.

Claim 1.4. *There is a vertex $w \in V_3$ such that the pair v_1, w does not dominate V_4.*

Proof. Suppose every pair v_1, w with $w \in V_3$ dominates V_4. Again since $d_{V_4}(v_1) < n/3$, the crossing-degree sum of w and any non-neighbor of v_1 in V_4 is larger than $5n/3$, again a contradiction to Corollary 1.1. □

Therefore there is a pair of vertices v_1, w with $w \in V_3$ that do neither dominate V_2 nor V_4, contradicting the fact that G contains no matching transversal (c.f. Lemma 1.2). □

Suppose G contains no matching transversal, then Lemma 1.5 implies the assumptions for Lemma 1.6, Lemma 1.6 implies the assumptions for Lemma 1.7 and finally Lemma 1.7 implies that $\Delta(G) \geq \lceil 4n/3 \rceil$.

For the upper bound we want to show the existence of a graph $G \in \mathcal{G}_4(n)$ with $\Delta(G) = 10n/7$ with no matching transversal if n is divisible by 7. Let us look at the graph G in Figure 1.4. We claim that G contains no matching transversal. For that suppose that the four vertices v_1, v_2, v_3, and v_4 form a matching transversal of G with $v_i \in V_i$, for $i \in [4]$. Since all vertices (including v_4) from V_4 have the same neighborhood we conclude that the following is not possible:

(i) $v_1 \in X_1$ and $v_3 \in X_3$,

(ii) $v_1 \in X_1$ and a vertex from $v_3 \in Y_3$, or

(*iii*) $v_1 \in Y_1$ and a vertex from $v_3 \in X_3$.

Thus $v_1 \in Y_1$ and $v_3 \in Y_3$. Now it is easy to see that any choice for a vertex from V_2 would result in a transversal T with $\Delta(G[T]) > 1$, a contradiction.

We assumed that 7 divides n. For an optimal choice of the order of the vertex classes, for instance $|X_1| = 3n/7, |Y_1| = 4n/7, |X_{2,1}| = n/7, |X_{2,3}| = n/7, |Y_2| = 5n/7, |X_3| = 3n/7, |Y_3| = 4n/7$, we observe that no vertex in the graph G has degree larger than $\Delta(G) = 10n/7$. If 7 does not divide n, then let $n' > n$ denote the smallest integer that is divisible by 7. Let G be the graph as in Figure 1.4 with partite sets containing n' vertices each and set $r = n \mod 7$. Delete from V_4 r many vertices, from X_1 and X_3 $\lfloor r/2 \rfloor$ many vertices, from Y_1 and Y_3 $\lceil r/2 \rceil$ many vertices, from $X_{2,1}$ $\lfloor r/4 \rfloor$, from $X_{2,3}$ $\lceil r/4 \rceil$, and from Y_2 $\lceil r/2 \rceil$ many vertices, resulting in a new graph G'. Now it is not hard to check that $\Delta(G') \leq \lfloor 10n/7 \rfloor + 9$. □

The graph G has the special property that for every pair of vertices u and v from V_4, $N(u) = N(v)$. For graphs $G \in \mathcal{G}_4(n)$ with this special property the following strengthening of Theorem 1.5 is true.

Lemma 1.8. *Every graph $G \in \mathcal{G}_4(n)$ with $N(v) = N(u)$ for all $u, v \in V_i$, for some $i \in [4]$ and $\Delta(G) < 10n/7$ contains a matching transversal.*

Proof. For a vertex $v \in V_i$ we define $X_j(v) = N_{V_j}(v)$ and $Y_j(v) = V_j \setminus X_j(v)$, for $j \in [4] \setminus \{i\}$. Further we define $X_{j,k}(v) \subseteq X_j(v)$ and $Y_{j,k}(v) \subseteq Y_j(v)$ as follows:

$$X_{j,k}(v) = \bigcap_{u \in Y_k(v)} N_{X_j(v)}(u) \quad \text{and} \quad Y_{j,k}(v) = \bigcap_{u \in Y_k(v)} N_{Y_j(v)}(u).$$

Let us now fix $i = 4$. We define $X(v) = \bigcup_{i=1}^{3} X_i(v)$ and $Y(v) = \bigcup_{i=1}^{3} Y_i(v)$. (In case it is obvious from the context, we might skip the parameter v.) Let in the following W either stand for X or Y. Note here that $W_{i,j}(v)$ and $W_{i,k}(v)$ for $j \neq k$ are not necessarily disjoint. Let us denote the cardinality of these sets by the corresponding small letters, e.g. $w_{i,j}(v) = |W_{i,j}(v)|$.

Proposition 1.4. *Let $G \in \mathcal{G}_4(n)$ be a graph with no matching transversal, then*

(*i*) $X_{i,j} \cup X_{i,k} = X_i$, *and*

(*ii*) $Y_{i,j} \cup Y_{i,k} = Y_i$,
for all i, j, k *such that* $\{i, j, k\} = [3]$.

Proof. Suppose that $W_{i,j} \cup W_{i,k} \subset W_i$ (with again W attaining either X or Y). There is a vertex $v_j \in Y_j$ and a vertex $v_k \in Y_k$ that do neither dominate V_i nor V_4. Hence by Lemma 1.2 $\{v, v_i, v_j, v_k\}$ forms a matching transversal in G. $\square$

Fix a vertex $v \in V_4$. According to Corollary 1.1(*ii*) $d_{V_i}(v) < n$, for all $i \in [3]$. Due to Proposition 1.4 we know that the partite sets V_i, $i \in [3]$ of G are dominated by classes $W_{i,j}$. For each of the $W_{i,j}, W_{i,k}$ pairs at least one of them has to be non-empty.

Choose $\{i, j\} \subset [3]$. According to the definition of the vertex classes the following holds:

$$d(u) \geq \begin{cases} |V_4| + y_j & \text{if } u \in X_{i,j}, \text{ and} \\ y_j + x_{j,i} + y_{k,i} + x_{k,i} & \text{if } u \in Y_{i,j}. \end{cases}$$

Hence for each vertex class we can lower bound the degree of the vertices contained in it in terms of the order of other vertex classes. These inequalities yield a set of constraints, such that we can define a linear programm that optimizes for the smallest maximum degree. To do so we also have to add the constraints that $x_i + y_i = n$, $w_{i,j} + w_{i,k} \geq w_i$, and the inequalities $\Delta(G) \geq d(w_{i,j})$ and finally the objective $\Delta(G)$. Since some of the sets $W_{i,j}$ could be empty and thus could not contain any single vertex, we minimize over total of at most 3^6 possibilities of the 6 pairs $X_{i,j}, X_{i,k}$ and $Y_{i,j}, Y_{i,k}$ being empty or not, and the corresponding linear programs. For each possibility we solve the corresponding linear program (with the inequalities removed that correspond to empty sets) such that $\Delta(G)$ gets minimized. The MAPLE code solving these linear programs can be found in the Appendix.

We verified that $\Delta(G) = 10n/7$ is the optimal value and a solution consists of the values $x_{1,2} = 0, x_{1,3} = 4n/7, y_{1,2} = 3n/7, y_{1,3} = 0$, and $x_{2,1} = n/7, x_{2,3} = n/7, y_{2,1} = 5n/7, y_{2,3} = 5n/7$, and $x_{3,2} = 0, x_{3,1} = 4n/7, y_{3,1} = 0, y_{3,2} = 3n/7$ (compare also Figure 1.4).

$\square$

I'm Not Normally a Praying Man,
But If You're Up There,
Please Save Me, Superman!

Homer Simpson

Chapter 2

Algorithmic Aspects

There seems to be no greedy-like algorithm for finding bounded transversals of multipartite graphs similar to the one for (list-)colorings. Also the proof of Theorem 1.1 shown in Chapter 1 resisted attempts to turn it into a polynomial-time algorithm for finding a bounded transversal of a multipartite graph. Similarly, the original proof of Theorem 1.1 by Haxell, Szabó, and Tardos in [39] and the topological proof by Szabó and Tardos in [60] are purely existential.

Throughout this chapter we restrict ourselves to independent transversals. We call the running-time of an algorithm with input graph G polynomial if it is polynomial in the order of G, assuming that $\Delta(G)$ is constant. Let G be a multipartite graph with partite sets containing at least n vertices. The goal of this chapter is to derive a polynomial-time algorithm that finds an independent transversal of G. In order to do so we are forced to strengthen the condition on the partite set sizes from $n \geq 2\Delta(G)$ (for which G is guaranteed to contain an independent transversal according to Theorem 1.1) to $n = \Omega(\Delta(G)^3)$. Note here that Alon [8] mentions that a deterministic polynomial-time algorithm exists that finds an independent transversal of G even if only $n \geq C\Delta(G)$ holds, C being a large constant. We believe that in a future work our results can be combined with the techniques of Alon to obtain an algorithm that finds an independent transversal in every multipartite graph G with parts containing at least $C'\Delta(G)$ many vertices, for some constant $C' \geq 2$ that is much smaller than the constant C implicitly given in [8].

In Section 2.1 we start with a simple application of the Lovász Local Lemma in order to show the existence of independent transversals in certain multipartite graphs. Then we argue that unfortunately an algorithmic variant of the Lovász Local Lemma (which would lead to an algorithm as wanted) cannot be directly applied for finding independent transversals.

In Section 2.2 we are finally going to present the simple deterministic algorithm that efficiently finds an independent transversal for every multipartite graph G with each partite set of G containing at least $20\Delta(G)^3$ many vertices.

2.1 No Algorithmic Local Lemma

We start by stating the famous Lovász Local Lemma.

Lemma 2.1 (Lovász Local Lemma [29]). *Let $A_1, \ldots, A_k$ be events in an arbitrary probability space. Suppose each A_i is mutually independent of all but at most d other events A_j and suppose the probability of each A_i is at most p. If $ep(d+1) < 1$ then with positive probability none of the events A_i hold.*

A *random transversal* of an m-partite graph G is defined to be a transversal $T = \{t_1, \ldots, t_m\}$ of G where each t_i is chosen uniformly and independently at random from V_i, for $i \in \{1, \ldots, m\}$.

Let T be a random transversal of an m-partite graph G with partite sets of size $n > 2e\Delta(G)$. For an edge $e = \{u, v\} \in E(G)$ we call A_e the event that $u \in T$ and $v \in T$. Obviously $\Pr(A_e) = 1/n^2$, and $p = \max_{e \in E(G)} \Pr(A_e) = 1/n^2$. Let $u \in V_i$ and $v \in V_j$. It is not hard to see that an event $A_{\{u,v\}}$ is mutually independent of all events $A_{\{u',v'\}}$ with $\{u', v'\} \cap (V_i \cup V_j) = \emptyset$, in particular $A_{\{u,v\}}$ depends on at most $d = 2n\Delta(G)$ many other such events.

Lemma 2.1 asserts that if $ep(d+1) < 1$, then there is a choice of elementary events such that none of the events A_e, $e \in E(G)$, occurs and hence there is an independent transversal of G. Indeed for $n > 2e\Delta(G)$, $ep(d+1) < 1$ holds. Hence we obtain a short proof that every graph G with parts of size larger than $2e\Delta(G)$ contains an independent transversal. (This is a slightly weaker results than Theorem 1.1 applied with $f = 1$.)

An algorithmic counterpart of the Lovász Local Lemma was shown to exist if we require stronger conditions by Beck [14] and Alon [7]. In a version by Molloy and Reed [53] among many additional requirements the condition $ep(d+1) < 1$ should be replaced by the new condition $pd^9 < 1/512$. Since in our case $pd^9 = \Omega(n^7\Delta^9) = \omega(1)$ we cannot apply this algorithmic variant of the Lovász Local Lemma from Molloy and Reed [53]. Still some ideas of its proof can be used to derive the algorithm in the next section.

2.2 Finding Independent Transversals

Theorem 2.1. *Let Δ be a constant. There is a deterministic algorithm running in polynomial-time that finds an independent transversal of every multipartite graph G with $\Delta(G) \leq \Delta$ and parts V_i of cardinality $|V_i| \geq 20\Delta^3$.*

In order to prove Theorem 2.1 we are first going to show the following useful lemma.

Lemma 2.2. *Let Δ be a constant. For every m-partite graph G with $\Delta(G) = \Delta$ and parts V_i of cardinality $|V_i| = n \geq 5\Delta$ there exists a $(5\log(5m\Delta))$-bc transversal T of G. Moreover there is a deterministic polynomial-time algorithm that finds T.*

Proof. First we want to bound the number of connected induced subgraphs on r vertices of G. For that we fix an unlabeled tree S on r vertices. Also we want to fix an ordering of the vertices of S such that for every $j = 2, \ldots, r$ the jth vertex v_j of S is adjacent to some vertex v_i of S with $i < j$. Let us now map the r vertices of S, starting with v_1, onto G. There are $|V(G)|$ many choices for v_1. For every subsequent vertex v_j of S we have already specified a vertex of G which maps onto a neighbor of v_j. Hence there remain only at most $\Delta(G)$ many vertices of G to map v_j onto. There are then a total of at most $|V(G)|\Delta^r$ many possibilities to map S onto G.

The total number of distinct unlabeled trees on r vertices is less than 4^r (c.f. Harary and Palmer [34]). Thus the number of trees on r vertices in G, and also the number of connected induced subgraphs on r vertices in G is at most $|V(G)|(4\Delta)^r$. Let us denote the set of all connected induced subgraphs on at most r vertices in G by $\mathcal{S}_r$. As we have seen $|\mathcal{S}_r| \leq |V(G)|(4\Delta)^r = mn(4\Delta)^r$.

We choose a random transversal T. Now we are ready to compute the expected number of components on r vertices contained in T.

$$\begin{aligned} E[|\{S \in \mathcal{S}_r \ : \ S \subset T\}|] &\leq \sum_{S \in \mathcal{S}_r} Pr(S \subseteq T) \\ &= \sum_{S \in \mathcal{S}_r} \prod_{v \in V(S)} Pr(v \in T) \\ &\leq mn(4\Delta)^r \left(\frac{1}{n}\right)^r \\ &\leq 4m\Delta \left(\frac{4}{5}\right)^{r-1}. \end{aligned}$$

We want to show that there is a $(5\log(5m\Delta))$-bc transversal by showing that for $r = (5\log(5m\Delta)) + 1$ the above expectation is less than 1.

$$\begin{aligned} E[|\{S \in \mathcal{S}_{5\log(5m\Delta)+1} \ : \ S \subset T\}|] &\leq 4m\Delta \left(\frac{4}{5}\right)^{5\log(5m\Delta)} \\ &= 4m\Delta (5m\Delta)^{5\log(4/5)} \\ &< \frac{4m\Delta}{5m\Delta} < 1. \end{aligned}$$

Hence there exists a set of vertices $T^* = \{t_1^*, \ldots, t_m^*\}$ such that $G[T^*]$ contains no tree with at least $5\log(5m\Delta) + 1$ vertices. Thus T^* forms a $(5\log(5m\Delta))$-bc transversal.

Let us now proceed by showing how to find T^* deterministically. The algorithm proceeds as follows: After having determined $T_i^* = \{t_1^*, \ldots, t_i^*\}$, that is, a partial transversal of the first i parts $V_1, \ldots, V_i$ of G, we choose $t_{i+1}^* \in V_{i+1}$ as follows:
Let $T = \{t_1, \ldots, t_m\}$ be such that t_j is chosen uniformly at random from V_j, for $j \in [m]$. For each vertex $v \in V_{i+1}$ we compute $E[|\{S \in \mathcal{S}_r \ : \ S \subset T_i^* \cup \{v\} \cup \{t_{i+2}, \ldots, t_m\}|]$, i.e., the expected number of connected trees on $r = 5\log(5m\Delta) + 1$ vertices in $T_i^* \cup \{v\} \cup \{t_{i+2}, \ldots, t_m\}$. Then we choose a vertex t_{i+1}^* for which this expectation is minimized. Thus,

$$\begin{aligned} &E[|\{S \in \mathcal{S}_r \ : \ S \subset \{t_1^*, \ldots, t_i^*, t_{i+1}^*, t_{i+2}, \ldots, t_m\}\}|] \\ &\qquad \leq E[|\{S \in \mathcal{S}_r \ : \ S \subset \{t_1^*, \ldots, t_i^*, t_{i+1}, t_{i+2}, \ldots, t_m\}\}|]. \end{aligned}$$

Therefore after having chosen t_m^* it holds that

$$|\{S \in \mathcal{S}_r \ : \ S \subset T^*\}| \leq E[|\{S \in \mathcal{S}_r \ : \ S \subset T\}|] < 1.$$

It only remains to show that the computation of $E[|\{S \in \mathcal{S}_r \ : \ S \subset \{t_1^*, \ldots, t_i^*, v, t_{i+2}, \ldots, t_m\}|]$ can be carried out fast. For a fixed tree $S \in \mathcal{S}_r$ it holds that $Pr[S \subset \{t_1^*, \ldots, t_i^*, v, t_{i+2}, \ldots, t_m\}] = \left(\frac{1}{n}\right)^{m-i-2}$, if $\{t_1^*, \ldots, t_i^*, v\} \subseteq S$ and 0 otherwise. Moreover we know that $|\mathcal{S}_r| \leq mn(4\Delta)^{5\log(5m\Delta)} = mn(5m\Delta)^{5\log(4\Delta)}$ – a polynomial in n and m. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $l = 10\Delta^2$ and let G be a graph with $\Delta(G) \leq \Delta$, $V(G) = V_1 \cup \cdots \cup V_m$, and $|V_i| \geq l2\Delta$, for $i \in \{1, \ldots, m\}$.

We partition an arbitrary subset U_i of each partite set V_i, with $|U_i| = l2\Delta$ into $|U_i|/(2\Delta) = l$ many parts $U_{i,j}$, $j \in \{1, \ldots, l\}$, each containing 2Δ many vertices. Also we define $U_i^* = \{U_{i,j} \mid j \in \{1, \ldots, l\}\}$.

Further we define another m-partite graph H as follows.

$V(H) = \{U_{i,j} \mid i \in \{1, \ldots, m\}, j \in \{1 \ldots, l\}\}$ and vertex partition $V(H) = U_1^* \cup \cdots \cup U_m^*$. Two vertices $U_{i,j}$ and $U_{i',j'}$ are adjacent in H if there is a vertex $v \in U_{i,j}$ and a vertex $v' \in U_{i',j'}$ of G such that $\{v, v'\} \in E(G)$. Obviously $\Delta(H) \leq 2\Delta \cdot \Delta = 2\Delta^2$.

Since $|U_i^*| = l = 10\Delta^2 \geq 5\Delta(H)$ we can apply Lemma 2.2 to H to obtain a $(5\log(5m\Delta(H)))$-bc transversal T' of H. Let us call V' the set of all vertices of G contained in the selected parts of T'. Since the $U_{i,j}$'s were chosen to contain 2Δ vertices of G we conclude that the graph $G[V']$ with partite sets imposed by T' contains an independent transversal T (according to Theorem 1.1). Since T' is an $(5\log(5m\Delta(H)))$-bc transversal of H, every component of $G[V']$ contains at most $5\log(5m\Delta(H))$ partite sets. Thus we can find an independent transversal T by an exhaustive search on each component of $G[V']$. For each component of $G[V']$ there are at most $(2\Delta)^{(5\log(5m\Delta(H)))} = (5m\Delta(H))^{(5\log(2\Delta))}$ candidates for independent transversals – a polynomial in m. □

I Wish...
...Robert Was Dead the Next Morning.

Prof. Emo Welzl, while playing "Mafia"

Chapter 3

Application - Linear Arboricity

Bounded transversals show many applications in graph theory and theoretical computer science. In this chapter we present an application of bounded transversals to a special variant of edge-colorings of graphs, the so-called linear arboricity. Although this is an application not related to bounded monochromatic component colorings, it's worth noting that bounded monochromatic component colorings have been first introduced in the context of linear arboricity (see for instance [62, 10]).

3.1 Linear Arboricity

We call an edge k-coloring $\chi : E(G) \to \{1, \ldots, k\}$ *linear* if every monochromatic component of χ forms a path. The linear arboricity $la(G)$ of a graph G is defined to be the smallest integer k such that there exists a linear edge k-coloring of G.

In [4] Akiyama, Exoo, and Harary made the following conjecture.

Conjecture 3.1 ([4]). *For any r-regular graph G, $la(G) = \lceil \frac{r+1}{2} \rceil$.*

The lower bound is easy to see. Consider an r-regular graph G with n vertices. The maximum length of each path in G is $n - 1$. Since G has $\frac{n \cdot r}{2}$ edges it follows

$$la(G) \geq \frac{n \cdot r}{2 \cdot (n-1)} > \frac{r}{2},$$

that is $la(G) \geq \lceil \frac{r+1}{2} \rceil$.

Let us note that every graph G of maximum degree $\Delta(G)$ can be embedded into a $\Delta(G)$-regular graph $H \supseteq G$ by adding edges and vertices to G. Hence the following upper bound is an equivalent formulation of the upper bound of Conjecture 3.1.

Conjecture 3.2 ([4]). *For every graph G, $la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.*

Observe that in every linear k-coloring χ of a graph G with maximum degree $\Delta(G)$ every vertex $v \in V(G)$ is incident to edges of at least $\lceil d(v)/2 \rceil$ many colors of χ, hence $la(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil$.

Conjecture 3.1 is obviously true for $r = 1, 2$. Further the conjecture has been proved affirmatively for $r = 3, 4$ by Akiyama, Exoo, and Harary ([4, 5]) and for $r = 5, 6, 8$ by Enomoto and Péroche ([28]). In [31] the case $r = 10$ has been proved by Guldan.

Attacking Conjecture 3.1 in general seems to be a very hard problem. Many variants of the linear arboricity problem have been studied. On one hand, restrictions to smaller classes of graphs have been considered, for instance to graphs with large girth.

Conjecture 3.1 has been partially verified by Alon [6] for graphs of large girth.

Theorem 3.1 ([6]). *For every r-regular graph G with*

(i) r even and girth $g(G) \geq 50r$, $la(G) = \frac{r+2}{2}$,

(ii) r odd, girth $g(G) \geq 100r$, and having a perfect matching, $la(G) = \frac{r+1}{2}$.

It was mentioned in [6] that the bounds on the girth could be reduced. In [11] Alon, Teague, and Wormald show using Theorem 3.1 that Conjecture 3.1 holds asymptotically:

Theorem 3.2 ([11]). *There is an absolute constant $c > 0$ such that for every r-regular graph G, $la(G) \leq \frac{r}{2} + cr^{\frac{2}{3}}(\log r)^{\frac{1}{3}}$.*

The linear arboricity of planar graphs has also been considered. Wu [66] almost completely proves the linear arboricity conjecture for planar graphs.

Theorem 3.3 ([66]). *Let G be a planar graph.*

If $\Delta(G) \neq 7$, then $la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.

If $\Delta(G) = 7$ and $g(G) \geq 4$, then $la(G) = 4$.

Hence, even for planar graphs, the 7-regular case remains open, and again a constraint on the girth had to be added.

In Section 3.1.1 we show that the restriction of Theorem 3.1 in G having a perfect matching (a 1-factor) can be weakened to having a 3-factor. Note here that every r-regular graph G, with r being an odd integer, that contains a perfect matching M, does also contain a 3-factor. Indeed the $(r-3)$-regular graph $G - M$ contains a 2-factor (c.f. Petersen's Theorem).

Theorem 3.4. *Let $r \geq 7$ be an odd integer. For every r-regular graph G such that $g(G) \geq 4r(r-1)$ and G contains a 3-factor, $la(G) = (r+1)/2$.*

In case G is a 7-regular graph we can completely omit the factor condition. In Section 3.1.2 we prove the following result.

Theorem 3.5. *For every 7-regular graph G with girth $g(G) \geq 253$, $la(G) = 4$.*

3.1.1 Odd Regular Graphs with a 3-Factor

We start this subsection with some useful lemmas.

Subsequently (in this chapter) we define $V^i(G)$ to be the vertices of degree i in G (or just by V^i if the G is obvious from the context). Similarly we define $V^{<i}(G) = \{v \in V(G) \mid d(v) < i\}$. The key lemma of the proof by Enomoto and Péroche used in [28] for r-regular graphs, with $r = 5, 6, 8$, is the following.

Lemma 3.1 ([28]). *Suppose G is a connected graph with $\Delta(G) \leq 4$, $\delta(G) \leq 3$, and $\Delta(G[V^4]) \leq 1$. Then $la(G) = 2$.*

We say that two edges e and e' of G are *dependent* if either e is incident to e' or there is an edge $f \in E(G)$ such that both e and e' are incident to f. The proof of the following lemma applies the existence of independent transversals.

Lemma 3.2. *Let G and H be two graphs such that $G \supseteq H$, $\Delta(G) \leq r$ and H can be factored into k graphs, $F_1, \ldots, F_k$ with $\Delta(F_i) \leq 2$, $i \in [k]$. Additionally let $E_{in}, E_{out} \subseteq E(H)$ with $E_{in} \cap E_{out} = \emptyset$, and such that no two edges in E_{in} are dependent in G. If for every graph F_i every* *cycle of* F_i *consists of at least* $g^* = 4r(r-1) + 2r(r-1)|E_{in}| + |E_{out}|$ *many vertices, then there exists a linear* $(k+1)$*-coloring of* H *such that one color-class* T *fulfills the following properties:*

(*i*) *no two edges in* T *are dependent in* G*,*

(*ii*) $E_{in} \subseteq T$*, and*

(*iii*) $E_{out} \cap T = \emptyset$*.*

Proof. From each cycle in $F_i, i \in [k]$ we choose one edge such that the resulting set of selected edges fulfills conditions (*i*)-(*iii*). We can assume without loss of generality that each F_i consists of disjoint cycles, define f_i to be the number of cycles in F_i. Moreover each cycle contains at least g^* many vertices (or edges). After numbering all cycles in F_i, we denote the edge set of the jth cycle in F_i by $E_{i,j}$ and define $E^*_{i,j} = E_{i,j} \setminus (\{e \in E_{i,j} \mid e \text{ and } e' \text{ are dependent in } G,\ e' \in E_{\text{in}}\} \cup E_{\text{out}})$. We now construct a new multipartite graph H^* as follows:

$V(H^*) = \bigcup_{i\in\{1,\dots,k\}, j\in\{1,\dots,f_i\}} V_{i,j}$, with $V_{i,j} = \{e \in E_{i,j}\}$, and

$e_1e_2 \in E(H^*) \iff e_1 \neq e_2$, and e_1 and e_2 are dependent in G.

We claim that $\Delta(H^*) \leq 2(r-1)r$. Indeed, since $\Delta(G) \leq r$ every edge in G is dependent to at most $2(r-1) + 2(r-1)(r-1) = 2(r-1)r$ other edges.

We aim to choose an independent transversal T of H^* such that its removal leaves only path components in H. All the edges of E_{in} are forced to be contained in T, hence no edge dependent to an edge of E_{in} in G can be chosen to be contained in T. Also none of E_{out} can be chosen to be contained in T. Hence only the edges $E^*_{i,j}$ of a partite set $V_{i,j}$ of H^* can be chosen for T. Since

$$\begin{aligned} |E^*_{i,j}| &= |E_{i,j} \setminus (\{e \in E_{i,j} \mid e \text{ and } e' \text{ are dep. in } G, e' \in E_{\text{in}}\} \cup E_{\text{out}})| \\ &\geq g^* - 2r(r-1)|E_{\text{in}}| - |E_{\text{out}}| \\ &\geq 4r(r-1) \geq 2\Delta(H^*), \end{aligned}$$

we can apply Lemma 1.1 to obtain an independent transversal $T^* \subseteq V(H^*)$ of H^* with partite sets restricted to $E^*_{i,j}$ only. We set $T = T^* \cup E_{\text{in}}$. Therefore $F_i - T$ contains only paths and no two edges of T are dependent in G with $E_{\text{in}} \subseteq T$ and $E_{\text{out}} \cap T = \emptyset$. We conclude that the k color-classes $F_i - T$, $i \in \{1, \dots, k\}$ and the special color-class T form a linear $(k+1)$-coloring as required. □

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. Let F be a 3-factor of G. Set $H = G - F$ and let $k = (r-3)/2$. Since H is $2k$-regular, H can be factored into k many 2-factors (c.f. Petersen's Theorem). Then by Lemma 3.2 applied to the two graphs G and H, $G \supset H$, and with $E_{\text{in}} = E_{\text{out}} = \emptyset$ and the fact that $g(H) \geq g(G) \geq 4r(r-1)$, H has a linear $(k+1)$-coloring with special color-class $M \subseteq E(H)$. Let $G' = F \cup M$. We see that $\Delta(G') \leq 4$ and, since M contains no pair of dependent edges, $\Delta(G'[V^4]) \leq 1$. So, by Lemma 3.1 (note that $\delta(G') \leq 3$ directly follows from the fact that M contains no dependent edges) we can conclude that $la(G') = 2$. Hence the linear $(k+1)$-coloring of H without the special color-class M together with the linear 2-coloring of G' is a linear $(k+2) = ((r+1)/2)$-coloring, as wanted. □

3.1.2 7-Regular Graphs

The following lemma proved by Bollobás, Saito, and Wormald is going to be helpful.

Lemma 3.3 ([23]). *Every 2-edge-connected 7-regular graph contains a 3-factor.*

In order to prove Theorem 3.5 we are going to color each maximal 2-edge-connected component (an edge-block) of G separately. Since an edge-block may contain vertices of degree less than seven, we first prove the following statement.

Lemma 3.4. *For every 2-edge-connected graph G with $\Delta(G) \leq 7$ and girth at least 253, $la(G) \leq 4$.*

Proof. If $V^{<7}(G) = \emptyset$, in other words if G is 7-regular, then due to Lemma 3.3 G contains a 3-factor. Also since $g(G) \geq 253 \geq 4 \cdot 7 \cdot 6$ we can apply Theorem 3.4 to G and obtain that $la(G) = 4$.

Hence we can subsequently assume that $V^{<7}(G) \neq \emptyset$. If $|V^{<7}(G)| \geq 2$ or $V^{<7}(G) = \{v\}$ and $d(v) < 6$, then we construct a new graph $G' \supseteq G$ such that G' is as well 2-edge-connected and either $V^{<7}(G') = \emptyset$ or $V^{<7}(G') = \{v\}$ and $d(v) = 6$, as follows. We set $k = 8 - \delta(G)$ and construct a new graph G' from k many copies of G ($G^{(i)}$ with $V(G^{(i)}) = \{v_j^{(i)}\}_{j \in \{1,\ldots,V(G)\}}$, $i \in \{1, \ldots, k\}$) with every set of k corresponding vertices $\{v^{(i)}\}_{i \in \{1,\ldots,k\}}$ and $d(v) = \delta(G)$ identified with a copy of K_k, see also Figure 3.1 for an example with $\delta(G) = 5$ and $V^5(G) = \{v_1, v_2, v_3\}$.

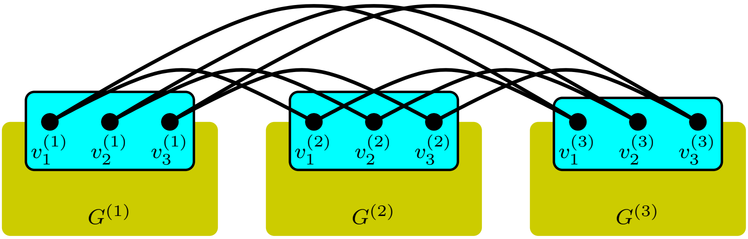


Figure 3.1: New graph G' with $V^{<7}(G) = \{v_1, v_2, v_3\}$ and $d(v_i) = 6$.

Iteratively apply this construction by setting $G = G'$ until we arrive to a graph G' with either $V^{<7}(G') = \emptyset$ or $V^{<7}(G') = \{v\}$ and $d(v) = 6$ (in this case applying the construction once more would result in a graph that is not 2-edge-connected, since a single K_2 is not 2-edge-connected).

Subsequently suppose that $V^{<7}(G') = \{v\}$, $d(v) = 6$ and $N(v) = \{u_1, \dots, u_6\}$. Split the vertex v into two vertices v_1 and v_2 such that $N(v_1) = \{u_1, u_2, u_3\}$ and $N(v_2) = \{u_4, u_5, u_6\}$ and moreover there is a path $u_i, \dots, u_j$ in $G' - v$ with $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$ (it is not hard to see that such a path has to exist), see Figure 3.2. Let H be the

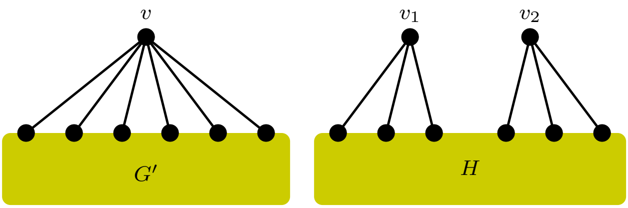


Figure 3.2: Split vertex v of G'.

resulting graph. Further we construct a new graph I from two copies of H, denoted by H' and H'', respectively, by identifying the two copies of the vertices v_1 and v_2 respectively and adding the edge $\{v_1, v_2\}$ to I, see Figure 3.3. The new graph I is 2-edge-connected and 7-regular. Apply Lemma 3.3 to I, and let F be a 3-factor of I. Either H' or H'' contains at most 3 edges of F incident to either v_1 or v_2 (but not to both), say H'. Moreover call f' the number of such edges in H'.

Observe that $|V(H')|$ is even, since all vertices but v_1 and v_2 of H'

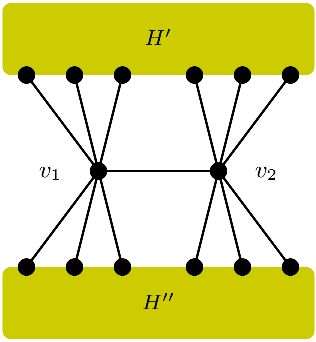


Figure 3.3: New graph I, 2-edge connected and 7-regular.

are odd and in every graph the number of vertices with odd degree is even. Suppose that f' is odd, then either $d_{F[V(H')]}(v_1)$ or $d_{F[V(H')]}(v_2)$ is odd and all other vertex degrees of $F[V(H')]$ are odd. Hence the number of vertices with odd degree in $F[V(H')]$ is odd, a contradiction.

Let us look at the remaining two cases $f' = 0$ and $f' = 2$.

- $f' = 0$
  We construct a new graph J from $H' - F$ as follows. Let u_1 and u_2 be two arbitrary non-adjacent neighbors of v. Let J be the graph $H' - F$ without the edges $\{v, u_1\}$ and $\{v, u_2\}$ but with the new edge $\{u_1, u_2\}$. It holds that $d(v) = 4$ and hence J is 4-regular. Hence J can be factored into two 2-factors F_1 and F_2. Also let $E_{\text{in}} = \{\{u_1, u_2\}\}$ and $E_{\text{out}} = \{\{u, v\} \mid u \in (N_G(v) \setminus \{u_1, u_2\})\}$. We define J_G to be the graph J restricted to the edges contained in G and to the edge $\{u_1, u_2\}$ if at least one of the edges $\{v, u_1\}$, $\{v, u_2\}$ is contained in G. Since $\Delta(G) \leq 7 = r$ and $4r(r-1) + 2r(r-1)|E_{\text{in}}| + |E_{\text{out}}| \leq 4\cdot7\cdot6 + 2\cdot7\cdot6\cdot1 + 4 = 253 - 1 \leq g(J) - 1$ (because of the new edge $\{u_1, u_2\}$) we can apply Lemma 3.2 to the two graphs G and J_G, and with E_{in} and E_{out} in order to obtain a linear 3-coloring of J_G with color-classes S_1, S_2 and special color-class $T \subseteq E(J_G)$. Then we apply Lemma 3.1 to $F \cup T$ (since no edges in T are dependent, clearly $\Delta(G[V^4]) \leq 1$) and obtain a linear 2-coloring T_1, T_2 of $F \cup T$. We color the edges $\{v, u_1\}$ and $\{v, u_2\}$ with the same color as $\{u_1, u_2\}$ and restrict to only the edges of G to obtain two new color-classes T_1' and T_2'. Therefore

the four color-classes S_1, S_2, T_1' and T_2' form a linear 4-coloring of G.

- $f' = 2$
  Hence $H' - F$ is a 4-regular graph. Let F_1 and F_2 be two 2-factors that factor $H' - F$. We apply Lemma 3.2 to the two graphs H' and $H' - F$ (restricted to G), and with $E_{\text{in}} = E_{\text{out}} = \emptyset$ and obtain a linear 3-coloring of $H' - F$ (restricted to G) with color-classes S_1, S_2 and T (with no two edges of T dependent in H'). The graph $F' = F \cup T$ has therefore maximum degree at most 4, moreover $\Delta(F'[V^4]) \leq 1$ and all vertices but possibly v have degree at least 3. Then we apply Lemma 3.1 to F' and obtain a linear 2-coloring of F' with color-classes T_1 and T_2 and hence also obtain a linear 4-coloring of G.

□

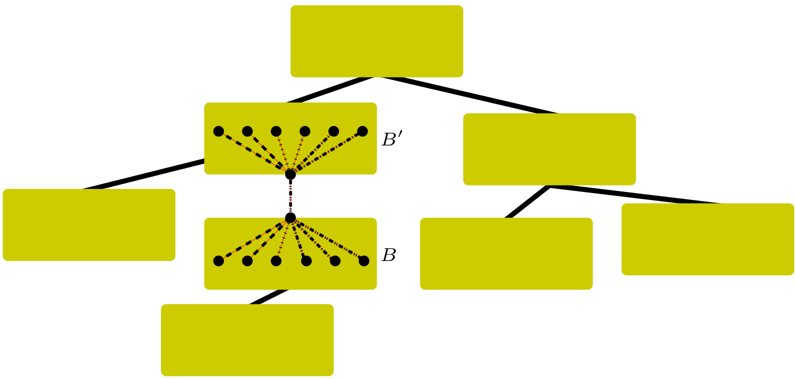


Figure 3.4: Color G edge-block by edge-block.

Let $B_1, \dots, B_m$ be the edge-blocks of G. We define the graph $B(G)$ with $V(B(G)) = \{B_i \mid i \in [m]\}$ and $E(B(G)) = \{\{B, B'\} \mid v \in B, v' \in B', \{v, v'\} \in E(G)\}$. The graph $B(G)$ forms a tree by the definition of edge blocks. We color $B(G)$ edge-block by edge-block in a preorder traversal. Let B be the currently processed edge-block and B' the parent edge-block that has already been colored. Let χ' be a linear 4-coloring of B'. Apply Lemma 3.4 to B and let χ be the linear 4-coloring of B. It remains to color the edge $e = \{v, v'\}$ with $v \in B$ and $v' \in B'$. Let c be one of the four colors such that c appears on at most one edge incident to v'. The color c exists since $d_{B'}(v') \leq 6$ (actually $d(v') \leq 7$ suffices as well). Permute the colors of χ such that c also appears on at most one edge incident to v and color $e = \{v, v'\}$ with the color c. See also Figure 3.4.

Part II

Bounded Monochromatic Component Colorings

Let us recall the definition of bmc colorings. We say that a k-coloring of a graph is $(C_1, C_2, \ldots, C_k)$-bmc if every monochromatic component induced by the vertices of the ith color-class is of order at most C_i, for $i \in [k]$. Note that a $(1, \ldots, 1)$-bmc k-coloring corresponds to a proper k-coloring. We mainly deal with the two most natural cases of bmc k-colorings. We say *C-symmetric bmc k-coloring* (also C-sbmc k-coloring) when $C_i = C$, for $i \in [k]$. Similarly we say *C-asymmetric bmc (k, l)-coloring* (also C-abmc (k, l)-coloring) when $C_i = 1$, for $1 \leq i \leq k$ and $C_j = C$, for $k + 1 \leq j \leq k + l$. For 2-colorings refer to a C-abmc $(1, 1)$-coloring also by a C-abmc 2-coloring.

In this part of the thesis we investigate bmc colorings of graphs with bounded maximum degree. Symmetric bmc colorings were first studied by Kleinberg, Motwani, Raghavan, and Venkatasubramanianby in [46]. In Chapter 4 we focus on bmc 2-colorings. Symmetric bmc 2-colorings have been studied by Alon, Ding, Oporowski, and Vertigan [10] and implicitly, even earlier, by Thomassen [62] who resolved the problem for the line graph of 3-regular graphs initiated by Akiyama and Chvátal [3]. Asymmetric bmc 2-colorings were first introduced in a joint paper with Tibor Szabó [19].

In Chapter 5 we investigate bmc k-colorings of bounded degree (planar) graphs, $k > 2$.

Also Haxell, Szabó, and Tardos in [39], Linial, Matoušek, Sheffet, and Tardos in [49], and Matoušek and Přívětivý in [52] investigated (symmetric) bmc colorings.

Related Relaxations of Proper Colorings There are several other types of coloring concepts related to our relaxation of proper coloring.

Independently Andrews and Jacobson [12], Harary and Jones [32, 33], and Cowen, Cowen, and Woodall [25] introduced and investigated the concept of *improper colorings* over various families of graphs. A k-coloring is called l-improper if none of the at most k colors induces a monochromatic component containing vertices of degree larger than l. Hence in an improper coloring the amount of error is measured in terms of the *maximum degree* of monochromatic components rather than in terms of their order. Several papers on the topic have since appeared; in particular, two papers, by Eaton and Hull [26] and Škrekovski[57], have extended the work of Cowen et al. to a list-coloring variant of improper colorings.

Linial and Saks [50] studied low diameter graph decompositions, where the quality of the coloring is measured by the *diameter* of the monochromatic components. Their goal was to color graphs with as few colors as possible such that each monochromatic connected component has a small diameter.

Haxell, Pikhurko, and Thomason [37] study the *fragmentability* of graphs introduced by Edwards and Farr [27], in particular for bounded degree graphs. A graph is called (α, f)-fragmentable if one can remove α fraction of the vertices and end up with components of order at most f. For comparison, in a C-abmc 2-coloring one must remove an independent set and end up with small components.

The so-called *relaxed chromatic number* (sometimes also called *generalized chromatic number*) was introduced by Weaver and West [65]. They used "relaxation" in a much more general sense than us, requiring that each color-class is the member of a given family of graphs. Naturally, our version also fits into this model.

HHHHRRRRRROOOOOOOAAAAAN!!!!!

Chewbacca the Wookiee

Chapter 4

Bmc 2-Colorings

We study bmc 2-colorings of bounded degree graphs from three points of view, extremal graph theory, complexity theory and algorithmic graph theory, and find that the first two points eventually meet for asymmetric bmc 2-colorings. We also make first steps for a similar connection in the symmetric case. To demonstrate our problems, in the next few paragraphs we restrict our attention to asymmetric bmc 2-colorings (as in Section 4.1); the corresponding questions are asked and partially answered for symmetric bmc 2-colorings in Section 4.2, but there our knowledge is much less satisfactory.

On the one hand, there is the purely graph theoretic question:

> For a given maximum degree Δ what is the smallest component order $f(\Delta) \in \mathbb{N} \cup \{\infty\}$ such that every graph of maximum degree Δ is $f(\Delta)$-abmc 2-colorable?

On the other hand, for fixed Δ and C one can study the computational complexity question:

> What is the complexity of the decision problem: Given a graph of maximum degree Δ, is there a C-abmc 2-coloring?

Obviously, for the critical component order $f(\Delta)$ which answers the extremal graph theory question, the answer is *trivial* for the complexity question: every instance is a "YES"-instance. Note also, that for $C = 1$ the complexity question is polynomial-time solvable, as it is equivalent to testing whether a graph is bipartite.

Moreover we consider the following algorithmic question.

> For a given maximum degree Δ is there a polynomial-time algorithm that finds a $g(\Delta)$-abmc 2-coloring of every graph with maximum degree at most Δ, for some integer $g(\Delta)$?

Section 4.1 is on one hand devoted to the extremal graph theory question. Mainly we show that $f(3)$ is finite and determine the following bound, $6 \leq f(3) \leq 22$. Moreover we show that every graph of maximum degree at most three can be 22-abmc 2-colored in polynomial-time. Also we show that for $\Delta \geq 4$, $f(\Delta)$ is not finite. On the other hand we look at the complexity theoretic question in the range between 2 and the critical component order $f(\Delta)$. We establish the monotonicity of the hardness of the problem in the interval $C \geq 2$ and prove a very sharp "hardness jump". By this we mean that the problem is NP-hard for every component order $2 \leq C < f(\Delta)$, while, of course, the problem becomes trivial (i.e. all instances are "YES"-instances) for component order $f(\Delta)$. It is maybe worthwhile to note that at the moment we do not see any *a priori* reason why the hardness of the decision problem should even be monotone in the component order C, i.e. why the hardness of the problem for component order $C+1$ should imply the hardness for component order C. In fact the problem is obviously polynomial-time decidable for $C = 1$, while for $C = 2$ we show NP-completeness.

In Section 4.2 we make similar investigations for sbmc 2-colorings. For a given maximum degree Δ we determine lower and upper bounds for the smallest component order $g(\Delta)$ such that every graph of maximum degree at most Δ admits a sbmc 2-coloring. We obtain that $5 \leq g(5) \leq 1908$ and moreover we derive an efficient algorithm that C-sbmc 2-colors every graph of maximum degree at most 5, for some large constant $C = 94371840$. Although we are not able to show a "hardness jump" in the symmetric case, we obtain some preliminary results and conjecture that indeed such a "hardness jump" occurs.

4.1 Asymmetric Bmc 2-Colorings

Recall that a C-abmc 2-coloring is a $(1, C)$-bmc 2-coloring. To formalize our theorems we need further definitions.

Let us define (Δ, C)-ABMCCol to be the decision problem whether a given graph G of maximum degree at most Δ allows a C-abmc 2-coloring. Note here that $(\Delta, 1)$-ABMCCol is simply testing whether a graph of maximum degree Δ is bipartite.

It is not hard to see that every graph of maximum degree at most two admits a 2-abmc 2-coloring. Therefore already $(2,2)$-ABMCCol is trivial. For $\Delta = 3$, we showed together with Tibor Szabó in [19] that every graph of maximum degree at most three admits a 189-abmc coloring, making $(3,189)$-ABMCCol trivial.

In the proof the vertex set of the graph was partitioned into a triangle-free and a triangle-full part (every vertex is contained in a triangle), then the parts were colored separately, using the following two results: *Every triangle-free graph of maximum degree at most three admits a* 6*-abmc* 2*-coloring.* The proof of this statement can be found in Subsection 4.1.1. And *Every triangle-full graph of maximum degree at most three admits a* 21*-abmc coloring.* Finally the two colorings were assembled amid some technical difficulties.

Here we present a completely different approach from [18] which avoids the separation. While we still deal with our share of technical difficulties, we greatly improve on the previous bound on the component order and the running time of the algorithm involved.

A variant of the new method is first presented for triangle-full graphs of maximum degree at most three. One facet of our technique is much simpler to present in this scenario and gives an improved and optimal result. *Every triangle-full graph G of maximum degree at most three admits a* 6*-abmc coloring.* We prove this statement in Section 4.1.2 and show the existence of a triangle-full graph, see Figure 4.5 for which monochromatic component order 6 is best possible.

The method is then enhanced in Subsection 4.1.3 to work for all graphs of maximum degree at most three. It also implies a quasilinear time algorithm (as opposed to the $\Theta(n^7)$ algorithm implicitly contained in [19]).

Theorem 4.1. *Any graph of maximum degree at most three is* 22*-abmc two-colorable. Moreover there is an $O(n \log^4 n)$ algorithm which finds such a* 22*-abmc* 2*-coloring.*

In our next theorem we show that $(3, C)$-ABMCCol exhibits the promised hardness jump.

Theorem 4.2. *There is an integer f, $6 \le f \le 22$ such that*

(i) $(3, C)$-ABMCCol is NP*-complete for every $2 \le C < f$, and*

(ii) every graph of maximum degree at most three admits an f-abmc

2-*coloring.*

The proof of Theorem 4.2 can be found in Subsection 4.1.5.

In [19] it was shown that for any $\Delta \geq 4$ and positive C, (Δ, C)-ABMCCol never becomes "trivial", i.e., for every finite C there is a "NO" instance. We show here, however, that the monotonicity of the hardness of $(4, C)$-ABMCCol still exists for $C \geq 2$ by proving that $(4, C)$-ABMCCol *is* NP-*complete for every* $2 \leq C < \infty$. The proof of this statement can also be found in Subsection 4.1.5, Theorem 4.4. Obviously, this implies that (Δ, C)-ABMCCol is NP-complete for every $\Delta > 4$ and $2 \leq C < \infty$.

4.1.1 Abmc 2-Colorings of Triangle-Free Graphs G with $\Delta(G) \leq 3$

In this subsection we prove that Theorem 4.1 (with a better constant) holds if G has maximum degree at most three and is triangle-free.

Lemma 4.1. *Every triangle-free graph G of maximum degree at most three admits a* 6-abmc 2-*coloring.*

Proof. We are going to construct an abmc 2-coloring with color-class I, forming an independent set, and color-class B, inducing components of order at most 6. As a first approximation let us take a maximum edge-cut (U_1, U_2) (i.e., there is no other vertex-partition with more edges going across), with $|U_1|$ minimal (among all maximum edge-cuts).

Since (U_1, U_2) is a maximum edge-cut, every vertex has degree at most one within its own part. That is, $G[U_1]$ and $G[U_2]$ consist of disjoint edges and isolated vertices. Eventually, our goal is to select one of the endpoints of each edge in $G[U_1]$ and move it to the other side such a way, that we do not create too large components.

First we make a few observations about the impossibility of certain configurations. For $i = 1, 2$ and $j = 0, 1$ let $U_{i,j} = \{x \in U_i : d_{U_i}(x) = j\}$. For $i \in \{1, 2\}$, we refer to i' as the other element of $\{1, 2\}$, i.e., $i' \in \{1, 2\}$ and $i' \neq i$.

Proposition 4.1. *Let $x \in U_{i,1}$ and $x', x'' \in U_{i',1}$, for some $i = 1, 2$. Then x is not adjacent to both x' and x''.*

Proof. Switching the sides of x, x', x'' increases the number of edges in the cut and thus contradicts the maximality of (U_1, U_2). □

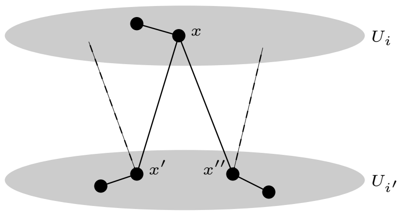


Figure 4.1: Configuration 1.

Proposition 4.2. *Let $x \in U_{2,0}$ and $x', x'', x''' \in U_{1,1}$. Then x is not adjacent to all of x', x'', and x'''.*

Proof. Switching the sides of x, x', x'', x''' would not decrease the number of edges in the cut, but would decrease the cardinality of $|U_1|$, a contradiction. □

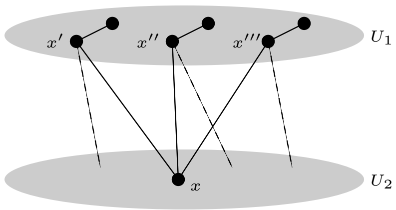


Figure 4.2: Configuration 2.

Proposition 4.3. *Let $x \in U_{i,0}$, $x', x'' \in U_{i,1}$ and $y', y'' \in U_{i',1}$, for some $i = 1, 2$. Then it is not possible that x is adjacent to both y' and y'', y' is adjacent to x', and y'' is adjacent to x''.*

Proof. Switching the sides of x, x', x'', y', y'' increases the number of edges in the cut and thus contradicts the maximality of (U_1, U_2). □

Note that Propositions 4.1, 4.2 and 4.3 fail to be true if G contains triangles.

We define an auxiliary graph H on the vertex set $V(H) = U_{1,1}$. Two vertices x and y of H are adjacent if they have a neighbor in the same component of $G[U_2]$.

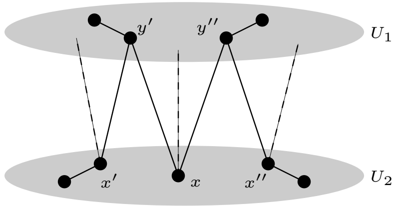


Figure 4.3: Configuration 3.

Claim: $\Delta(H) \leq 2$

Proof. Let $x \in V(H)$. By the definition of $V(H)$, x has at most two neighbors in U_2. Let y be one of them.

If y is an isolated vertex of $G[U_2]$, then by Proposition 4.2, y has at most one more neighbor (besides x) in $U_{1,1}$. So y does not account for more than one H-neighbor of x. If y is in an edge-component of $G[U_2]$, let w be its unique neighbor in U_2. By Proposition 4.1 y has no other neighbor in $U_{1,1}$ but x. Similarly, w has at most one neighbor in $U_{1,1}$. So y is responsible for at most one H-neighbor of x.

We showed that each of the (at most two) U_2-neighbors of x can produce at most one H-neighbor for x. That is, the degree of x in H is at most 2. □

Since $\Delta(H) \leq 2$, we can apply Lemma 1.4 for H, with a partition of $V(H)$ imposed by the edges e_i of $G[U_1]$. (Remember $G[V(H)]$ is a perfect matching!) We select a matching transversal T and move it over; That is, we define $I = U_1 \setminus T$ and $B = U_2 \cup T$.

Clearly I is an independent set.

How large could a component be in $G[B]$? Note that T is an independent set in G and since T induces a matching in H, any component of $G[B]$ can contain at most two vertices from T. If a component of $G[B]$ contains exactly one vertex of T, then its size is at most 5. Suppose now that a component C of $G[B]$ contains two vertices $t_1, t_2 \in T$. There must be a component C' of $G[U_2]$ in which both t_1 and t_2 has a neighbor. Since both t_1 and t_2 have at most two neighbors in U_2, C contains at most three components of $G[U_2]$. If there are at most two components of $G[U_2]$ in C, then the cardinality of C is at most 6. Assume now that there are three components C', C_1, C_2 of $G[U_2]$ glued together in C. By Proposition 4.1, neither t_1 nor t_2 is adjacent to two components of order two. So if $|C'| = 2$, then $|C_1| = |C_2| = 1$ and thus $|C| = 6$. Similarly, if $|C'| = |C_i| = 1$ for some $i = 1, 2$, then $|C| = 6$. Finally, the case $|C'| = 1$ and $|C_1| = |C_2| = 2$ is impossible because of Proposition 4.3.

Concluding, we proved that all components of $G[B]$ are of order at most 6. □

4.1.2 Abmc 2-Colorings of Triangle-Full Graphs G with $\Delta(G) \leq 3$

Recall that graphs for which every vertex is contained in a triangle are called "triangle-full".

In our investigation of C-abmc 2-colorings we will encounter two color-classes I and B, where I denotes an independent set and B denotes the color-class which induces components of order at most C. We say that the color-class B and I are *opposite*s of each other.

For a color-class R (which is a subset of the vertices of G), we often say that *we color a vertex v with color R*, when in fact *we place v into R*.

It about the right moment to justify a couple of simplifying assumptions for abmc 2-colorings of graphs with maximum degree at most three.

Diamond-freeness Let G be a graph with maximum degree at most three. No two triangles in G share exactly one vertex. Two triangles sharing an edge form a *diamond*. We argue that, without loss of generality, we can assume that our graph is diamond-free. Indeed, let D be a diamond in G and let G' be the graph obtained from G by deleting the vertices of D. By induction (on the number of diamonds) we obtain a partitioning of G' into sets I' and B' satisfying the properties of Lemma 4.2. Let the two vertices of D sharing the common edge be v_1, v_2, the remaining two vertices are called u_1, u_2 and the unique neighbor of u_i outside of D by u'_i (u'_i might not exist). Now let us define a partition of $V(G)$ into sets I and B by letting $I' \subseteq I$ and $B' \subseteq B$ and putting u_i into I if and only if u'_i is in B'. The vertices v_1, v_2 are

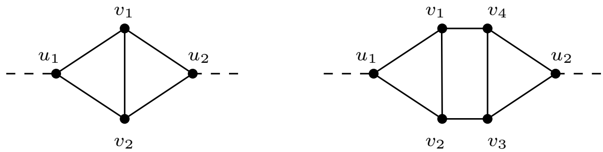


Figure 4.4: A diamond and two triangles connected by at least two edges

put into B regardless. Since u_1 is not adjacent to u_2, I is independent by definition. Also, the vertices of D put into B are separated from B' by a vertex of I, thus the largest newly introduced component of $G[B]$ could be the diamond itself. Thus the order of the largest component of $G[B]$ is $\max\{C', 4\}$, where C' is the order of the largest component of $G[B']$.

By a similar argument, we can assume that G does not contain two triangles connected by two edges (see Fig. 4.4).

Lemma 4.2. *Every triangle-full graph G of maximum degree at most three admits a* 6-abmc 2*-coloring.*

The following proof and moreover the algorithm shown in the proof serves as a good introduction to the much more involved algorithm in Subsection 4.1.3 for that 22-abmc 2-colors every graph with maximum degree at most three.

Proof of Lemma 4.2. As seen we can assume that G is diamond-free. Hence from now on we suppose that every vertex is contained in exactly one triangle. Let M be the set of edges of G not contained in triangles of G. Obviously, M forms a matching. Further $G - M$ consists of disjoint triangles covering all vertices of G. The Algorithm `PA_TF`(G) (a pseudo-code for `PA_TF` can be found in Algorithm 1) constructs a 6-abmc 2-coloring (I, B) of G by coloring the vertices triangle after triangle. It colors the currently processed vertex v with I if it can, i.e., if v has no neighbor which is colored with I already. The main point of the algorithm is how to select the next vertex to color when all vertices in the current triangle are colored. In particular we make sure that the first vertex we color from each triangle gets a color opposite to its partner.

Let's first introduce some notation used in Algorithm 1. For a vertex v and an oriented triangle C in $G - M$ containing v we define v^- to be the predecessor of v in C, by v^+ it's successor in C and by v^* its unique neighbor in M (if it exists). We call v^* the *partner* of v.

Algorithm 1: PA_TF(G)

Input: Graph G; $\Delta(G) \leq 3$, triangle-full and diamond-free.
Output: Vertex-partition (I, B); I independent set, no component in $G[B]$ larger than 6.

```
  I ← ∅, B ← ∅
  give an arbitrary cyclic orientation to each triangle in G
  choose arbitrary vertex v in G
  while not all vertices of G are colored do
      while not all vertices of the triangle containing v are colored
      do
1         if v⁻ ∈ I or v* ∈ I or v⁺ ∈ I then Add(v, B)
2         else Add(v, I)
          v ← v⁺
      if not all vertices of G are colored then
          v ← v⁻      // now v is the last vertex we colored
3         if v* is uncolored then v ← v*
4         else if v⁻* is uncolored then v ← v⁻*
5         else  v ← w, where w is arbitrary uncolored vertex with
          w* colored
  return (I, B)
```

We immediately see that I forms an independent set. Indeed, only in Line 2 we color a vertex with I, where no neighbor of it is colored I already.

Suppose that there is a B-component C larger than 6.

First observe that if a triangle T of G is completely contained in C then according to Line 1 in PA_TF(G) the partner of each vertex in T must be contained in I. Thus C consists of only the vertices from T, a contradiction.

Hence we assume that C does not contain any triangle from G completely. Such a component C intersects with at least four triangles T_1, T_2, T_3, T_4 in G. Suppose, without loss of generality, that T_i is incident to T_{i+1}, for $i \in \{1, 2, 3\}$ and that T_2 gets colored before T_3 during the execution of `PA_TF`(G). We denote by $v_{i,j}$ the vertex contained in $T_i \cap C$ incident to triangle T_j.

Which vertex of T_2 is colored first? It can be neither $v_{2,1}$ nor $v_{2,3}$, since the first vertex of any triangle gets color opposite to its partner's. (In Lines 3, 4, 5 we select the first vertex of the next triangle, such that its partner is colored. This is true for the first colored vertex of every triangle except the very first one. Then Lines 1, 2 make sure that the first vertex receives a color different from its partner. This is even true for the very first vertex, since it is colored I in Line 2 and its partner will receive color B in Line 1.)

So either $v_{2,1}$ or $v_{2,3}$ is the last vertex we color in T_2 After all vertices of T_2 have been colored, `PA_TF`(G) chooses either $v_{1,2}$ or $v_{3,2}$ to be colored next, according to Line 3 and Line 4 (note that $v_{3,2}$ is not yet colored according to our assumption). This is a contradiction since, again, the first vertex in any triangle has color opposite to its partner. □

We want to note here that our proof is constructive and yields a 6-abmc 2-coloring of triangle-full graphs. It is not hard to see that the running time of `PA_TF`(G) is linear in the order of G.

Consider the graph H' whose triangle-graph H (the graph where every triangle is contracted to a single vertex) is shown in Figure 4.5. It is not hard to observe that the removal of any independent set of H' leaves at least one component of order at least 6. Hence Lemma 4.2 is in this sense optimal.

4.1.3 Abmc 2-Colorings of Graphs G with $\Delta(G) \leq 3$

Again in our investigation of C-abmc 2-colorings we will encounter two color classes I and B, where I denotes an independent set and B denotes the color-class which induces components of order at most C. We say that the color-class B and I are *opposite*s of each other. In one of the main auxiliary lemmas, we encounter a third color-class X. We will also use the term *opposite* in relation to X and say that B and X are *opposite.*

Proof of Theorem 4.1. We prove the statement of Theorem 4.1 by induction on the number of vertices in G. A *generalized diamond* D is a subgraph of G induced by four vertices of G such that $d_{V(G)-V(D)}(v) \leq$

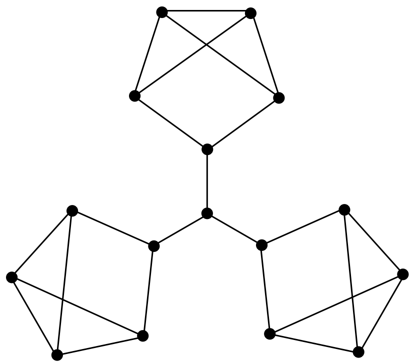


Figure 4.5: The "triangle-graph" of H

1 for all $v \in V(D)$ and the vertices of D with degree 1 into $V(G) - V(D)$ form an independent set in G.

The core of the proof is the case when G is generalized diamond-free. Otherwise let D be a generalized diamond in G and proceed similarly to the diamond case. By the induction hypothesis, $G - V(D)$ has an I/B-coloring such that the I-vertices form an independent set and the B-vertices induce monochromatic components of order at most 22. We extend this coloring to an I/B-coloring of G. We color the vertices of D with B unless the vertex has a neighbor in $G - V(D)$, in which case we use the color *opposite* to the color of this neighbor. This is always possible since such vertices of D form an independent set in G. Hence all the B-components of $G - V(D)$ remain the same, while the vertices in D will be part of a B-component of order at most four.

It is now left to prove Theorem 4.1 when G is generalized diamond-free. One of the main ingredients of the proof is the following lemma:

Lemma 4.3. *Let G be a generalized diamond-free graph with maximum degree at most three on n vertices. Further let $v_{\mathit{fix}} \in V(G)$ and $c \in \{I, B\}$. There exists a vertex partition (I, X, B) of G such that*

- *(i) $I \cup X$ induces a graph where each I-vertex has degree 0 and each X-vertex has degree 1,*
- *(ii) no triangle contains two vertices from X,*
- *(iii) every B-component is of order at most 6, and*

(iv) if $d(v_{fix}) = 2$ then either v_{fix} is contained in c, or $c = I$ and v_{fix} is contained in X.

Moreover, this vertex-partition can be found in time $O(n \log^4 n)$.

First let us see how Lemma 4.3 implies Theorem 4.1. We note that property *(iv)* is only needed for the inductive proof of Lemma 4.3.

Let I, X and B be such as promised by Lemma 4.3. We do a post-processing in two phases, during which we distribute the vertices of X between I and B: for each adjacent pair v, w of vertices in X we put one of them to B and the other into I. When this happens we say that we *distributed* the X-edge $\{v, w\}$. We specify how we distribute an X-edge $\{v, w\}$ by the operation *Distribute*(v, c), where $c \in \{I, B\}$. *Distribute*(v, c) puts v into c while w is put into the opposite color-class. Note that if property *(i)* is valid at some point then it is still valid after the distribution of any X-edge. During the first phase some vertices contained in B will be moved to I, but once a vertex is in I, it stays there during the rest of the postprocessing.

For the first phase let us say that a vertex v is *ready for a change* if $v \in B$ and all the neighbors of v are in $B \cup X$. Once we find a vertex v ready for a change we move v to I, and distribute each X-edge which contains a neighbor u of v by *Distribute*(u, B). We iteratively make this change until we find no more vertex ready for a change, at which point the first phase ends. Property *(ii)* ensures that the rules of our change are well-defined: It is not possible that an X-neighbor of v is instructed to be placed in B, while it could also be the X-neighbor of another X-neighbor of v which would instruct it to be in I.

Property *(i)* remains valid during the first phase, since besides X-edges being distributed (which preserves property *(i)*) only such B-vertices are moved to I whose neighbors will all be in B.

Let us now look at how property *(iii)* changes during the first phase. Crucially, at the end of the first phase every B-component is a path, since any B-vertex with three B-neighbors is ready for a change. As a result of one change no two B-components are joined, possibly a vertex u from X which just changed its color to B is now stuck to an old B-component. In case this happens both of the other neighbors of u are in I (and stay there). Let C be a B-component after the first phase. We claim that all vertices adjacent to C are in I except possibly two: one-one at each endpoint of C. Indeed, if an interior vertex of C had an X-neighbor, it would have been ready for a change. By *(iii)* there is a path C' in C containing at most 6 vertices which used to be part of a B-component before the first phase. So we can distinguish three cases in terms of how many X-neighbors C has besides its I-neighbors.

Observation 4.1. *After the first phase every B-component is one of the following:*

(a) C is a path containing at most 6 vertices with one X-neighbor at each of its endpoints or

(b) C is a path containing at most 7 vertices with one X-neighbor at one of its endpoints or

(c) C is a path containing at most 8 vertices with no X-neighbors.

In the second phase we distribute between I and B those vertices which are still in X. The vertices of color I or B preserve their color during this phase. Property (i) ensures that the set I we obtain at the end of the second phase is an independent set. We have to be very careful though that the connected components in $G[B]$ don't grow too much during the second phase. We guarantee this via finding a *matching transversal* in an auxiliary graph H. The graph H is defined on the vertices of X, $V(H) = X$. There is an edge between two vertices u and v of H if u and v are incident to the same component of $G[B]$.

Claim 4.1. $\Delta(H) \leq 2$.

Proof. Let us pick a vertex y from $V(H) = X$. We aim to show that each edge e incident to y which is not an X-edge (there are at most two of these) is "responsible"for at most one neighbor of y in H. That is, the component of $G[B]$ adjacent to y via such edge e is incident to at most one other vertex from X. Indeed, by Observation 4.1 above, each B-component is a path, possibly adjacent to X-vertices through its endpoints, but not more than to one at each. □

Theorem 1.4 guarantees a transversal inducing at most matching. We note that the proof of Theorem 1.4 by Haxell, Szabó, and Tardos in [39] involves a linear time algorithm which constructs the matching transversal.

We apply Theorem 1.4 for H with the partition $\mathcal{P}$ defined by the edges of $G[X]$ (i.e., $\mathcal{P} = E(G[X])$) and find a matching transversal T.

The second phase of our postprocessing consists of moving all vertices of T into B and moving $X \setminus T$ into I.

Since $\Delta(H[T]) \leq 1$ we connect at most three connected components Q_1, Q_2 and Q_3 of $G[B]$ by moving an edge $\{u, v\}$ of H into B, with u incident to Q_1 and Q_2 and v incident to Q_2 and Q_3. Obviously, Q_1 and Q_3 are incident to at least one vertex of H (u and v respectively) and Q_2 is incident to at least two vertices from H (u and v) before moving the vertices of T. According to Observation 4.1, the largest B-component created this way is of order at most $7+1+6+1+7=22$. Lemma 4.3(i) guarantees that I is independent so the defined coloring is a 22-abmc 2-coloring.

We note that both phases of this proof could be turned into an algorithm whose running time is linear in the number of vertices of G □

Proof of Lemma 4.3. We use induction on the number of vertices of G. By induction we can of course assume that G is connected. If G is not 2-connected then there is a cut-vertex u in G. Let $G_0 \subseteq G$ be a component of $G - u$, such that $d_{V(G_0)}(u) = 1$ and let u' be the unique neighbor of u in G_0. Define $G_1 = G - G_0$. Then $d_{V(G_1)}(u) \leq 2$. Suppose that $v_{\text{fix}} \in V(G_i)$ for $i = 0$ or 1. By induction, we can find a (I_i, X_i, B_i)-partition of G_i such that v_{fix} receives its prescribed color. Depending on whether $u \in V(G_i)$, either u or u' has a color assigned to it by the partition (I_i, X_i, B_i); say, u is part of the partition. Then we find a partition $(I_{1-i}, X_{1-i}, B_{1-i})$ of G_{1-i} by induction, such that the vertex u' receives the color *opposite* to the color of u. This implies that the partition of G defined by the partition $(I_0 \cup I_1, X_0 \cup X_1, B_0 \cup B_1)$ is as required by Lemma 4.3.

All these steps can be done quickly. Standard techniques involving a depth first search tree of G enable to find a cut-vertex of G in linear time in the number of edges plus number of vertices of G (since we only consider graphs with maximum degree at most three this is certainly also linear in the number of vertices of G).

The essence of the proof of Lemma 4.3 is the case when G is 2-connected. We start proving this case by finding an appropriate matching in G.

Proposition 4.4. *Every n-vertex, 2-edge-connected graph G with maximum degree at most three contains a matching M such that*

(i) $\Delta(G - M) \leq 2$, *and*

(ii) $G - M$ *is triangle-free.*

Moreover, M can be found in time $O(n \log^4 n)$.

Proof. Let us first assume that G contains an even number of vertices of degree exactly two. We pair each vertex of degree 2 with another vertex of degree 2 and add one edge between the vertices of each such pair. We denote the new graph by H. Obviously H is a 3-regular, 2-edge connected multigraph.

Secondly, suppose that G contains an odd number of vertices of degree 2. We pick one vertex v with $d(v) = 2$ from G, remove v from G and connect its two neighbors via an edge e_v. The new graph contains an even number of vertices of degree 2. Then we proceed as above to obtain the graph H.

Assume first that H is triangle-free. By Petersen's theorem, H contains a perfect matching M_H. Moreover, if the number of vertices of degree 2 was odd, i.e., if e_v is defined, then M_H can be chosen such that $e_v \notin M_H$. In Biedl, Bose, Demaine, and Lubiw [21] it is shown that such a matching M_H can be found in time $O(n \log^4 n)$. Let M consist of those edges of M_H which are also edges of G. Then the requirements of Proposition 4.4 are satisfied (if e_v is defined, then the neighbors of v have degree at most 2 in $G - M$, since $e_v \notin M_H$.)

Let us now consider the general case, when H might contain triangles. In order to obtain a perfect matching M such that $H - M$ is triangle-free we iteratively contract every triangle of H into a vertex, yielding a new triangle-free graph H'. Then we apply the above procedure to H' instead of H and get a perfect matching M' of H'. We observe that this perfect matching M' can easily be extended to a perfect matching M_H of H where each triangle of H contains exactly one edge of M_H. Thus $H - M_H$ is triangle-free. Also, even if e_v is contained in a triangle T, we can force $e_v \notin M_H$ by simply forcing that the unique edge incident to T, but not to e_v, is not contained in M'.

□

The algorithm that partitions the vertices of G will be called `PA`(G, v_{fix}, c) (see Algorithm 2 for the pseudo-code) with v_{fix} being the vertex of G that will be colored c according to Lemma 4.3 (*iv*).

Let us first discuss informally the main ideas of our algorithm. `PA`(G, v_{fix}, c) chooses a matching M of G as in Proposition 4.4. This is in fact the bottleneck of our algorithm, all other parts are done in linear time. The graph $G - M$ consists of path- and cycle-components.

Algorithm $\texttt{PA}(G, v_{\text{fix}}, c)$ colors the vertices of G, one component of $G-M$ after another, by traversing each component in a predefined orientation.

$\texttt{PA}(G, v_{\text{fix}}, c)$ starts the coloring with the vertex v_{fix} and color c. We will sometimes also refer to this vertex as the *very first vertex*.

For each component the algorithm chooses one of its two orientations. For the component of v_{fix} this is done according to a special rule. The orientation of other components is arbitrary. Recall that v^+ (v^-) denotes the vertex following (preceding) v according to the fixed orientation of its component. To simplify the description of our algorithm we introduce the following conventions. For the source v of a path component, we denote by v^- the sink of the path. Similarly for the sink u of a path component we denote by u^+ the source of the path. If a vertex v is saturated by M, then the vertex v^* adjacent to v in M is called the *partner* of v.

As a default $\texttt{PA}(G, v_{\text{fix}}, c)$ tries to color the vertices of a component of $G - M$ with the colors I and B alternatingly. Its original goal is to create a proper two-coloring this way. Of course there are several reasons which will prevent $\texttt{PA}(G, v_{\text{fix}}, c)$ from doing so. One main obstacle is when the partner (if it exists) of the currently processed vertex u is already colored, and it is done so with the same color we would just want to give to u. If the conflict would be in color I then the algorithm resolves this by changing both u and its partner to X. The algorithm generally decides not to care if the conflict is in B. Of course there is a complication with this rule when the partner is within the same triangle as u, since Lemma 4.3 does not allow two X-vertices in the same triangle. This and other anomalies (like the coloring of the last vertex of a cycle when the first and next-to-last vertex have distinct colors) are handled by a well-designed set of exceptions in place. In fact the design of such a consistent set of exceptions poses a major challenge.

Subsequently a vertex which is colored first in a component of $G-M$ is referred to as a *first vertex*. Similarly, a *last vertex* is just a vertex colored last in a component of $G - M$.

After having colored the last vertex v of component C the algorithm $\texttt{FirstVertex}(G, v, I, X, B)$ chooses the partner v^* of v unless v^* is already colored or v^* does not exist. In that case $\texttt{FirstVertex}(G, v, I, X, B)$ looks for a vertex with color B whose partner is uncolored by stepping backwards along the order in which the vertices of C have been colored and eventually starts to color such a partner. If all of the B-colored vertices of C have an already colored partner or no partner, then `FirstVertex`(G, v, I, X, B) selects an arbitrary uncolored vertex with an already colored partner. The selection of first vertices according to `FirstVertex` coupled with `PA` makes sure that every first vertex has a color opposite to its partner.

For some subset U of the vertices, the operation $Add(U, c)$, as used in `PA`, first uncolors those vertices of U which were colored before and colors all vertices in U with c. $Add(v, c)$ will be written for $Add(\{v\}, c)$. In case a vertex that has been referenced (for instance v^*) does not exist, then $Add(v^*, c)$ does not change anything. To simplify the description of the algorithm, by saying, for example "$v^* \in I$" we mean "v^* exists and $v^* \in I$".

Analysis of `PA`$(G, v_{\mathbf{fix}}, c)$ In the following we make a couple of observations about first vertices. The proof of conclusion (ii) of Observation 4.2 does depend on Corollary 4.1 whose proof only depends on part (i) of Observation 4.2.

Observation 4.2. *Let v be a first vertex (but not the very first vertex).*

- (i) *The partner of v exists and v^* is colored before v. In particular, v and v^* are contained in distinct components of $G - M$.*
- (ii) *v and v^* receive opposite colors.*

Proof. (i) A new first vertex is chosen by `FirstVertex` when each component of $G - M$ has either all or none of its vertices colored. If there are still uncolored vertices in G, then there must be one which has a colored partner (since G is connected) and `FirstVertex` will select such a first vertex. The last claim then follows since a first vertex by definition is colored first within its component, so its partner cannot be in it.

(ii) When `FirstVertex` selects the next first vertex v, then we know that v^* exists and is colored. Then Line 4 or 5 of `PA` will color v to the opposite color, either I or B. If this color changes later during the execution of `PA` then, according to part (i) and (ii) of Corollary 4.1, this change must be from I to X, which does not effect the validity of (ii). By part (iii) of Corollary 4.1, an X-vertex can change its color to B only if it is the very first vertex v_{fix}. □

Observation 4.3. *If Algorithm `PA` recolors a previously colored vertex then one of the following three cases hold.*

Algorithm 2: PA(G, v_{fix}, c)

Input: 2-edge-connected, generalized diamond-free graph G with $\Delta(G) \leq 3$; vertex $v_{\text{fix}} \in V(G)$; color-class $c \in \{I, B\}$;
Output: Vertex partition (I, X, B); according to Lemma 4.3(i)-(iv).

$I \leftarrow \emptyset$, $X \leftarrow \emptyset$, $B \leftarrow \emptyset$
choose matching M according to Proposition 4.4
while *not all vertices of G are colored* **do**
  if $I \cup X \cup B = \emptyset$ **then**
    $v \leftarrow v_{\text{fix}}$
    Orient the component of v such that $\{v^{--}, v^{-}, v\}$ does not form a triangle and $\{v, v^{+}\} \in E(G)$
    if $d(v) = 3$ **then** *Add*(v, I)
    else *Add*(v, c) // rule ‘‘very first’’
  else
    $v \leftarrow$ FirstVertex(G, v, I, X, B)
    Orient the component of v arbitrarily
    if $v^{*} \in I \cup X$ **then** *Add*(v, B) // rule ‘‘first’’
    else *Add*(v, I)
  while *not all vertices of the component containing v are colored* **do**
    $v \leftarrow v^{+}$
    if $v^{-} \in I \cup X$ **and** $\{v^{-}, v\} \in E(G)$ **then**
      Add(v, B) // rule ‘‘standard’’
    else // that is, $v^{-} \in B$ or $\{v^{-}, v\} \notin E(G)$
      if v^{+} *is not colored* **or** $v^{+} \in B$ **or** $\{v, v^{+}\} \notin E(G)$ **then**
        if $v^{*} \in B$ **or** v^{*} *is not colored* **or** v^{*} *does not exist* **then**
          Add(v, I)
        else // that is, $v^{*} \in I \cup X$
          if $\{v, v^{*}\}$ *in a triangle* **then** *Add*(v, B) // rule ‘‘triangle’’
          else if $v^{*} \in X$ **then** *Distribute*(v^{*}, B), *Add*(v, I) // rule ‘‘special’’
          else *Add*$(\{v, v^{*}\}, X)$ // move partners into X
      else // color the last vertex of a cycle if the first is in $I \cup X$
        if $v^{*} \in I \cup X$ **or** v^{*} *does not exist* **or** $\{v, v^{*}\}$ *in a triangle* **then**
          Add(v, B) // rule ‘‘last’’
        else // that is, $v^{*} \in B$ or uncolored, $\{v, v^{*}\}$ not in a triangle
          if $v^{+} \in X$ **then** *Distribute*(v^{+}, B), *Add*(v, I) // rule ’’special’’
          else *Add*$(\{v, v^{+}\}, X)$ // move non-partners into X

return (I, X, B)

Algorithm 3: `FirstVertex`(G, v, I, X, B)

Input: G, I, X, and B as defined in Algorithm `PA`(G, v_{fix}, c), vertex $v \in V(G)$ colored last.
Output: First vertex of an uncolored component C to be colored.

if v^* *is uncolored* **then return** v^*
else
  $u \leftarrow v^-$
  while $u \neq v$ **and** ($u \notin B$ **or** u^* *is colored*) **do**
    $u \leftarrow u^-$
  if $u \neq v$ **then return** u^*
  else return w, where w is arbitrary uncolored vertex with w^* colored.

(*i*) *A color I is changed to X either in Line 10 or 14. In Line 10 we move partners to X, in Line 14 we move the last and first vertex of a component into X.*

(*ii*) *In Line 9 the previously uncolored vertex v^*_{fix} receives color I. Vertex v_{fix} changes its color from X to B and v^-_{fix} changes its color from X to I.*

(*iii*) *In Line 13 the previously uncolored vertex v^-_{fix} receives color I. Vertex v_{fix} changes its color from X to B and v^*_{fix} changes its color from X to I.*

Proof. It is easy to check that `PA` always assigns colors to the currently processed vertex v, except in those lines stated in the Observation.

Note that there are only two lines, Line 10 and 14, when vertices are placed into X. Part (*i*) is then immediate.

Let v be the currently processed vertex which is eventually colored I in Line 9. Its partner, v^* was colored to X at a point when v was not yet colored. Hence v^* was not colored with X in Line 10, where partners together are colored with X, but it had to be colored in Line 14 where the first and last vertex of a component is colored with X. Thus v^* is either a first or a last vertex. If v^* was a last vertex, then, since its partner, v, is uncolored at the time, `FirstVertex` would select v as the next first vertex and `PA` would color v in Line 4 and not in Line 9. So

v^* must be a first vertex. Unless v^* is the very first vertex, according to Observation 4.2(i), its partner, v, should have been colored already, which it is not, a contradiction. Hence v^* is the very first vertex and part (ii) follows.

For part (iii), suppose that v is the currently processed vertex which is eventually colored I in Line 13. We know that v^+ is a first vertex, which has color X right before v is processed. v^+ had to receive its color X in Line 10 together with its partner. This is a contradiction unless v^+ is the very first vertex, since, according to `FirstVertex` and Lines 4 or 5, a first vertex gets colored right after its partner with the *opposite color*. Hence v^+ is the very first vertex and part (iii) follows. □

Let us collect some direct implications of Observation 4.3.

Corollary 4.1. (i) *A B-vertex is never recolored.*

(ii) *An I-vertex can only change its color to X. In this case it had an uncolored neighbor.*

(iii) *An X-vertex can be recolored to B only if it is the very first vertex v_{fix} and $d(v_{fix}) = 3$.*

(iv) *An X-vertex can be recolored to I only if its X-neighbor is v_{fix} and $d(v_{fix}) = 3$.*

After these preparations we are ready to start the actual proof of Lemma 4.3.

Property (i) The first property of Lemma 4.3 is certainly true at the initialization of `PA`, we must check that the algorithm maintains it. A vertex v can be added to I in Lines 3, 5, 7, 9, or 13. In each of these cases it is easy to check that all the neighbors of v are in B or uncolored. For Lines 9 and 13 note that first we distribute an X-edge between B and I such that the neighbor of v in this X-edge gets color B. (That is we call $Distribute(v^*, B)$ for the X-edge $\{v^*, v^{*-}\}$ in Line 9 and $Distribute(v^+, B)$ for the X-edge $\{v^+, v^{+*}\}$ in Line 13). Distributing an X-edge does not create any conflict with property (i), provided the property was true up to that point. Then we put v into I knowing that *all* its neighbors are in B or uncolored.

Vertices are put into X in Lines 10 and 14; always an uncolored vertex v, together with one of its neighbors z. It is easy to check that in both of these lines all neighbors of v except z are in B or uncolored.

To maintain property (i) it is enough to verify that before processing v, z was in I. In Line 10 we know that z is the partner of v and is colored I or X, in fact Line 9 excludes that $z \in X$. In Line 14 we know that z is equal to v^+ and is colored I or X, and Line 13 excludes that $z \in X$.

In conclusion, property (i) is valid throughout the algorithm.

Property (ii) Why is property (ii) valid? The "triangle rule" on Line 8 ensures that the vertices we move to X in Line 10 are not part of the same triangle. In Line 14 we move the last and first vertices v and v^+, respectively, of a component of $G - M$ into X. We must check that neither $\{v, v^+, v^{++}\}$ nor $\{v^-, v, v^+\}$ induces a triangle in G. If $\{v, v^+, v^{++}\}$ was a triangle then, since no component of $G - M$ is a triangle, v^{++} has to be the partner of v. Then Line 11 ensures that $v^* = v^{++}$ and v are not in the same triangle. Suppose now that $\{v^-, v, v^+\}$ induces a triangle. Again, since no component of $G - M$ is a triangle, v^+ has to be the partner of v^-. Unless v^+ is the very first vertex, v^- cannot be the partner of v^+, since, according to Observation 4.2(i), v^+ and its partner has to be in a different component of $G - M$. Finally, if v^+ is the very first vertex, then according to the orientation of v^+'s component (see Line 1) $\{v^-, v, v^+\}$ does not form a triangle. Hence property (ii) is valid.

Property (iii) To derive the bound on the order of the B-components we list the six reasons a vertex u is colored B. In the following we emphasize some property of each, which follow immediately from `PA` and Corollary 4.1. We will implicitly refer to these properties throughout the remainder of this section.

- "very first"-B: it is given in Line 3; u is the very first vertex v_{fix}, $u^+ \in I \cup X$.
- "first"-B: it is given in Line 4; u is the first vertex colored in its cycle, $u^+, u^* \in I \cup X$
- "triangle"-B: it is given in Line 8; u and u^* are in the same triangle and u^* is already colored with an I (by the end u^* might change its color to X).
- "last"-B: it is given in Line 12; u is the last vertex colored in its cycle, whose coloring started with I or X, $u^+ \in I \cup X$.

- "special"-B: it is given in Lines 9 and 13; u is the very first vertex v_{fix}. $u^-, u^* \in I$, $u^+ \in B$, $u^{++} \in I \cup X$.

- "standard"-B: it is given in Line 6; $u^- \in I \cup X$ unless u^- is a "special"-B and $u^+ \in I \cup X$.

Every B-colored vertex has a exactly one of these six reasons why it is colored a B. Note that a B-colored last vertex is not necessarily a "last"-B, it could be a "standard"- or "triangle"-B. Also, a B-colored very first vertex is not necessarily a "very first"-B, but can also be a "special"-B.

We call a B-component of a component C of $G - M$ a *segment.* Let $\tilde{C}$ be the component C together with the edges of G of the form $\{v, v^{++}\}$ for $v \in V(G)$ (such edges we call *extended edges*). Note that every triangle contains an extended edge. We call a B-component of $\tilde{C}$ an *extended segment.*

Proposition 4.5. *Any extended segment contains at most* 4 *vertices.*

Proof. First let us show the following facts.

Claim 4.2. (*i*) *Suppose u^-, u, and u^+ are all colored B for some $u \in V(G)$. Then u is a "triangle"-B. In particular its partner is in $I \cup X$.*

(*ii*) *Let v_1, v_2, v_3, v_4, v_5 be five distinct, consecutive vertices along some component C in $G - M$ which are colored $B, B, I/X, B, B$, in this order. Then v_2 cannot be adjacent to v_4.*

Proof. (*i*) For a vertex v which is a "standard"-B, "first"-B, "very first"-B, "last"-B, or "special"-B, either v^- or v^+ is in $I \cup X$.

(*ii*) Let us suppose that v_2 is adjacent to v_4 and the orientation of the cycle is passing through these vertices from left to right (with possibly starting/ending among them).

The vertex v_2 is not a "triangle"-B since $v_2^* = v_4$ is not in $I \cup X$. If v_2 is a "standard"-B, then v_1 has to be a "special"-B, since $v_1 \notin I \cup X$. In any case, the first vertex colored in C is either v_1, v_2 or v_3. This implies that v_5 is neither a "first"-B nor a "very first"-B nor a "special"-B. If v_5 was a "last"-B, then $v_5^+ \in I \cup X$. Also, v_5^+ is the first vertex of C so $v_5^+ = v_1$ which has color B, a contradiction. If v_5 was a "standard"-B, then v_4 should be in $I \cup X$ or should be a "special"-B, neither of which is the case. Hence v_5 is a "triangle"-B. Its partner cannot be v_3, since then $\{v_2, v_3, v_4, v_5\}$ would induce a generalized diamond. So its partner is v_7 (the other vertex distance two away from v_5 along C) which then must have been colored already when we arrive to v_5. Hence the first vertex colored in C had to be either v_6 or v_7. Since v_7, as the partner of a "triangle"-B, is in $I \cup X$, $v_7 \neq v_1, v_2, v_4$. Also, $v_7 \neq v_3$ since our assumption about the v_i's being distinct. This contradicts that the first vertex of C is among v_1, v_2, and v_3. □

Part (i) immediately implies that a segment of length 5 does not exist.

Let S be an extended segment and classify the cases according to a longest segment S' it contains.

If S' is of order 1 then obviously S is of order at most 2.

If S' is of order two, then by part (ii) of Claim 4.2 S cannot contain more segments of order two, only possibly two more segment of order one. Hence its order is at most $1 + 2 + 1 = 4$.

If S' is of order 3, then again by part (ii) it cannot be joined to a segment of order at least two. Moreover it cannot be joined to segments of order one both ways, because, by part (i), at least one way it is closed by a triangle (no generalized diamonds!).

If S' is of order 4 then by part (i) both endpoints participate in a triangle and they cannot extend the segment further, because G contains no generalized diamonds. □

A vertex v of an extended segment S is called a *potential connector* if its partner v^* exists, $\{v, v^*\}$ is not an extended edge, and v^* either has color B or is uncolored at the time when the coloring of the component of $G - M$ containing S is concluded. Observe that two extended segments can be connected only via their respective potential connectors.

Proposition 4.6. (i) *If v is a potential connector of extended segment S which does not contain a "special"-B then $v^- \notin S$.*
Every extended segment contains at most one potential connector. In particular, every extended segment is adjacent to at most one other extended segment in G.

(ii) *No extended segment of order at least three is adjacent to another extended segment of order at least three.*

Proof. Let v be a potential connector of extended segment S, $|S| \geq 2$. We claim that v is a "standard"-B.

If v was a "first"-B, "triangle"-B, or "special"-B, then v^* is in $I \cup X$ right after we colored v with B, so v is not a potential connector.

If v was a "last"-B, then it is colored in Line 12. Since v^* exists and $\{v, v^*\}$ is not part of a triangle, we have that $v^* \in I \cup X$ at the time of the coloring. Hence v is not a potential connector.

If $v = v_{\text{fix}}$ was a "very-first"-B, then $v^+ \in I \cup X$. Since $\{v, v^+\} \in E(G)$ (see the orientation rule in Line 1), v^*_{fix} exists, and $d(v_{\text{fix}}) = 2$ (see Line 2), we have that $\{v^-, v\}$ is not an edge of G. Since $\{v, v^*\}$ is not an extended edge, S consists only of a single vertex.

Let us now show Part (i) of Proposition 4.6. Let S be an extended segment not containing a "special"-B with a potential connector v. Since v is a "standard"-B and v^- is not a "special"-B, $v^- \in I \cup X$ and in particular is not in S.
Suppose now that an arbitrary extended segment S contains two potential connectors u and w. In particular $u^*, w^* \notin S$. Then either u^- or w^- has to be in S (otherwise u and w could not be in the same extended segment). Assume that, say, $u^- \in B$. In accordance with the above u is a "standard"-B. Hence u^- must be a "special"-B and $u^+ \in I \cup X$. Moreover u^{-*} and u^{--} are both contained in $I \cup X$. Thus $S = \{u, u^-\}$ and u^- is not a potential connector, a contradiction.
Let us now proceed with the proof of part (ii). Suppose there are two distinct extended segments S and S', each of order at least 3, contained in the same B-component C of G. If S contained a "special"-B vertex v (which is the very first vertex) then v^+ is the only neighbor of v which is in B. Also, since $v^{++} \in I \cup X$ and $|S| \geq 3$, the partner of v^+ has to be v^{+++} and have color B. It is easy to see that $v^{++++} \in I \cup X$, so C is equal to $S = \{v, v^+, v^{+++}\}$.

Hence we can assume that neither S nor S' contains a "special"-B vertex. Suppose further that `PA` colors S prior to S'. According to (i), C does not contain any other vertex besides the vertices of S and S'. Let us denote the potential connectors of S and S' by w and w', respectively. Hence $w^* = w'$, $w'^* = w$ and $\{w, w'\} \in E(G)$.

We will derive a contradiction by showing that $w' \in I \cup X$.

Claim 4.3. *Let S be an extended segment of order at least three, which does not contain a "special"-B vertex. Then S contains a last vertex v_l.*

We postpone the proof of this Claim 4.3 a little and continue with the proof of (*ii*).

After having colored the last vertex $v_l \in S$ of a component of $G-M$ containing the extended segment S, `FirstVertex`(G, v_l, I, X, B) searches for a vertex u with an uncolored partner to continue the coloring with u^*. The potential connector w has an uncolored partner, w', and we claim that `FirstVertex`(G, v_l, I, X, B) will arrive to w and will output $w^* = w'$ as the new first vertex. If v_l^* is uncolored then v_l is the unique potential connector of S, $v_l = w$. Otherwise the algorithm `FirstVertex`(G, v_l, I, X, B) starts stepping backwards on C looking for a vertex of color B with an uncolored partner (c.f. Line 1 of `FirstVertex`). We claim that the first such vertex is w. By Proposition 4.6(i) we have that $w^- \notin S$, and $\{w, w^{--}\} \notin E(G)$, since w is a potential connector, so $w^{--} \notin S$. Hence there is a $v_l w$-path $v_l = p_1 \cdots p_m = w$ in S such that $p_{i+1} = p_i^-$ or p_i^{--} for every $i = 1, \ldots, m-1$. `FirstVertex`(G, v_l, I, X, B) will consider all vertices of C in a backward direction from v_l to w. Vertices p_i with $i < m$ are not eligible since they have a colored partner. Other vertices between v_l and w are outside of S and thus are contained in $I \cup X$. Eventually `FirstVertex`(G, v_l, I, X, B) reaches vertex w. According to our assumption $w' \in S'$ has not yet been colored, thus `FirstVertex`(G, v_l, I, X, B) chooses w' to be colored next. Then w' is colored I according to Line 5 of `PA`, a contradiction.

We thus concluded the proof of Proposition 4.6. □

Proof of Claim 4.3. Suppose S with $|S| \geq 3$ does neither contain a "special"-B nor v_l.

Then S certainly does not contain a "last"-B vertex.

If S contained a "very-first"-B vertex v, then $v^- = v_l \notin S$ and $v^+ \in I \cup X$. Since $|S| \geq 3$, $v^* \in S$ and at least one of v^{*+} and v^{*-} is in S. First assume that $v^* = v^{++}$. It is easy to check, that then $v^{*+} \in I \cup X$, which is a contradiction since $v^{*-} = v^+ \in I \cup X$. Now assume that $v^* = v^{--}$. Obviously, v^{--} is not a "very-first"-B, not a "first"-B, not a "special"-B and not a "last"-B. Also, v^{--} is not a "triangle"-B since its partner, v, is not in $I \cup X$. Therefore v^{--} has to be a "standard"-B. Then v^{---} is in $I \cup X$ since it is certainly not a "special"-B (it is not the very first vertex). This is then a contradiction to $|S| \geq 3$ since by our assumption $v^{--+} = v^- \in I \cup X$. We can thus conclude that S does not contain a "very-first"-B.

S does not contain a "first"-B vertex v either, otherwise $S = \{v\}$.

Indeed, $v^- = v_l$ and $v^+ \in I \cup X$ and, according to Observation 4.2(i), v^* is contained in a different component of $G - M$.

From now on we assume that every vertex of S is either a "triangle"-B or a "standard"-B. Suppose S contains a "triangle"-B vertex u, such that $u^* = u^{++}$. Then $u^{++} \in I \cup X$ and u^+ has to be in B because property (i) and (ii) of Lemma 4.3 hold. It follows that $u^+ \in S$, but u^+ neither can be a "standard"-B since its predecessor is not in $I \cup X$ nor can be a "triangle"-B because $\{u, u^+, u^{++}, u^{+*}\}$ would form a generalized diamond. We conclude that S does not contain a "triangle"-B vertex u, such that $u^* = u^{++}$. Suppose now that S contains a "triangle"-B vertex v, such that $v^* = v^{--}$. Then $v^* \in I \cup X$. Vertex v^{-*} is not in S otherwise $\{v, v^-, v^{--}, v^{-*}\}$ would be a generalized diamond. Since $|S| \geq 3$, vertex v^+ has to be in B. It cannot be a "standard"-B because its predecessor is not in $I \cup X$. Vertex v^+ also cannot be a "triangle"-B since we already saw that its partner cannot be v^{+++} and if its partner was v^- then $\{v^{--}, v^-, v, v^+\}$ would form a generalized diamond.

Thus the vertices in S are all "standard"-B vertices, each forming a (not extended) segment of order 1. Each such segment can connect to at most one other such segment via an extended edge. Thus $|S| \leq 2$, a contradiction. □

Proposition 4.5 and Proposition 4.6 immediately imply part (iii) of Lemma 4.3.

Property (iv) We can assume that $d(v_{\text{fix}}) = 2$. The vertex v_{fix} is contained in c after Line 3. If $c = B$, then according to Corollary 4.1(i), v_{fix} is not recolored at all. If $c = I$, then according to Corollary 4.1(ii) and (iii), v_{fix} can be recolored to X, but not to B. □

Note again that there are graphs with no 5-abmc 2-colorings, for instance the graph in Figure 4.5.

4.1.4 Abmc 2-Colorings of Graphs G with $\Delta(G) > 3$

Let abmc(Δ, n) be the smallest integer f such that every n-vertex graph of maximum degree Δ is f-abmc 2-colorable. While every graph of maximum degree at most three admits a C-abmc 2-coloring, with $C = 22$ being finite, the graph $G_{k,\Delta}$, which consists of k copies of $K_{\Delta-1}$

with the each vertex of the ith copy of $K_{\Delta-1}$ being connected with the corresponding vertex of the $(i+1)th$ copy of $K_{\Delta-1}$, show that such a statement is not possible for graphs with maximum degree larger than three. Figure 4.6 provides an example for $G_{k,4}$. The graph $G_{k,\Delta}$ shows in a strong sense that the monochromatic component order is non-finite, for $\Delta > 3$: in any asymmetric abmc 2-coloring of $G_{k,\Delta}$ there is a monochromatic component whose order is linear in the number of vertices.

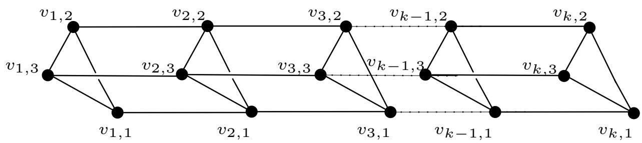


Figure 4.6: The graph $G_{k,4}$.

Proposition 4.7. *For every independent set I of $G_{k,\Delta}$ it holds*

(i) $|I| \leq k$, and

(ii) the graph $G_{k,\Delta} - I$ is connected.

Therefore $\text{abmc}(\Delta, n) \geq \frac{\Delta-2}{\Delta-1}n$, *for $n = k(\Delta - 1)$.*

Proof. We simply observe that from each of the k copies of $K_{\Delta-1}$ we can put at most one vertex into the independent set I. Thus $G - I$ contains one single component with at least $|V(G)| - |I| \geq k\Delta - k = \frac{\Delta-2}{\Delta-1}n$ many vertices. □

Note that for $\Delta = 4$ this is in sharp contrast with Theorem 5.1 applied with $k = 1$ and $l = 1$ for which the monochromatic component order is only logarithmic in the number of vertices. It would be interesting to determine the exact asymptotics of the function $\text{abmc}(\Delta, n)$; we only know of the trivial upper bound $\text{abmc}(\Delta, n) \leq \frac{\Delta-1}{\Delta}n$ (since every graph of maximum degree Δ, except $K_{\Delta-1}$, has an independent set containing at least n/Δ many vertices).

4.1.5 Hardness Results for Abmc 2-Colorings

In this subsection we show the existence of the "hardness jump" of the decision problem whether a graph of maximum degree at most three admits a C-abmc 2-coloring.

Related work Similar hardness jumps of the k-SAT problem with limited occurrences of each variable was shown by Tovey [63] for $k = 3$ and Kratochvíl, Savický, and Tuza [48] for arbitrary k. Let k, s be positive integers. A Boolean formula in conjunctive normal form is called a (k, s)-formula if every clause contains *exactly* k distinct variables and every variable occurs in *at most* s clauses. Tovey showed that every $(3, 3)$-formula is satisfiable while the satisfiability problem restricted to $(3, 4)$-formulas is NP-complete. Kratochvíl, Savický, and Tuza [48] generalized this by establishing the existence of a function $f(k)$, such that every $(k, f(k))$-formula is satisfiable while the satisfiability problem restricted to $(k, f(k)+1)$-CNF formulas is NP-complete. By a standard application of the Local Lemma they obtained $f(k) \geq \left\lfloor \frac{2^k}{ek} \right\rfloor$. After some development [48, 56] the most recent upper estimate on $f(k)$ is only a log-factor away from the lower bound and is due to Hoory and Szeider [43]. Recently new bounds were also obtained on small values of the function $f(k)$ [42]. Observe that the monotonicity of the hardness of the satisfiability problem for (k, s)-formulas is given by definition.

0/1-Colorings

In this subsubsection we present the main ingredient of our hardness reduction, which is common to all our hardness proofs. Our plan is to reduce our problems to 3-SAT. Given a 3-SAT formula F, we construct (in polynomial time) a graph G_F together with a constraint function $c = c_F$, such that (G_F, c) has a so-called 0/1-*coloring* if and only if the formula F is satisfiable.

Let G be a graph and $c : V(G) \to \mathbb{N} \cup \{\infty\}$ be a constraint function. Then a mapping χ from $V(G)$ to $\{0, 1\}$ is called a *0/1-coloring* of (G, c) if the vertices with χ-value 1 induce an independent set and the order of each component C induced by vertices of χ-value 0 is not larger than the constraint of any of its vertices, that is, $c(v) \geq |C|$ for all $v \in C$.

We will assemble G_F from various building blocks, pictured in Figure 4.7 and Figure 4.8. In the following, if the constraint of a vertex is not specified than it is taken to be ∞.

The *not-gadget* NG is just a path $v\bar{v}$ of length one, where v has constraint 1.

The *copy-gadget* $CG(1)$ consists of just one vertex v_1, which is called both the *root* and the *leaf* of the gadget. Let P be a path of length two, where the interior vertex is given constraint 1. For $i \geq 2$, a copy-gadget $CG(i)$ is constructed from $CG(i-1)$ by identifying an arbitrary leaf v_{i-1} of $CG(i-1)$ with one endpoint of each of two copies of P. Note that v_{i-1} is no longer a leaf and we gained two new leaves - the other endpoints of the two copies of P. Thus $CG(i)$ contains exactly i leaves. The root of $CG(i)$ is the vertex v_1 for every i. For more insight see Figure 4.7. Let's collect some simple facts about these gadgets.

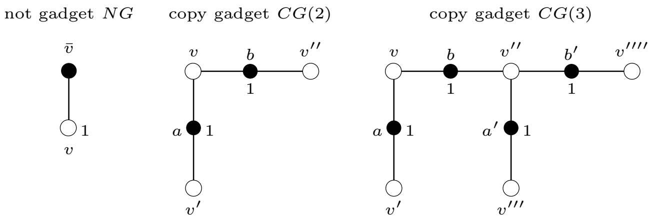


Figure 4.7: Basic building blocks of the graph G_F.

Proposition 4.8. (*i*) *The not-gadget NG is 0/1-colorable. Moreover in any 0/1-coloring of the not-gadget the vertex* $\bar{v}$ *is colored with a different color than vertex* v.

(*ii*) *The copy gadget* $CG(i)$ *is 0/1-colorable. Moreover in any 0/1-coloring of* $CG(i)$*, all i leaves have identical colors with the root of the gadget.*

Proof. For each gadget a 0/1-coloring is indicated on Figure 4.7. All the statements are easily verified. □

For every clause $D = (l_{i_1} \vee l_{i_2} \vee l_{i_3})$ in F we also construct a gadget. The clause-gadget G^*_D as shown in Figure 4.8 contains vertices a_D, b_D, c_D, d_D and a vertex $l_{i,D}$ corresponding to each literal l_i appearing in the clause D. The constraints of $l_{i_1,D}$ and $l_{i_2,D}$ are 2 and the constraints of $l_{i_3,D}$ and b_D are 1.

Proposition 4.9. *An* 0/1*-coloring* χ *of the vertices* $l_{i_1,D}, l_{i_2,D}, l_{i_3,D}$ *of the clause-gadget* G^*_D *is extendable to a 0/1-coloring of* G^*_D *if and only if at least one of* $l_{i_1,D}, l_{i_2,D}, l_{i_3,D}$ *received the color* 1.

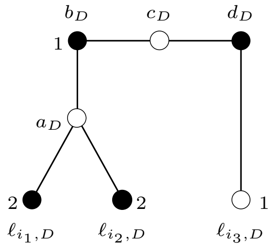


Figure 4.8: The clause-gadget G^*_D for clause $D = (l_{i_1} \vee l_{i_2} \vee l_{i_3})$.

Proof. Let us first suppose that $\chi(l_{i_j,D}) = 0$, for all $j \in \{1, 2, 3\}$ and try to extend χ to a 0/1-coloring of G^*_D. Then a_D must be colored 1, since $l_{i_1,D}$ and $l_{i_2,D}$ have constraint 2. Since 1-vertices form an independent set, $\chi(b_D) = 0$. The constraint of b_D implies that $\chi(c_D) = 1$, which then implies that $\chi(d_D) = 0$. Hence $l_{i_3,D}$ is contained in a 0-component of order at least 2, which contradicts that its constraint is 1. We conclude that an extension to a 0/1-coloring of G^*_D is not possible.

Secondly, we show that an extension exists if some $l_{i_j,D}$ is colored 1 in χ.

First suppose that $\chi(l_{i_1,D}) = \chi(l_{i_2,D}) = 0$ and $\chi(l_{i_3,D}) = 1$. Then $\chi(a_D) = \chi(c_D) = 1$, $\chi(b_D) = \chi(d_D) = 0$ is a 0/1-coloring of G^*_D.

Now let $(\chi(l_{i_1,D}), \chi(l_{i_2,D})) \neq (0, 0)$. Then $\chi(a_D) = 0$, $\chi(b_D) = 1$, $\chi(c_D) = 0$ and either $\chi(d_D) = 1$ if $\chi(l_{i_3,D}) = 0$ or $\chi(d_D) = 0$ if $\chi(l_{i_3,D}) = 1$ again results in a 0/1-coloring of G^*_D.

□

Now we are ready to define the graph G_F together with its constraint function c_F. First for each clause D we construct the *extended clause-gadget* G_D by taking the clause-gadget G^*_D and identify each vertex $l_{i,D}$ corresponding to a negated variable $\bar{x}_i$ in the clause D with the leaf $x_{i,D}$ of a not-gadget. We call this the *extended clause-gadget* of the clause D. See Figure 4.9 for an example.

Proposition 4.10. *An assignment α satisfies the clause D if and only if there is a 0/1-coloring of the extended clause-gadget G_D such that the vertices corresponding to the variables receive the colors the assignment α gives them.*

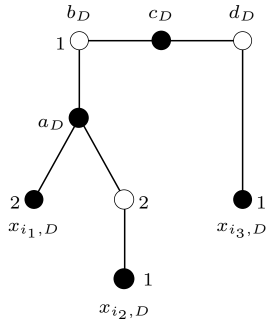


Figure 4.9: The extended clause-gadget G_D for the clause $D = (x_{i_1} \vee \bar{x}_{i_2} \vee x_{i_3})$.

Proof. It is easy to verify based on the properties of the not gadget and the properties of the clause-gadget discussed in the previous two proposition. □

The graph G_F is put together from these extended clause-gadgets of the clauses of F with the help of one copy-gadget for each variable of F. Formally G_F is constructed as follows. We take the disjoint union of one extended clause-gadget for each clause in F. Then we add one copy-gadget C_x for each variable x. If the variable x occurs in i_x clauses than the leaves of the copy-gadget $C_x \cong CG(i_x)$ are identified with the vertices corresponding to the same variable x in the extended clause-gadgets.

Obviously, the graph G_F can be constructed in polynomial time in the number of clauses and variables of F.

The main theorem of the section is now a simple consequence of the above.

Theorem 4.3. (*i*) *G_F is 0/1-colorable if and only if F is satisfiable.*

(*ii*) *$\Delta(G_F) \leq 3$ and every vertex v of G_F with $c(v) < \infty$ has degree at most* 2.

Proof. Let α be a satisfying assignment of F. Then we start defining a 0/1-coloring of G_F by assigning color $\alpha(x)$ to the root of the copy-gadget C_x corresponding to the variable x. This can be extended to an 0/1-coloring of the copy-gadgets by part (*ii*) of Proposition 4.8 where the leaves receive the same color as their respective roots. All these leaves are identified with a vertex of an extended clause-gadget. Since α satisfies all the clauses of F, these partial colorings of the extended clause-gadgets can be extended to a 0/1-coloring of the whole gadget (cf. Proposition 4.10) and thus the whole graph G_F is 0/1-colored.

Let now χ be a 0/1-coloring of G_F. We claim that the colors given to the roots of the copy-gadgets corresponding to the variable is a satisfying assignment of F. By part (*ii*) of Proposition 4.8 all the leaves are the same color as their roots in the copy-gadget. By Proposition 4.10 every extended copy gadget has a satisfying assignment, so we are done.

Part (*ii*) is straightforward. □

Hard $(3, C)$-ABMCCol

We will use the core graph G_F defined above to construct in polynomial time a graph *ABMCColGraph*(F) which is C-bmc 2-colorable if and only if the formula F is satisfiable.

For a C-abmc 2-coloring we denote the color-class forming an independent set by I and the color-class spanning components of order at most C by B.

Definition 4.1. *Let $C \geq 2$ and $\Delta \geq 1$ be integers. A graph G is called* (Δ, C)-forcing *with forced vertex $f \in V(G)$ if*

(*i*) *$\Delta(G) \leq \Delta$ and f has degree at most $\Delta - 1$,*

(*ii*) *G is C-abmc 2-colorable, and*

(*iii*) *f is contained in I for every C-abmc 2-coloring of G.*

Lemma 4.4. *For any integer $\Delta \geq 1$ and integer $C \geq 2$ the decision problem (Δ, C)-ABMCCol is* NP*-complete provided a (Δ, C)-forcing graph exists.*

Proof. We assume the existence of a (Δ, C)-forcing graph H, hence $\Delta \geq 3$. We will show that there is a polynomial time algorithm which, given a 3-CNF formula F, produces a graph *ABMCColGraph*(F) of maximum degree at most Δ such that F is satisfiable if and only *ABMCColGraph*(F) has a C-abmc 2-coloring.

The *base-gadget* BG_l contains l disjoint copies $H_1, \ldots, H_l$ of the (Δ, C)-forcing graph H, the forced vertex f_i of copy H_i is joined to a new vertex t_i for $i \in [l]$, and the vertices $t_1, t_2, \ldots, t_l$ form a path. The vertex t_1 (of degree two) is called the *sink* of the base-gadget.

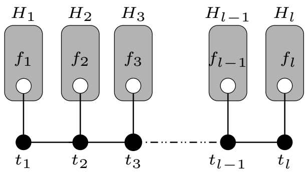


Figure 4.10: The base gadget BG_l

Proposition 4.11. *The base gadget BG_l is C-abmc 2-colorable for every $l \leq C$. Moreover in any C-abmc 2-coloring of BG_l, $l \leq C$, the sink is contained in a B-component of order l.*

Proof. A C-abmc 2-coloring of the base-gadget is indicated on Figure 4.10. In any C-abmc 2-coloring χ of the base-gadget BG_l, $\chi(t_i) = B$, since f_i is forced to be contained in I. Thus the vertices t_i for $i \in [l]$ form a B-component of order exactly l. □

Now $ABMCColGraph(F)$ is obtained from G_F by connecting each vertex with constraint 1 to the sink of a base-gadget BG_{C-1}, and connect each vertex with constraint 2 to the sink of a base-gadget BG_{C-2}. Note that the obtained graph has maximum degree Δ, according to part (ii) of Theorem 4.3. Note also that G_F is 0/1-colorable if and only if $ABMCColGraph(F)$ has a C-abmc 2-coloring. A C-abmc 2-coloring of $ABMCColGraph(F)$ restricted to $V(G_F)$ is a 0/1-coloring if we exchange the color I to 1 and the color B to 0. Conversely a 0/1-coloring of G_F can be extended to a C-abmc 2-coloring of $ABMCColGraph(F)$ by identifying 1 with I, and 0 with B, and extending this coloring to the base-gadgets appropriately (such coloring exists by Proposition 4.11). □

$(3, C)$-Forcing Graphs

Let $\mathcal{G}_C$ denote the family of graphs with maximum degree at most three that are not C-abmc 2-colorable.

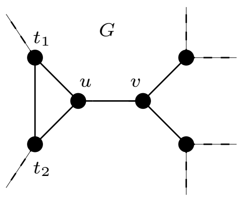
Lemma 4.5. *For all $C \geq 2$, if $\mathcal{G}_C \neq \emptyset$ then there is a $(3, C)$-forcing graph.*

Proof. Let us assume first that $C \geq 6$. By Lemma 4.1 we can assume that any member of $\mathcal{G}_C$ contains a triangle.

Let us fix a graph $G \in \mathcal{G}_C$ which is minimal with respect to deletion of edges. Let T be a triangle in G with $V(T) = \{t_1, t_2, u\}$ and $e = \{u, v\}$ be the unique edge incident to u not contained in T. We split e into e_1, e_2 with $e_1 = \{u, f\}$ and $e_2 = \{f, v\}$ and denote this new graph by H (cf. Figure 4.11). We claim that H is $(3, C)$-forcing graph with forced vertex f. H is C-abmc 2-colorable since the minimality of G ensures that $G - e$ has a C-abmc 2-coloring while the non C-abmc 2-colorability of G ensures that the colors of u and v are the same on *any* C-abmc 2-coloring of $G - e$. So *any* C-abmc 2-coloring χ of $G - e$ can be extended to a C-abmc 2-coloring of H by coloring f to the opposite of the color of u and v. Moreover, any such extension is unique. If $\chi(u) = \chi(v) = I$, then obviously $\chi(f) = B$. If $\chi(u) = \chi(v) = B = \chi(f)$ and χ is a C-abmc 2-coloring of H, then χ restricted to $V(G)$ is a C-abmc 2-coloring of G, a contradiction.

Thus in any C-abmc 2-coloring χ_H of H, $(\chi_H(u), \chi_H(f), \chi_H(v))$ is either (I, B, I) or (B, I, B).

We denote by v_1, v_2 the neighbors of t_1 and t_2, respectively, not contained in T (might be $v_1 = v_2$). Suppose the vertices (u, f, v) of H can be colored with (I, B, I). But then $\chi_H(t_1) = \chi_H(t_2) = B$.



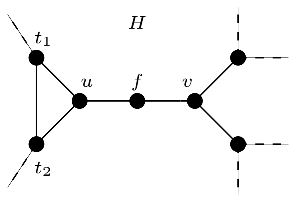


Figure 4.11: Splitting $e = \{u, v\}$ into $e_1 = \{u, f\}$ and $e_2 = \{f, v\}$

Case (i)**:** If $\chi_H(v_1) = \chi_H(v_2) = I$ then we define a C-abmc 2-coloring χ_G for G as follows:
$\chi_G(x) = \chi_H(x)$ for all $x \in V(G) \setminus \{u\}$ and $\chi_G(u) = B$.
Case (ii)**:** Without loss of generality $\chi_H(v_1) = B$. We define a C-abmc 2-coloring χ_G for G as follows:
$\chi_G(x) = \chi_H(x)$ for all $x \in V(G) \setminus \{t_1, u\}$, $\chi_G(t_1) = I$, and $\chi_G(u) = B$.
Indeed, the B-component containing t_2 did not increase, since $\chi_G(t_1) = \chi_G(v) = I$ and in H $\chi_H(t_1) = B$.

In both cases G would be C-abmc 2-colorable, a contradiction. Thus in any C-abmc 2-coloring of H the vertices (u, f, v) are colored (B, I, B). The vertex f is contained in I and is of degree 2, hence H is a $(3, C)$-forcing graph with forced vertex f.

For $2 \leq C \leq 5$ we explicitly construct $(3, C)$-forcing graphs. The graph G in Figure 4.12 is $(3, C)$-forcing for $C \in \{2, 3\}$. First we observe that G is indeed 2-abmc 2-colorable: just take $I = \{f, t'_2, t''_3\}$ and $B = V(G) \setminus I$. It is also not hard to check that there is no 3-abmc 2-coloring where vertex f is contained in B. Suppose there is a 3-abmc 2-coloring of G in which f is contained in B. If t'_1, t''_1 are contained in I then no other vertex is contained in I and we have a B-component of order four. On the other hand if t'_1, t''_1 are both contained in B then we have a B-component of order at least five. So without loss of generality t'_1 is contained in I and t''_1 is contained in B. The B-components on both triangles are connected, thus we have a B-component of order five again.

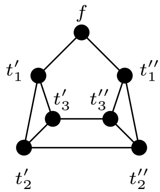


Figure 4.12: $(3, C)$-forcing graph for $C \in \{2, 3\}$

Next we construct a graph H which is $(3, C)$-forcing for $C \in \{4, 5\}$. First let us show that for the graph H^* in Figure 4.13, (i) there is a 4-abmc 2-coloring and (ii) there is no 5-abmc 2-coloring where u is contained in I.

(i) The vertex-partition defined by $I = \{t_{1,2}, t_{2,4}, t_{3,1}, t_{4,5}, t_{5,3}\}$ and $B = V(H^*) \setminus I$ is a 4-abmc 2-coloring of H^*,

Note that in this coloring $u = t_{1,1}$ is contained in a B-component of order two.

(ii) The key observation is that in any 5-abmc 2-coloring of H^*, for a triangle T_i with $V(T_i) = \{t_{i,j}, t_{i,k}, t_{i,l}\}$, if $t_{i,j}$ is contained in I then at least one of $t_{k,i}, t_{l,i}$ is contained in I. Suppose not, then the at least six B-vertices of the three triangles T_i, T_k, and T_l are contained in the same B-component.

Thus if $t_{1,1}$ is contained in I in a 5-abmc 2-coloring of H^*, then without loss of generality $t_{3,1}$ is contained in I as well. This then implies that one of $t_{4,3}$ and $t_{5,3}$, say $t_{5,3}$ is in I. Hence $t_{1,2}, t_{5,2} \in B$ and $t_{3,4}, t_{5,4} \in B$. These, together with the key observation imply that $t_{2,4} \in B$ and $t_{4,2} \in B$, respectively. Finally, all neighbors of triangle T_4 are in B, which together with the key observation imply that all vertices of T_4 are in B, so the B-component of T_4 has order at least six.

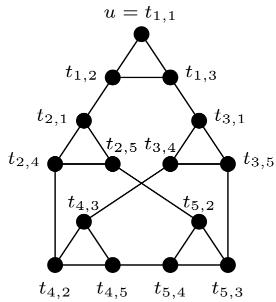


Figure 4.13: Graph H^*

The graph H is pictured on Figure 4.14. The subgraphs H_i, $i \in \{1, \ldots, 4\}$, are copies of the graph H^*, with u_i corresponding to vertex u of H^*.

The coloring of part (i) can easily be extended to a 4-abmc 2-coloring of H.

As we have seen, in any 5-abmc 2-coloring of H all $u_i \in B$. Thus, similarly to the key observation above, v and w are contained in B. Hence if f was in B, then its B-component would be of order at least seven, a contradiction. Thus in any 5-abmc 2-coloring of H the vertex

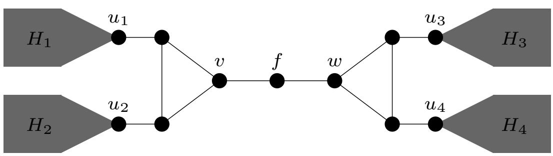


Figure 4.14: $(3, C)$-forcing graph for $C \in \{4, 5\}$

f is contained in I, so H is $(3, C)$-forcing for $C \in \{4, 5\}$. □

Note that $(3, C)$-ABMCCol is obviously trivial for all C with $\mathcal{G}_C = \emptyset$, so Theorem 4.2 follows immediately from Lemma 4.5 and Lemma 4.4.

$(4, C)$-Forcing Graphs

Theorem 4.4. $(4, C)$-ABMCCol is NP-complete for every $2 \leq C < \infty$.

In order to show the Theorem 4.4, we are going to show the existence of (Δ, C)-forcing graphs.

Lemma 4.6. *For all $\Delta \geq 4$ and all $C \geq 2$ there is a (Δ, C)-forcing graph.*

Proof. Suppose first that $C = 2k - 2$. Let us look at the graph $G_{k,4}$ in Figure 4.6. This graph is not $(2k - 1)$-abmc 2-colorable, since in any triangle $v_{i,1}, v_{i,2}, v_{i,3}$ at most one vertex is contained in the independent set I. The two other vertices are contained in B and since there are three edges connecting this triangle to a neighboring triangle the components in $G_{k,4}[B]$ of all triangles of $G_{k,4}$ are connected and form one big component in $G_{k,4}[B]$. Removing the edge $e = \{v_{1,1}, v_{1,2}\}$ makes $G_{k,4}$ $(2k-2)$-abmc 2-colorable and in any such coloring χ, $\chi(v_{1,1}) = \chi(v_{1,2}) = I$. Thus $G_{k,4} - e$ is $(4, 2k - 2)$-forcing, with forced vertex $v_{1,1}$ (or $v_{1,2}$).

Similarly, $G_{k,4}$ with an additional vertex v adjacent to $v_{k,1}, v_{k,2}, v_{k,3}$, denote this graph by H, is not $(2k)$-abmc 2-colorable, hence $H - e$ is $(v, 2k - 1)$-forcing again with forcing vertex $v_{1,1}$ or $v_{1,2}$.

□

Combining Lemma 4.6 and Lemma 4.4 concludes the proof of Theorem 4.4.

4.2 Symmetric Bmc 2-Colorings

Recall that a C-sbmc 2-coloring is a (C, C)-bmc 2-coloring.

Analogously to the asymmetric case we define (Δ, C)-SBMCCol to be decision problem whether a given graph G of maximum degree at most Δ allows a C-sbmc 2-coloring. Note here that $(\Delta, 1)$-SBMCCol is simply the problem of deciding whether a graph of maximum degree Δ is bipartite.

We claim that every graph with maximum degree at most three admits a 2-sbmc 2-coloring. In fact the two vertex classes V_1 and V_2 of a maximum edge-cut $\mathcal{V} = (V_1, V_2)$ of a graph G with $\Delta(G) \leq 3$ immediately yield a 2-sbmc 2-coloring of G. For that suppose there is a vertex v, without loss of generality contained in V_1, with $d_{V_1}(v) \geq 2$. This contradicts the maximality of (V_1, V_2), since the edge-cut $(V_1 \setminus \{v\}, V_2 \cup \{v_1\})$ obviously contains more edges.

Investigations about bmc colorings for the symmetric case were first studied by Kleinberg, Motwani, Raghavan, and Venkatasubramanianby in [46] and independently by Alon, Ding, Oporowski, and Vertigan [10]. The authors of [10] showed that any graph of maximum degree 4 has a 2-coloring such that each monochromatic component is of order at most 57. This was improved by Haxell, Szabó, and Tardos [39], who showed that a 2-coloring is possible even with monochromatic components of order 6, and such a 6-sbmc 2-coloring can be constructed in polynomial time (the algorithm of [10] is not obviously polynomial). In [39] it is also proved that the family of graphs of maximum degree 5 is 17617-sbmc 2-colorable. Unfortunately the proof does not imply an efficient algorithm for finding such a sbmc 2-coloring.

In Subsection 4.2.1 we improve on this results in two ways. On the one hand we decrease the component order from $17,617$ to 1908, on the other hand we derive a polynomial-time algorithm that finds a C-sbmc 2-colorings with some large constant C.

Theorem 4.5. *For every n-vertex graph G of maximum degree 5 the following holds:*

(*i*) *G is 1908-sbmc 2-colorable, and*

(*ii*) *a C-sbmc 2-coloring of G can be found in polynomial-time in n, $C \leq 94371840$.*

The authors of [10] showed that a similar statement cannot be true for the family of graphs of maximum degree 6, as for every constant C there exists a (planar) 6-regular graph G_C such that in any 2-coloring of $V(G_C)$ there is a monochromatic component of order larger than C.

For the problem (Δ, C)-SBMCCol we make progress in the direction of establishing a sudden jump in hardness. We just saw that $(3, C)$-SBMCCol is trivial already for $C = 2$, so the first interesting maximum degree is $\Delta = 4$. From the result of [39] mentioned earlier it follows that (4, 6)-SBMCCol is trivial. Here we show that $(4, C)$-SBMCCol is NP-complete for $C = 2$ and $C = 3$, and that $(6, C)$-SBMCCol is NP-complete for $C \geq 2$. We do not know about the hardness of the problem $(4, C)$-SBMCCol for $C = 4$ and $C = 5$. Again, we do not know any *direct* reason for the monotonicity of the problem. I.e., at the moment it is in principle possible that (4, 4)-SBMCCol is in P while (4, 5)-SBMCCol is again NP-complete.

Theorem 4.6. *The problems $(4, C)$-SBMCCol, for $C \in \{2, 3\}$ and $(6, C)$-SBMCCol, for $C \geq 2$ are* NP*-complete.*

The proof of the theorem appears in Section 4.2.3.

4.2.1 Sbmc 2-Colorings of Graphs G with $\Delta(G) \leq 5$

In this section we start by proving Theorem 4.5(i) reusing ideas of Haxell, Szabó, and Tardos [39]. We keep track of algorithmic aspects for finding the structures whose existence we are showing along the proof. On the way we improve the bound on the component order in Theorem 4.5(i) from 17617 to 1908. Also we are going to make use of independent transversals (instead of the Local Lemma as in [39]). Ultimately this enables us to apply Theorem 2.1 instead of Theorem 1.1 in order to find a 94371840-sbmc 2-coloring as promised in Theorem 4.5(ii).

Before we prove Theorem 4.5 we start with some initial observations and propositions. An edge-cut is called *k-maximal* if switching the sides of any at most k vertices does not increase the size of $\mathcal{U}$.

Let G be a graph of maximum degree 5, and let $\mathcal{U} = (U_1, U_2)$ be a 3-maximal edge-cut of G. Let $G'(\mathcal{U}) = G[U_1] + G[U_2]$, and let $C_1, \ldots, C_s$ be the components of $G'(\mathcal{U})$. Further we define $W(\mathcal{U}) = \{v \in V(G) \mid d_{G'(\mathcal{U})}(v) = 2\}$ and we denote by $H(\mathcal{U})$ the bipartite subgraph of G consisting of the vertices in $W(\mathcal{U})$ and all the edges of G with one endpoint in $W(\mathcal{U}) \cap U_1$ and the other endpoint in $W(\mathcal{U}) \cap U_2$. The vertex sets of the components of $H(\mathcal{U})$ are called *ladders*. If the edge-cut $\mathcal{U}$ is obvious from the context, then we write G', H and W instead of $G'(\mathcal{U})$, $H(\mathcal{U})$ and $W(\mathcal{U})$.

The following proposition states some properties of ladders in 3-maximal edge-cuts. The same proposition but for maximum edge-cuts has already been shown in [39].

Proposition 4.12 ([39]). *Let $\mathcal{U} = (U_1, U_2)$ be a 3-maximal edge-cut. Then we have* (*i*) $\Delta(G') \leq 2$, *hence each component either forms a cycle or a path,*

(*ii*) $\Delta(H) \leq 3$,

(*iii*) *any two H-neighbors of a vertex $w \in W$ are adjacent in G.*

(*iv*) *for each ladder L, $L \cap U_j$ consists of consecutive elements of some (path or cycle) component C_k of G', for each $j = 1, 2$. In particular ladders, unless they consist of one vertex, have nontrivial intersection with exactly one component of each side of the partition (U_1, U_2).*

(*v*) *if $d_H(w) = 3$, and $w \in U_j \cap L$ for some ladder L, then $U_{3-j} \cap L$ consists only of the three H-neighbors of w.*
Furthermore $|L| \leq 6$.

Proof. (*i*) If the degree of a vertex in G' were at least 3, then putting the vertex into the other class would increase the number of edges going across.

(*ii*) This follows immediately from the definition of the vertex set W and the maximum degree of G being at most five

(*iii*) Suppose on the contrary that $w', w'' \in W$ are two H-neighbors of w that are not adjacent in G. Then switching the classes for w', w'', w would increase the number of edges going across the partition.

(*iv*) Follows directly from (*i*) and (*iii*).

(*v*) by (*iii*), the three H-neighbors of w need to form a triangle in G', which is already a complete component of G'. Thus $U_j \cap L$ can only contain 2 more vertices besides U_i, since any vertex in $U_j \cap L$ is a neighbor of a neighbor if w, thus (again by (*iii*)) a neighbor of w in G' as well. But w has only two G'-neighbors in U_j. □

A main ingredient of the proof of Theorem 4.5 is the following lemma:

Lemma 4.7. *Let G be an n-vertex graph with $\Delta(G) \leq 5$. Then there is a 3-maximal edge-cut $\bar{\mathcal{U}} = (\bar{U}_1, \bar{U}_2)$ of the vertex set of G containing no ladder of size larger than 12. Moreover, $\bar{\mathcal{U}}$ can be found in polynomial-time in n.*

Proof. Let us consider a 3-maximal edge-cut $\mathcal{U}$. The edge-cut $\mathcal{U}$ can obviously be found in polynomial-time in n. We can assume that $\mathcal{U}$ contains a ladder L with $|L| > 12$, otherwise we are done.

By Proposition 4.12(*ii*) and (*v*), $\Delta(H[L]) \leq 2$. If $H[L]$ forms a path, then let v_1 be a endpoint of the path. Otherwise $H[L]$ is a cycle, and we arbitrarily choose $v_1 \in L$. Hence we find vertices $v_2, \ldots, v_k \in L, k \geq 9$ such that v_i is adjacent to v_{i+1} in H for $i \in [k-1]$. We may assume that $v_i \in U_1$ for odd i and $v_i \in U_2$ for even i. By Proposition 4.12(*iii*) we have that v_i and v_{i+2} are adjacent in G (and thus also in G') for $i \in [k-2]$.

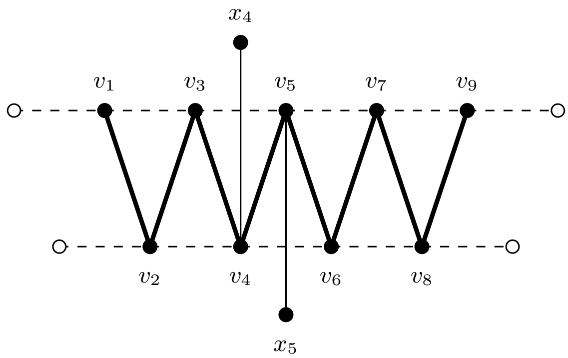


Figure 4.15: The ladder L in $\mathcal{U}$.

We define the edge-cut $\mathcal{U}^{L,i}$ by switching the vertices v_i and v_{i+1} of ladder L, for $i = 4, 5$. For example for $\mathcal{U}^{L,4} = (U_1^{L,4}, U_2^{L,4})$:

$U_1^{L,4} = (U_1 \setminus \{v_5\}) \cup \{v_4\}$ and $U_2^{L,4} = (U_2 \setminus \{v_4\}) \cup \{v_5\}$.

Subsequently the ladder L is always obvious from the context and therefore we often write $\mathcal{U}^i$ when we actually mean $\mathcal{U}^{L,i}$. If $\mathcal{U}^{L,i}$ is again 3-maximal, then on the one hand Proposition 4.12 applies to $\mathcal{U}^{L,i}$ as well, and on the other hand $\mathcal{U}^{L,i}$ contains at least as many edges as $\mathcal{U}$. The latter follows from the fact that $\mathcal{U}^{L,i}$ contains all edges of $\mathcal{U}$ but a subset of the at most four edges of $W(\mathcal{U})$ incident to either v_i or v_{i+1} and not to both. Additionally $\mathcal{U}^{L,i}$ contains the four edges of $G'(\mathcal{U})$ incident to v_i and v_{i+1}.

We denote by `3-MaxCut`$(\mathcal{U})$ an efficient procedure that finds a 3-maximal edge-cut starting at the edge-cut $\mathcal{U}$. A possible implementation for `3-MaxCut`$(\mathcal{U})$ proceeds iteratively as follows. If there is an edge-cut $\mathcal{U}'$ that originates from $\mathcal{U}$ by switching the sides of at most three vertices such that $|\mathcal{U}'| > |\mathcal{U}|$, then replace $\mathcal{U}$ by $\mathcal{U}'$ and start over. Since in each iteration the number of edges in the edge-cut increases. Moreover in each step one has to check for at most a polynomial number of edge-cuts $\mathcal{U}'$. We require that if $\mathcal{U}$ is not provided, then `3-MaxCut`() outputs an arbitrary 3-maximal edge-cut of G.

Subsequently we call a ladder *long* if it contains more than 8 vertices. For each 3-maximal edge-cut $\mathcal{U}$ we define $l(\mathcal{U}) = \sum_{L \text{ is long}}(|L| - 8)$.

Let us analyze the Procedure `EdgeCut`($\mathcal{U}$).

Procedure `EdgeCut(`$\mathcal{U}$`)`

Input: $\mathcal{U}$; 3-maximal edge-cut of G
Output: 3-maximal edge-cut of G with no ladder larger than 12

if $\mathcal{U}$ *contains no ladder larger than* 12 **then** **return** $\mathcal{U}$;
else
  Select a ladder L with $|L| > 12$;
  if *exists* $i \in \{4, 5\}$ *such that* $\mathcal{U}^{L,i}$ *is* not 3-*maximal* **then**
1   **return** `EdgeCut(3-MaxCut(`$\mathcal{U}^{L,i}$`))`;
  else
    Determine $i \in \{4, 5\}$ such that $l(\mathcal{U}^{L,i}) < l(\mathcal{U})$;
2   **return** `EdgeCut(`$\mathcal{U}^{L,i}$`)`;

Obviously the edge-cut returned by `EdgeCut`($\mathcal{U}$) contains no ladder larger than 12. We say that an edge-cut $\mathcal{V}$ is larger than $\mathcal{V}'$ if and only if either $\mathcal{V}$ contains more edges in the cut than $\mathcal{V}'$ does or both contain the same number of edges, but $l(\mathcal{V}) < l(\mathcal{V}')$. More formally

$$\mathcal{V} \succ \mathcal{V}' \iff |\mathcal{V}| > |\mathcal{V}'| \text{ or } |\mathcal{V}| = |\mathcal{V}'| \text{ and } l(\mathcal{V}) < l(\mathcal{V}').$$

First we show that the recursive procedure `EdgeCut`($\mathcal{U}$) terminates in polynomial-time because in each recursive call `EdgeCut`() is called with input $\mathcal{U}'$ such that $\mathcal{U}' > \mathcal{U}$. Suppose that $\mathcal{U}$ contains a ladder L with $|L| > 12$. We have seen that $|\mathcal{U}^{L,i}| \geq |\mathcal{U}|$, for $i \in \{4, 5\}$. If therefore `EdgeCut`() is recursively called in Line 1, then with input `3-MaxCut`($\mathcal{U}^{L,i}$), an edge-cut containing strictly more edges than the edge-cut $\mathcal{U}$. If `EdgeCut`() is recursively called in Line 2, then with input $\mathcal{U}^{L,i}$ an edge-cut with $l(\mathcal{U}^{L,i}) < \mathcal{U}$, whose existence is guaranteed under these circumstances by the forthcoming lemma, Lemma 4.8. From the two rather obvious facts that $|\mathcal{V}| \in \{0, \dots, 5/2n\}$ for every edge-cut $\mathcal{V}$ of G and that $l(\mathcal{V}) \in \{0, \dots, n\}$ we indeed conclude that `EdgeCut`($\mathcal{U}$) terminates after polynomially many steps.

□

Lemma 4.8. *Let $\mathcal{U}$ be a 3-maximal edge-cut of G containing a ladder L with $|L| > 12$ and suppose that $\mathcal{U}^{L,4}$ and $\mathcal{U}^{L,5}$ are 3-maximal, then* $\min_{i\in\{4,5\}} l(\mathcal{U}^{L,i}) < l(\mathcal{U})$.

Proof of Lemma 4.8. We assume that $\mathcal{U}$ contains a ladder L with $|L| > 12$ and all three edge-cut $\mathcal{U}, \mathcal{U}^4$, and $\mathcal{U}^5$ are 3-maximal.

We need some more observations about ladders in $\mathcal{U}^i$, for $i \in \{4,5\}$. Let us agree upon the following notation. We add a superscript to a ladder L, e.g. L^i, to denote that L "lives" in the edge-cut $\mathcal{U}^i$. We add a subscript, e.g L_v, to denote the ladder L containing the vertex v. The vertices v_i and v_{i+1} have degree 2 in $G(\mathcal{U}^i)'$. We conclude that v_i and v_{i+1} are contained in one ladder in $H(\mathcal{U}^i)$. Let us define this ladder in $H(\mathcal{U}^i)$ to be $L^i = L^i_{v_i}$. The two vertices v_{i-1} and v_{i+2} of L are contained in L^i as well. Here we note that L^i contains at least 4 vertices. The vertex adjacent to $v_j, j \in \{3,\ldots,k-2\}$ on the other side of $\mathcal{U}$ and not contained in L (we know that $d_{H(\mathcal{U})}(v_i) \leq 2$ from Proposition 4.12(v)) is denoted by x_i (if it exists). Since $x_j \notin W, d_{G(\mathcal{U})'}(x_j) \leq 1$. If x_j has such a neighbor in $G'(\mathcal{U})$, then we denote it by y_j.

Note that the two vertices v_{i-2} and v_{i+3} of L are not contained in $W(\mathcal{U}^i)$ but in $W(\mathcal{U})$. On the other hand x_i and x_{i+1} gained one neighbor (v_i and v_{i+1}, respectively) in $G(\mathcal{U}^i)'$ and are thus candidates to be contained in $W(\mathcal{U}^i) \setminus W(\mathcal{U})$, indeed $W(\mathcal{U}^i) \subseteq W(\mathcal{U}) \cup \{x_i, x_{i+1}\} \setminus \{v_i, v_{i+1}\}$

Switching the sides of v_i and v_{i+1} will change the set of ladders.

Let us call a ladder L' *new* in $\mathcal{U}^i$ if there is no ladder L in $\mathcal{U}$ such that $L' = L$. The following holds for each new ladder L': either L' contains a vertex from $W(\mathcal{U}^i) \setminus W(\mathcal{U})$ or is incident to a vertex from $W(\mathcal{U}) \setminus W(\mathcal{U}^i)$. The set of new ladders thus consists of a subset of the not necessarily distinct ladders $L^i, L^i_{x_i}, L^i_{x_{i+1}}$, and the two ladders $L^i_{v_{i-3}}$ and $L^i_{v_{i+4}}$.

Similarly we call a ladder L *old* in $\mathcal{U}$ if there is no ladder L' in $\mathcal{U}^i$ such that $L' = L$. The following holds for each old ladder L: either L contains a vertex from $W(\mathcal{U}) \setminus W(\mathcal{U}^i)$ or is incident to a vertex from $W(\mathcal{U}^i) \setminus W(\mathcal{U})$. The set of old ladders consists certainly of L, and possibly also of L_{y_i} and $L_{y_{i+1}}$.

Proposition 4.13. *Let $\mathcal{U}$ be a 3-maximal edge-cut of G containing a ladder L with $|L| > 12$. Suppose that $\mathcal{U}^{L,4}$ and $\mathcal{U}^{L,5}$ are 3-maximal as well, then the following is true:*

(*i*) $|L^i| \leq 4 + 2|\{x_i, x_{i+1}\} \cap L^i|$ *for at least one* $i \in \{4, 5\}$.

(*ii*) *If* $L^i \neq L^i_{x_j}$, *then* $|L^i_{x_j}| \leq |L_{y_j}| + 2|\{x_j \cap L^i_{y_j}|$, *for all* $i \in \{4, 5\}$ *and* $j \in \{i, i+1\}$.

Let us postpone the proof of Proposition 4.13 to the end of this section. Also we want to observe that all ladders of $\mathcal{U}^{L,i}$ besides $L^i, L^i_{x_i}$ and $L^i_{x_{i+1}}$ are either not new or contained in L. We now want to conclude the proof of Lemma 4.7 by showing the existence of $i^* \in \{4, 5\}$ with $l(\mathcal{U}^{L,i^*}) < l(\mathcal{U})$. We choose $i^* \in \{4, 5\}$ such that $|L^{i^*}| \leq 4 + 2|\{x_{i^*}, x_{i^*+1}\} \cap L^{i^*}|$ (c.f. Proposition 4.13(i)). The ladder L^{i^*} does not contribute to $l(\mathcal{U}^{L,i^*})$ since it is not long. The contribution of $L^{i^*}_{x_j}$ to $l(\mathcal{U}^{L,i^*})$ is at most $2|\{x_j\} \cap L^{i^*}_{y_j}|$ more than L_{y_j} contributes to $l(\mathcal{U})$, $j \in \{i^*, i^*+1\}$. Finally, the contribution to $l(\mathcal{U}^{L,i^*})$ of ladders subset of L is at least 5 less than the contribution of L to $l(\mathcal{U})$, as $|L| - 8 \geq 5$. We conclude with

$$\begin{aligned} l(\mathcal{U}^{L,i^*}) &\leq l(\mathcal{U}) + 2|(\{x_{i^*}\} \cap L^{i^*}_{x_{i^*}}) \cup (\{x_{i^*+1}\} \cap L^{i^*}_{x_{i^*+1}})\}| - 5 \\ &\leq l(\mathcal{U}) + 2 \cdot 2 - 5 \\ &< l(\mathcal{U}). \end{aligned}$$

□

Proof of Proposition 4.13. Again for ease of notation we define $\mathcal{U}^i = \mathcal{U}^{L,i}$, for $i \in \{4, 5\}$.

(i) Suppose first that $|L^4| > 4 + 2|\{x_4, x_5\} \cap L^4|$. If $\{x_4, x_5\} \cap L^4 = \emptyset$, then L^4 neither extends via v_3 nor via v_6 and $L^4 = \{v_3, v_4, v_5, v_6\}$.

Suppose now that L^4 contains x_5 but not x_4. Thus L^4 extends by at least 3 vertices via v_3. It is not hard to check that x_5 has to be adjacent to v_3 and v_1 and that $\{x_5, v_1\} \subset L^4$. Suppose first that $H(\mathcal{U})[L]$ is a cycle. Hence also the vertices of $L \cap U_2$ induce a cycle component C in $G'(\mathcal{U})$, according to Proposition 4.12(iii). Therefore x_5 is not adjacent to any vertex of C and L^4 cannot extend via v_1. If $H(\mathcal{U})[L]$ is a path, then v_1 is an endvertex of it. The only neighbor of v_1 in $U^4_2 \cap L^4$ is x_5. Therefore L^4 can extend via x_5 by at most 2 vertices. A contradiction.

Next suppose that L^4 contains x_4 (and possibly x_5). Again according to our assumptions and the observations just made L^4 has to extend by

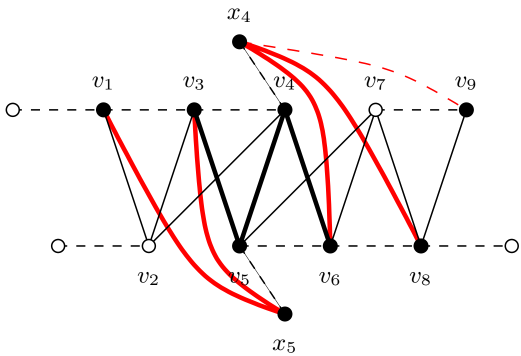


Figure 4.16: The ladder L^4 in $\mathcal{U}^4$.

at least 3 vertices via v_6. Hence x_4 is adjacent to v_6 and v_8. Moreover we assumed that $\mathcal{U}^4$ is 3-maximal, we conclude from Proposition 4.12(iii) and the fact that $v_9 \in L^4$ that $\{x_4, v_9\} \in E(G)$ as well, see Figure 4.16.

Let us now carry this information over to $\mathcal{U}^5$. Since $\{x_4, v_6\} \in E(G)$ and thus $x_4 = x_6$, we conclude that $x_4 \in L^5$. Moreover since the four vertices x_4, v_6, v_7, v_9 form a cycle component of U_1^5, $L^5 \cap U_1^5 \subseteq \{x_4, v_6, v_7, v_9\}$, see Figure 4.17. It remains to show that in this case $|L^5| \leq 6 + 2|x_5 \cap L^5|$. From Proposition 4.12($v$) it follows that $d_{H(\mathcal{U}^5)}(v) \leq 2$, for every vertex $v \in L^5$. Due to Proposition 4.12(iii), we observe that $|L^5 \cap U_2^5| \leq |L^5 \cap U_1^5|$ and thus $|L^5| \leq 8$. The vertex v_9 is contained in L^5 only if $x_5 \in L^5$.

(ii) If $L^4_{x_4} \neq L^i$, then $v_4 \notin L^4_{x_4}$. Thus $L^4_{x_4}$ can extend by at most one vertex beyond x_4. We conclude that the contribution of $L^4_{x_4}$ to $l(\mathcal{U}^4)$ is at most 2 more than the contribution of L_{y_4} to $l(\mathcal{U})$. A similar statement holds true for $L^i_{x_j}$ and L_{y_j} for $i \in \{4, 5\}$ and $j \in \{i, i+1\}$. □

We conclude the proof of Lemma 4.7.

Proof of Theorem 4.5. In order to finish the proof of Theorem 4.5 we define the following graph H_L according to an edge-cut $\mathcal{U}$ containing no ladders with more than 12 vertices. Note here that $\mathcal{U}$ can be efficiently obtained by a call to `EdgeCut(3-MaxCut())` as shown in Lemma 4.7. Let $\mathcal{L}_i, i = 1, 2$ be the set of ladders of $\mathcal{U}$ having at least on vertex in U_i and $V(H_L) = \mathcal{L}_1 \cup \mathcal{L}_2$. Thus for each ladder of H that touches both parts

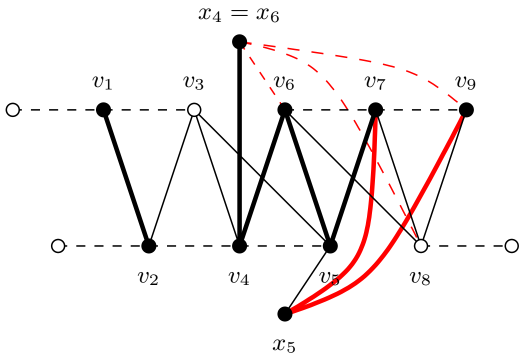


Figure 4.17: The ladder L^5 in $\mathcal{U}^5$.

of $\mathcal{U}$ there are two vertices $v_1^L, v_2^L \in V(H_L)$ corresponding to L. Let two vertices v_1, v_2 in H_L be adjacent if they correspond to two distinct ladders L_1 and L_2, such that they contain vertices $u \in L_1$ and $w \in L_2$ and either the two vertices u and w are adjacent in G or there is a path $P = u, v, w$ in G of length 2 with v being an endpoint of a path component C_i of G' (possibly of length 0).

Claim 4.4. $\Delta(H_L) \leq 64$.

Proof. Each ladder L in H is incident to at most $|L|+4$ many edges with exactly one endpoint in L, since an inner vertex of L is connected to at least 4 vertices of L and an endpoint of L is connected to at least two vertices of L in G. For a vertex $u \in L$ and vertex v as before there are at most 4 paths $P = \{u, v, w'\}$ of length 2 with $w' \in L'$ for some distinct ladder L'. We conclude that $\Delta(H_L) \leq 4(|L| + 4) = 4(16) = 64$. □

The crucial observation is that while switching several ladders simultaneously, vertices of degree 2 in G' that do not switch sides, do not receive any new neighbor. This is true simply because, if a vertex $v \in U_i$ has degree 2 in G' and a neighbor in U_{3-i} is selected for switching, then v (being in the same ladder as w) is also selected for a switch. Thus, in choosing a switch that breaks up large components of G', we just need to take care that the vertices of degree at most one in G' do not join up a lot of components via the newly switched vertices. This will be done with the help of an independent transversal as follows.

We define a vertex partition $\mathcal{P}$ of a subset of the vertex set of H_L as follows: Set $l_0 = 2 \cdot \Delta(H_L) = 128$. We partition each component C_i in G' into $\lfloor |C_i|/l_0 \rfloor$ many parts of l_0 consecutive ladders such that the $|C_i| \pmod{l_o}$ many remaining ladders of C_i are evenly distributed between these parts. Also these remaining vertices will not be contained in $\mathcal{P}$. For instance a component C_i with $|C_i| = k \cdot l_0 - 1$, for some $k > 2$ will be partitioned into $k-1$ parts each containing l_0 many ladders. Moreover between each part there are at most $\lceil \frac{l_0-1}{k-1} \rceil$ many ladders. Then we apply Theorem 1.1 to the multipartite subgraph of H_L defined on the vertices contained in $\mathcal{P}$. Let T be such an independent transversal. We switch the ladders in G corresponding to vertices of H_L contained in T and thereby, since vertices of degree 2 in G' do not receive a new vertex, breaking all large components of G' into components of at most $(l_0-1) + \lceil \frac{l_0-1}{k-1} \rceil + (l_0-1) = 127+64+127 = 318$ ladders, since we easily observe that $k = 3$ constitutes the worst case. Let $\mathcal{U}^*$ be the new edge-cut. According to Lemma 4.7 and Proposition 4.12(*iii*)-(*iv*) each ladder contains at most $12/2 = 6$ vertices in either U_1 or U_2. Note here that since T is a set of independent ladders in H_L, no degree 1 vertex of G' receives more than one ladder. Hence a ladder L that has been switched forms a component of order at most $(|L|+|L|+4)/2 \leq 14$ in either U_1^* or U_2^*. In total each component in the new graph $G[U_1^*] + G[U_2^*]$ contains at most $6(318) = 1908$ many vertices concluding the proof of Theorem 4.5(*i*).

Let us now finish the proof of Theorem 4.5(*ii*). Partitioning the components C_i of G' into parts of order $l_1 = 20\Delta(H_L)^3$ instead of order l_0 enables us to apply Theorem 2.1 instead of Theorem 1.1 in the proof of Lemma 4.5. Thus we can find an independent transversal in H_L and moreover find a C-sbmc 2-coloring as in Theorem 4.5(*ii*), $C \leq 6(3l_1) \leq 18 \cdot 4 \cdot 5 \cdot 64^3 = 94371840$ (here we plug the factor 3 in front of l_1 for simplicity, a more detailed analysis as above for the case of Theorem 4.5(*i*) would be possible as well). □

4.2.2 Sbmc 2-Colorings of Graphs G with $\Delta(G) > 5$

Let $\text{sbmc}(\Delta, n)$ be the smallest integer g such that every n-vertex graph of maximum degree Δ is g-sbmc 2-colorable. Motivated by the fact that $\text{sbmc}(5, n) = O(1)$ according to [39] or Theorem 2.1 and a result by Linial, Matoušek, Sheffet, and Tardos in [49] showing that $\text{sbmc}(7, n) = \Omega(n)$, we would be very curious to know the order of $\text{sbmc}(6, n)$.

The graph T_C defined in the following yields a lower bound on sbmc$(6, n)$.

Definition 4.2. *For a positive integer C, let T_C be the graph whose vertices are the triples (x, y, z) of nonnegative integers summing to C, with an edge connecting two triples if they agree in one coordinate and differ by one in the remaining two coordinates.*

Note first that T_C has maximum degree at most six. A Lemma by Hochberg, McDiarmid, and Saks [41] shows the following property of any 2-coloring of the graph T_C. (This is also a direct consequence of the so-called HEX-Lemma, see for instance [30].)

Lemma 4.9 ([41]). *T_C is not C-sbmc 2-colorable.*

Hence we conclude that sbmc$(6, n) = \Omega(\sqrt{n})$.

The value of sbmc(Δ, n) has also been investigated for more restrictive graph classes. Matoušek and Přívětivý [52] investigate sbmc colorings of supergraphs of the grid graph. In [49] the authors ask for the determination of sbmc$(6, n)$ if the graphs under consideration are restricted to being line graphs of 4-regular graphs. This problem is motivated by the construction of the line graphs for Theorem 5.1.

4.2.3 Hardness Results for Sbmc Colorings

Let us start this section with a conjecture that the "hardness jump" indeed also occurs for the symmetric case.

Conjecture 4.1. *For $\Delta \in \{4, 5\}$ there is an integer $g(\Delta) \in \mathbb{N}$ such that*

(i) (Δ, C)-SBMCCol is NP*-complete, for $2 \leq C < g(\Delta)$, and*

(ii) every graph with maximum degree at most Δ admits a $g(\Delta)$-sbmc 2-coloring.

Towards a proof of Conjecture 4.1 we prove Theorem 4.6 by constructing the appropriate base gadgets and then defining the graph SBMCColGraph(F) which can be C-sbmc 2-colored if and only if the formula F is satisfiable. We denote the two color-classes of a C-sbmc 2-coloring by B_1 and B_2.

Definition 4.3. *Let $C \geq 2$ and $\Delta \geq 4$ be integers. A graph G is called* (Δ, C)-sym-forcing *with a set $F \subseteq V(G)$ of at most two forced vertices if*

(i) $\Delta(G) \leq \Delta$ *and* $\sum_{f \in F}(\Delta - d(f)) \geq 2$,

(ii) G is C-sbmc 2-colorable, and

(iii) for every C-sbmc 2-coloring of G there is a color-class c such that every $f \in F$ is contained in a c-component of order at least C.

Lemma 4.10. *For any two integers $\Delta \geq 4$ and $C \geq 2$ the decision problem (Δ, C)-SBMCCol is* NP*-complete provided a (Δ, C)-sym-forcing graph exists.*

Proof. Suppose a (Δ, C)-sym-forcing graph H exists. We will reduce our problem to 3-SAT. As in the asymmetric problem, the graph we construct will be an extension of the core graph G_F. But the base-gadgets will be different from the ones in the previous subsection and some of them will be connected to each other unlike in the asymmetric problem.

We construct our base-gadget BG_l by taking l copies $H_1, \ldots, H_l$ of the (Δ, C)-sym-forcing graph H and l vertices $s_1, \ldots, s_l$ and connecting them in a path-like fashion as depicted in Figure 4.18.

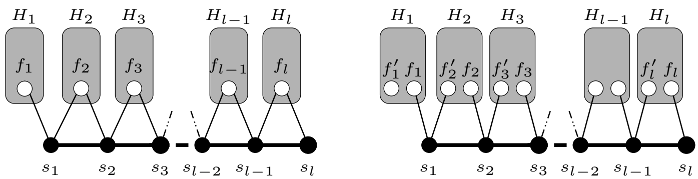


Figure 4.18: Base gadget BG_l using sym-forcing graphs with either one or two forced vertices.

By property (i) of Definition 4.3, $\Delta(B_l) \leq \Delta$. By property (iii), in any C-sbmc 2-coloring of the base gadget BG_l all the forced vertices f_i have the same color, which is different from the color of the vertices s_i. Thus the vertices s_i form a monochromatic component of order l. We call f_1 and s_l the *source* and *sink* of the base gadget, respectively.

We can conclude the following.

Proposition 4.14. *The base gadget BG_l is C-sbmc 2-colorable for every $l \leq C$. Moreover, in any C-sbmc 2-coloring of BG_l the sink is contained in a monochromatic component of order exactly l whose color is different from the color of the source.*

Suppose now that we are given a 3-SAT formula F. We construct the graph SBMCColGraph(F) by connecting various base gadgets to vertices of G_F via an edge. First, we connect the sink of a base gadget $B_w^{(1)} \cong BG_{C-2}$ to each vertex w of G_F which has constraint 2 and the sink of a base gadget $B_w^{(1)} \cong BG_{C-1}$ to each vertex w with a constraint 1. These we call the *base-gadgets of the first-type*. Further, we connect the sink of a base-gadget $B_w^{(2)} \cong BG_{C-1}$ to every vertex w of G_F. These we call the *base gadgets of the second-type*. Note that by part (ii) of Theorem 4.3, after adding these new edges the degree of each vertex of G_F is at most four. Also, the sink of each base-gadget now has degree three, and the source has degree at most $\Delta - 1$.

Then, by adding an edge between some sink and source vertices, we connect all base gadgets of the first-type in a path-like fashion, pictured on Figure 4.19. We act similarly for base gadgets of the second-type. Finally, we add an edge between the source of the first base gadget of the first-type and the source of the first base gadget of the second-type. Let us denote this new graph by SBMCColGraph(F). By the above, the maximum degree of SBMCColGraph(F) is clearly at most Δ. For an insight about the connections between the base-gadgets, see Figure 4.19.

The following is an immediate corollary of Proposition 4.14

Proposition 4.15. *In any C-sbmc 2-coloring of SBMCColGraph(F) the sinks of base-gadgets of the first-type are all colored with the same color, say B_1. On the other hand the sinks of base-gadgets of the second-type are all colored with the other color, B_2.*

We claim that there is a C-sbmc 2-coloring of SBMCColGraph(F) if and only if the core G_F has a 0/1-coloring. Via Theorem 4.3, this will conclude the proof of Lemma 4.10.

Suppose χ is a C-sbmc 2-coloring of SBMCColGraph(F). Suppose the sinks of the base-gadgets of the first-type all received color B_1. (We know they are all the same from Proposition 4.15) Then all sinks of the base-gadgets of the second-type must receive color B_2. Changing color B_1 to 0 and color B_2 to 1 gives us the the 0/1-coloring of G_F which observes all the constraints.

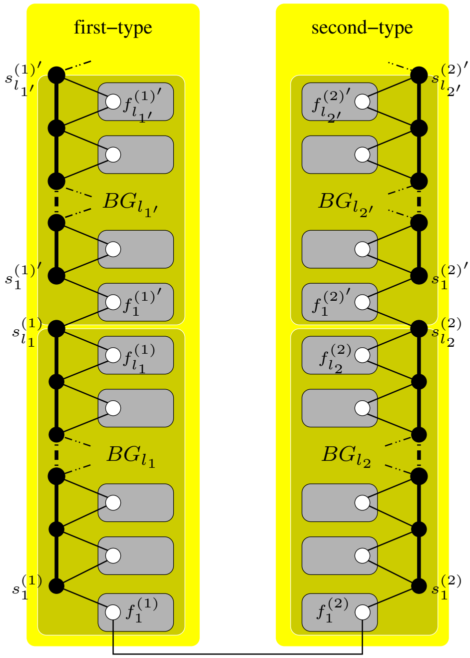


Figure 4.19: Connecting base gadgets of the first- and second-type

Conversely, suppose we are given a 0/1-coloring of G_F. We can extend this to a (C, C)-sbmc 2-coloring of SBMCColGraph(F) by arbitrarily selecting either 0 or 1 to color the forced vertices of the base-gadgets of the first-type and then extending this coloring to all vertices of all base-gadgets. □

(Δ, C)-Sym-Forcing Graphs

We are able to show the existence of $(4, C)$-sym-forcing graphs with $C \in \{2, 3\}$ and $(6, C)$-sym-forcing graphs, for $C \geq 2$. This will conclude the proof of Theorem 4.6.

Proposition 4.16. *The graph G in Figure 4.20 is $(4, 2)$-sym-forcing with a forced set $\{f', f''\}$.*

Proof. Adding the edge $\{f', v\}$ to G yields a graph that is not 2-sbmc

2-colorable. In order to not contradict this fact, f' and v must have the same color in any 2-sbmc 2-coloring G, whereas the two common neighbors of f' and v are contained in the other color-class. Hence also f'' is contained in the same color-class as f' and v. □

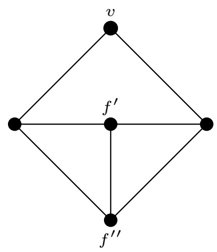


Figure 4.20: $(4, 2)$-sym-forcing graph

To construct a $(4, 3)$-sym-forcing graph we introduce a weakened concept of Definition 4.3 which makes it easier to construct (Δ, C)-sym-forcing graphs, and thus is interesting in its own right.

Definition 4.4. *Let $C \geq D \geq 2$ and $\Delta \geq 4$ be integers. A graph G is called* (Δ, C, D)-sym-forcing *with a set $F \subseteq V(G)$ of at most two forced vertices if*

(i) $\Delta(G) \leq \Delta$ and $\sum_{f \in F}(\Delta - d(f)) \geq 2$,

(ii) G is C-sbmc 2-colorable, and

(iii) for every C-sbmc 2-coloring of G there is a color class c such that every $f \in F$ is contained in a c-monochromatic component of order at least D.

Clearly, (Δ, C, C)-sym-forcing is the same as (Δ, C)-sym-forcing.

Proposition 4.17. *The existence of a $(\Delta, C, \lceil \frac{C+1}{2} \rceil)$-sym-forcing graph implies the existence of a (Δ, C, D)-sym-forcing graph for every D, $\lceil \frac{C+1}{2} \rceil \leq D \leq C$.*

Proof. Let G_1 and G_2 be two copies of an (Δ, C, i)-sym-forcing graph, $\lceil \frac{C+1}{2} \rceil \leq i \leq C - 1$. First assume that we have one forcing vertex in G_i. We connect the forcing vertex f_1 of G_1 to the forcing vertex f_2 of G_2. Also we add a new vertex v to the new graph, denote it by H, and connect it to f_1 and f_2, see Figure 4.21. Suppose f_1 and f_2 are contained in the same color-class in a C-sbmc 2-coloring of H, then the two adjacent vertices f_1 and f_2 are contained in one monochromatic component of order at least $2i \geq C+1$, a contradiction. Thus without loss of generality $f_1 \in B_1$ and $f_2 \in B_2$. We conclude that v is contained in a monochromatic component of order $i+1$. The construction for the case when the G_i's have two forcing f_i' and f_i'' vertices is depicted in Figure 4.21 as well. The proof is very similar to the former case. □

Figure 4.21: Weakly forcing graphs

Proposition 4.18. *The graph in Figure 4.22 is* $(4,3,2)$*-sym-forcing with forced vertex* f*.*

Proof. The graph is 3-sbmc 2-colorable. It is then sufficient to observe that in a 3-sbmc 2-coloring the neighbors v_1, v_2 of f cannot have the same color, so f participates in a monochromatic component of order at least two. Let us assume to the contrary that v_1 and v_2 are both contained in color-class B_2. Obviously at least three out of the four neighbors of v_1 and v_2 (not considering f) have to be contained in B_1 in order to not span a B_2-component of order 4. On the other hand not all of the four vertices can be contained in B_1. Hence the unique common neighbor v of those four vertices is incident to a B_1 component and a B_2-component, each of order 3. We conclude that v cannot be colored. Hence one of v_1 and v_2 has a color identical to that of f, that is, f is a forced vertex. □

The previous two propositions imply the existence of a $(4,3)$-sym-forcing graph.

Recall the definition of the graph T_C, see Definition 4.2. We denote by v, w and f the following three vertices, respectively: $v = (0, C, 0), w = (0, C-1, 1)$ and $f = (1, C-1, 0)$. Let H_{C-1} denote the graph T_C with the edge $\{v, f\}$ removed. The graph H_4 is shown in Figure 4.23

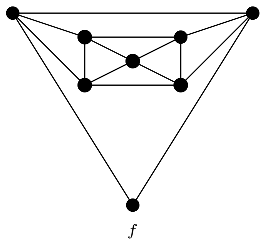


Figure 4.22: $(4, 3, 2)$-forcing

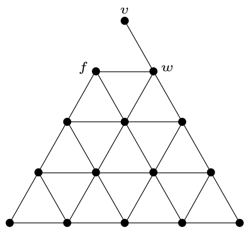


Figure 4.23: The graph H_4

Proposition 4.19. *H_{C-1} is $(6, C)$-sym-forcing for $2 \leq C$ with forced vertex f.*

Proof. It is not hard to check that H_{C-1} is C-sbmc 2-colorable. The following three properties of C-sbmc 2-colorings of H_{C-1} are immediate consequences of the Lemma 4.9.

(i) v and f are contained in the same color-class.

(ii) w is contained in the other color-class than v and f.

(iii) The order of the union of the monochromatic component containing v and containing f is at least $C + 1$.

According to (ii) and the fact that v has a unique neighbor w, v is contained in a monochromatic component of order exactly 1. We conclude due to (iii) that f is contained in a monochromatic component of order C always. □

Nichts ist Wahr, Alles ist Erlaubt.

Friedrich Nietzsche

Chapter 5

Bmc k-Colorings with $k \geq 3$

Most of the results presented in the previous chapter can be generalized to more than just two colors. In this chapter we will focus only on extremal graph theoretic aspects of bmc colorings. The first section deals with asymmetric bmc k-colorings of graphs with bounded maximum degree, the second section deals with symmetric bmc 2- and 3-colorings of planar graphs with bounded maximum degree.

5.1 Abmc (k, l)-Colorings

We have seen in Proposition 0.1 that the chromatic number of a graph G can be bounded by the maximum degree of G, i.e., $\chi(G) \leq \Delta(G)+1$. Or in other words every graph with maximum degree at most $k-1$ can be properly k-colored.

Recall that a $(k+l)$-coloring of a graph G is said to be a C-abmc (k, l)-coloring if every of the first k color-classes forms an independent set and every monochromatic component in the ith color-class contains at most C vertices, $i \in \{k+1, \dots, k+l\}$. We say that a family of graphs $\mathcal{F}$ is *abmc* (k, l)*-colorable* if there exists a constant $C > 0$ such that every graph $G \in \mathcal{F}$ admits a C-abmc (k, l)-coloring. We define $\Delta(k, l)$ to be the smallest integer Δ such that the family of graphs with maximum degree Δ is *not* abmc (k, l)-colorable.

Hence similarly to Proposition 0.1 we want to determine $\Delta(k, l)$.

Theorem 5.1. *Let $k, l > 0$. For any constant C there exists a graph of maximum degree $\Delta = 2(k+2l-1)$ which is not C-abmc (k,l)-colorable. That is, $\Delta(k,l) \leq 2k+4l-2$.*

Proof. Erdős and Sachs [54] proved the existence of a $(k+2l)$-regular graph G_C with girth $C+1$, for an arbitrary integer C. Our construction is the line graph H of $G = G_C$. Denote by n the number of vertices of G and e the number of edges of G.

Obviously H is $2(k+2l-1)$-regular and has $e = \frac{(k+2l)n}{2}$ vertices. Suppose we have a C-abmc (k,l)-coloring of H, and $V_1, \ldots, V_k$, $V_{k+1}, \ldots, V_{k+l}$ are the appropriate color-classes. Then either

(*i*) $\exists i \in \{1, \ldots, l\}$ with $|V_{k+i}| \geq n$, or

(*ii*) $\exists j \in \{1, \ldots, k\}$ with $|V_j| \geq \lfloor \frac{n}{2} \rfloor + 1$.

Case (i). The set V_{k+i} corresponds to n edges in G. G has n vertices, so some of these edges form a cycle K in G, whose length is at least $C+1$. The vertices of H corresponding to these edges in K also form a cycle of the same length. In particular they induce a component of H with order at least $C+1$.

Case (ii). The set V_j corresponds to $\lfloor \frac{n}{2} \rfloor + 1$ edges in G. That is, two of these edges will share an endpoint. The two vertices corresponding to these two edges are adjacent in H, a contradiction to the independence of V_j. □

A construction by Alon, Ding, Oporowski and Vertigan in [10] is a special case with $k = 0$ and $l = 2$. In Haxell, Szabó and Tardos [39] the following theorem has been proved:

Theorem 5.2 ([39]). *There exists a constant C such that the following holds. Given a graph of maximum degree $\Delta \geq 3$, it is possible to $\lceil (\Delta+1)/3 \rceil$-partition the vertex set such that each part induces components of size at most C.*

This statement has immediate implications for abmc (k,l)-colorings.

Corollary 5.1. *Let k, l be nonnegative integers. The family of graphs of maximum degree at most $k+3l-1$ is abmc (k,l)-colorable. That is, $\Delta(k,l) > k+3l-1$.*

Proof. First suppose that $l > 1$. By a lemma of Lovász [51] one can partition the vertex set of G into $k+1$ classes $V_0 \cup V_1 \cup \ldots \cup V_k = V(G)$

such that $\Delta(G[V_i]) = 0$, for all $i \in [k]$ and $\Delta(G[V_0]) \leq \Delta - k$. Then we apply Theorem 5.2 to l-partition V_0

Next we consider the case when $l = 1$. Again we apply the same lemma from [51] to partition the vertex set into k classes $V_1 \cup \ldots \cup V_k = V(G)$ such that $\Delta(G[V_i]) = 0$, for all $i \in \{2..k\}$ and $\Delta(G[V_1]) \leq 3$. Then we apply Theorem 4.1 to $(1, 22)$-bmc 2-color V_1. □

One would like to know more about the behavior of the function $\Delta(k, l)$ in general, or at least tighten the existing asymptotic gap. In the following, we discuss the most intriguing special cases. As we mentioned before the main theorem of [39] states that $\Delta(0, 2) = 6$. The value of $\Delta(0, 3)$ is not known and is certainly worth determining. It is known to be either 9 or 10 (see [39]). In other words, one has to decide whether there is a constant C such that it is possible to color the vertex set of any graph with maximum degree 9 by three colors such that every monochromatic component is bounded by C. Also in [39] it is shown that there exists $\delta > 0$ such that for large l, $3 + \delta < \Delta(0, l)/l < 4$. It would be of great interest to determine asymptotically $\Delta(0, l)$. Theorem 4.1 states that $\Delta(1, 1) = 4$. By the results in this section the value of $\Delta(2, 1)$ is either 5 or 6. Asymptotically, $\Delta(k, 1)$ is between k and $2k$. We conjecture the lower bounds are (closer to) the truth.

Finally, let us generalize here a problem raised in [39]. A natural way to weaken the maximum degree condition is by rather bounding the maximum average degree of the graph, which allows a few very large degree vertices. Let $\mu(G) = \max\{2|E(G[W])|/|W| : W \subseteq V(G)\}$. For non-negative integers k, l what is the supremum value $\alpha(k, l)$ such that every graph G with $\mu(G) < \alpha(k, l)$ has a C-abmc (k, l)-coloring with some constant C. Obviously $\alpha(k, l) \leq \Delta(k, l)$. In [39] the determination of $\alpha(0, 2)$ was raised as a question. The *wheel* graph shows that $\alpha(0, 2) \leq 4$, while Kostochka [47] proved a lower bound of 3. The greedy coloring implies that $\alpha(k, 0) = k$, for any k. We would be very much interested interested in the value of $\alpha(1, 1)$.

5.2 Sbmc Colorings of Planar Graphs

In this section we investigate symmetric bmc colorings of planar graphs. Due to the Four Color Theorem [13] (see Theorem 0.3) we know that every planar graph G admits a proper 4-coloring, hence $\mathrm{sbmc}_4(G) = 1$. On the other hand there are planar graphs which cannot be properly colored with only three colors. Moreover for a planar graph G the determination whether G is properly 3-colorable is NP-complete, see Theorem 0.1.

Nevertheless the following simple characterization of triangulations that are properly 3-colorable has been shown by Heawood [40].

Theorem 5.3 ([40, 58]). *The vertices of a triangulation G are properly 3-colorable if and only if G is Eulerian.*

In contrast to the long and computer aided proof of the Four Color Theorem, and even its simplification by Robertson, Sanders, Seymour, and Thomas [55], there is a rather compact proof by Cowen, Cowen, and Woodall [25] not assuming the truth of the Four Color Theorem that $\text{sbmc}_4(G) \leq 2$ for every planar graph G.

In this chapter we restrict ourselves to 2-colorings and 3-colorings of planar graphs and investigate $\text{sbmc}_2(G)$ and $\text{sbmc}_3(G)$ with G restricted to subclasses of planar graphs.

As mentioned earlier amongst the many results of [10], the authors show the existence of planar graphs H_1 with arbitrarely many vertices and with maximum degree at most six such that $\text{sbmc}_2(H_1) = \Omega(\sqrt{|V(H_1)|})$. Linial, Matoušek, Sheffet, and Tardos [49] study bounded monochromatic component colorings of minor closed classes of graphs. They show for instance that for every planar graph G, $\text{sbmc}_2(G) = O(|V(G)|^{2/3})$ and they construct planar graphs H_2 with $\text{sbmc}_2(H_2) = \Omega(|V(H_2)|^{2/3})$. For 3-colorings Kleinberg, Motwani, Raghavan, and Venkatasubramanianby in [46] construct planar graphs H_3 such that $\text{sbmc}_3(H_3) = \Omega(|V(H_3)|^{1/3})$. These graphs H_3 have large maximum degree, that is, $\Delta(H_3) = \Omega(|V(H_3)|)$. Motivated by this they ask for the following which we want to formulate as a conjecture.

Conjecture 5.1. *For any non-negative integer Δ there is an integer $f(\Delta)$ such that $\text{sbmc}_3(G) \leq f(\Delta)$ for every planar graph G of maximum degree at most Δ.*

As mentioned above a similar statement is not true if we restrict ourselves to 2-colorings (the graph H_1). According to the following proposition it suffices to prove Conjecture 5.1 for triangulations.

Proposition 5.1. *Every planar graph H of maximum degree $\Delta(H)$ can be triangulated by adding edges to H such that for the resulting triangulation H^*, $\Delta(H^*) \leq 3\Delta(H)$.*

Proof. We process every face of H iteratively as follows. Let f be a face of H with length $k \geq 4$ and let $v_1, \ldots, v_k$ be the vertices of f in a circular order. We connect the vertices $v_2, v_k, v_3, v_{k-1}, \ldots$ by a (zigzag-)-path. Obviously the degree of each vertex in f increases by at most two. We can conclude that $\Delta(H^*) \leq 3\Delta(H)$ and H^* is a triangulation. □

In Subsection 5.2.2 we make a first step towards Conjecture 5.1 by proving:

Theorem 5.4. *For every triangulation G with maximum degree Δ and at most k many vertices of odd degree,* $\mathrm{sbmc}_3(G) \leq 2k\Delta^3$.

Unfortunately Theorem 5.4 cannot directly be generalized from triangulations to planar graphs. There exist planar graphs G with no vertices of odd degree such that every triangulation of G contains a linear number of vertices of odd degree.

Complementing Conjecture 5.1 we can construct planar graphs H_4 (see Figure 5.3) for which $\mathrm{sbmc}_3(H_4)$ is large even with respect to $\Delta(H_4)$. We show that $\mathrm{sbmc}_3(H_4) = \Theta(\sqrt{\Delta(H_4)})$, thereby improving on the implicitly given bound $\mathrm{sbmc}_3(H_3) = \Theta(\Delta(H_3)^{1/3})$.

Our proofs rely on the embedding of the planar graph. Hence eventhough we state our results for planar graph we actually assume that an embedding of the graph into the plane is fixed. For every vertex $v \in V(G)$ the edges incident to v impose a circular ordering of the neighbors of v as follows. Let $e_1, e_2, \ldots, e_{d(v)}$ denote the set of edges incident to v in clockwise order with some arbitrary first edge e_1 and let $e_i = \{v, u_i\}$, for $i \in [d(v)]$. Then $N^c(v) = (u_1, u_2, \ldots, u_{d(v)})$ is the corresponding circular ordering (the first vertex u_1 is chosen arbitrarely from all neighbors of v).

Before we prove Theorem 5.4 let us once more consider sbmc 2-colorings. In the next subsection we show a statement similar to Conjecture 5.1 for sbmc 2-colorings of outerplanar graphs.

5.2.1 Outerplanar Graphs

Lemma 5.1. *For every integer $\Delta > 1$ and every outerplanar graph G with the property that all vertices have degree at most Δ except one vertex v with $d(v) \leq 2(\Delta - 1)$ it holds that* $\mathrm{sbmc}_2(G) \leq 2(\Delta - 1)$. *Moreover, there exists such a $(2(\Delta - 1))$-sbmc 2-coloring for which v is contained in a monochromatic component of order 1.*

Proof. We can assume that G is connected. The proof applies induction on $|V(G)|$. The base cases $|V(G)| \leq 2$ are trivial. Let $N^c(v) = (v_1, v_2, \ldots, v_{d(v)})$ be given by the outerplanar embedding of G. Define $V_i \subset V(G)$, $i \in [d(v)-1]$ to be the vertices $v' \in V(G) \setminus (N(v) \cup \{v\})$ such that *every* path from v' to v contains one of the vertices v_i, v_{i+1}. Observe here that by the outerplanarity of G it holds that $V_i \cap V_j = \emptyset$, for $i \neq j$ and $i, j \in [d(v)-1]$. Let G_i be the graph $G[V_i]$, for $i \in [d(v)-1]$, with one additional vertex v_i^* such that $N_{G_i}(v_i^*) = (N_G(v_i) \cup N_G(v_{i+1})) \cap V_i$. Note that $d_{G_i}(v_i^*) \leq 2(\Delta - 1)$. Let us show that G_i is outerplanar. Starting with the outerplanar embedding of G and moving the vertex v_i along the edge $\{v_i, v\}$ towards v and the same with v_{i+1} until the two vertices meet in one vertex v_i^* yields an outerplanar embedding of G_i.

By the induction hypothesis we can $2(\Delta - 1)$-sbmc 2-color every outerplanar graph G_i such that v_i^* is colored differently than all its neighbors in G_i and v_i^* gets, say, color 2. Let χ_i be the respective $2(\Delta - 1)$-bmc 2-coloring of G_i, for $i \in [d(v)-1]$. We extend these colorings to a $2(\Delta-1)$-sbmc 2-coloring of G in the following way. Color all vertices but $v, v_1, \ldots, v_{d(v)}$ of G with the same color as in the corresponding coloring χ_i. Color the vertices $v_1, \ldots, v_k$ by color 2 and finally, color the vertex v with color 1. □

Corollary 5.2. *For every outerplanar graph G it holds that* $\mathrm{sbmc}_2(G) \leq 2(\Delta(G) - 1)$.

In contrast to Corollary 5.2 we consider the graph H_k' in Figure 5.2.1. H_k' is a k-ary tree with all leaves at depth $k-1$ and such that all the children of each vertex are connected by a path. Hence $\Delta(H_k') = k+3$.

Proposition 5.2. $\mathrm{sbmc}_2(H_k') = k = \Delta(H_k') - 3$.

Proof. Suppose that there is a 2-coloring of H_k' with no monochromatic component containing at least k vertices. Obviously for every vertex w in H_k' it holds that not all children of w receive the same color. Hence there is a monochromatic path starting at the root of H_k' and ending at a leaf of H_k'. Since every leaf in H_k' has depth at least $k-1$, this monochromatic path contains at least k vertices, a contradiction.

On the other hand, we define a 2-coloring χ showing that $\mathrm{sbmc}_2(H_k') \leq k$ as follows. Set $\chi(v) = 1$ and for every other vertex $u \in V(H_k') \setminus \{v\}$ set

$$\chi(u) = \begin{cases} 1, & \text{if } dist(u,v) \text{ is even, and} \\ 2, & \text{if } dist(u,v) \text{ is odd.} \end{cases}$$

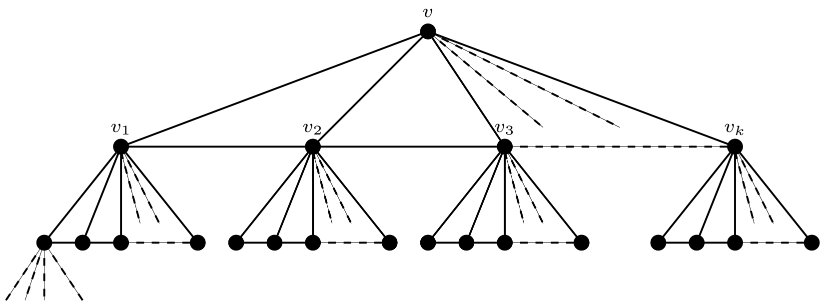


Figure 5.1: Outerplanar graph H'_k.

Obviously, the 2-coloring χ is a k-sbmc 2-coloring. □

5.2.2 Bounded Number of Vertices with Odd Degree

Lemma 5.2. *For every triangulation G with exactly two vertices u and v having odd degree it holds that u and v are non-adjacent in G.*

Proof. Suppose that $\{u, v\} \in E(G)$. Let H denote the graph G without the edge $\{u, v\}$. Obviously H contains exactly one 4-face and all other faces of H are 3-faces. Denote by H^* the dual graph of H. Since every vertex degree of H is even and H originates from the triangulation G, the vertices of every face of H^* induce an cycle of even length. Thus H^* does not contain any cycle of odd length. It is well known that this fact is equivalent to H^* being bipartite. Let V_1 and V_2 be the partite sets of H^* and assume without loss of generality that the vertex w of H^* corresponding to the unique 4-face of H is contained in V_2. Every vertex of $V(H^*) = V_1 \cup V_2$ has degree three except the vertex w, which is of degree four. The number of edges of H^* is on one hand $\sum_{x \in V_1} d_{H^*}(x) \equiv 0 \pmod 3$ and on the other hand $\sum_{x \in V_2} d_{H^*}(x) \equiv 1 \pmod 3$. A contradiction. □

Some Vertices with Odd Degree

Let $Odd(G)$ be the set of vertices of odd degree contained in G, and let $odd(G) = |Odd(G)|$. For two adjacent vertices $u, v \in V(G)$ we define

$N^c(v,u) = (u_j, \ldots, u_{d(v)}, u_1, \ldots, u_{j-1})$ such that $N^c(v) = (u_1, \ldots, u_{d(v)})$ and $u_j = u$. For a directed path $P : v_1, v_2, \ldots, v_k$ we define for every vertex v_i the set of left-neighbors $\mathrm{LN}_P(v_i) = \{u_2, \ldots, u_{l-1}\}$ with $N^c(v_i, v_{i-1}) = (u_1, \ldots, u_{d(v_i)})$ such that $u_l = v_{i+1}$, for $i \in \{2, \ldots, k-1\}$. Also we define $d_P(v_i) = |\mathrm{LN}_P(v_i)|$ and $\mathrm{LN}_P = \bigcup_{i\in\{2,\ldots,k-1\}} \mathrm{LN}_P(v_i)$.

A *left-shortcut* of a directed path $P : v_1, \ldots, v_k$ is an edge $\{v_i, v_j\}$, with $1 < i < j < k$, $(j - i) > 1$, and $v_i \in \mathrm{LN}_P(v_j)$ (hence also $v_j \in \mathrm{LN}_P(v_i)$). A directed path $P : v_1, \ldots, v_k$ of G with $k \geq 3$ and $d_P(v_{k-1}) > 0$ is called *propagating* if P contains no left-shortcut.

Note here that every directed shortest path of G with length at least two is a propagating path. For a propagating path P, we often refer to v_{k-2} as w_P, to v_{k-1} as x_P, to v_k as z_P and to the common neighbor of x_P and z_P in $\mathrm{LN}_P(x_P)$ as y_P such that the triangle formed by the vertices x_P, y_P and z_P contains no vertex of G. The set of the three vertices $\{x_P, y_P, z_P\}$ induce the *terminal* triangle T_P of the propagating path P (see Figure 5.2).

For a propagating path $P : v_1, \ldots, v_k$ we define its exterior $\mathrm{Ext}(P) = \mathrm{LN}_P \cup V(P)$, and let $ext(P) = |\mathrm{Ext}(P)|$. Also we define the set of missing edges $\mathrm{ME}(P) = \{\{v_i, u\} \mid u \in \mathrm{LN}_P(v_i), i \in \{2, \ldots, k-1\}\}$.

We now have all prerequesites at hand to state the key lemma for proving Theorem 5.4.

Lemma 5.3. *Let G be a triangulation and $P : v_1, \ldots, v_k$ be a propagating path of G. Then there is a triangulation G_P such that*

(i) $V(G_P) \supseteq V(G)$,

(ii) $E(G_P) \supseteq (E(G) \setminus \mathrm{ME}(P))$, *and*

(iii) $Odd(G_P) \subseteq (Odd(G) \setminus \mathrm{Ext}(P)) \cup T_P$.

Proof. Let us first observe that if $d_P(v_i) = 0$, for some $i \in \{2, \ldots, k-2\}$, then $v_{i-1} \in \mathrm{LN}_P(v_{i+1})$ and also if $d_P(v_{k-1}) = 0$, then $v_k \in \mathrm{LN}_P(v_{k-2})$.

The proof applies induction on $m(P) = |V(P)| + |ME(P)|$. Clearly since $|V(P)| \geq 3$ and $|ME(P)| \geq 1$ according according to the definition of propagating paths, $m(P) \geq 4$.

For the base case $m(P) = 4$ we first note that $d_P(v_2) = 1$ (let the vertex l be defined such that $\mathrm{LN}_P(v_2) = \{l\}$) and $T_P = \{v_2, l, v_3\}$. Also we note that if $d(v_1)$ is even, then $G_p = G$ fulfills all three conditions $(i) - (iii)$. Suppose therefore that $d(v_1)$ is odd. We subdivide the edge $\{v_2, l\}$ with a new vertex w and connect the three vertices v_1, w, v_3 with a path of length two. In the new graph G_P, $d(v_1)$ and $d(w)$ are even, hence G_P is as required.

Let us now assume that $m(P) > 4$.

If $d_P(v_2) = 0$, then $k > 3$ again according to the definition of propagating paths. For the propagating path $Q : v_2, v_3, \ldots, v_k$ it holds that $|ME(Q) = |ME(P)|$ and $|V(Q)| = |V(P)| - 1$. Applying the induction hypothesis to G with propagating path Q yields a triangulation $G_P = G_Q$ that fulfills all three conditions $(i) - (iii)$.

Suppose now that $d_P(v_2) > 0$. Let l_1 and l_2 be the first and second vertex of the sequence $N^c(v_2, v_1)$ (possibly including the vertex v_3). We distinguish three cases:

- $d(v_1)$ is odd and $d(l_1)$ is even.
  We subdivide the edge $\{v_2, l_1\}$ by adding a new vertex w. Then we connect v_1, w, l_2 by a path of length 2, resulting in a triangulation H. The parity of the degrees of v_1 and l_2 changed, the one of l_1 did not. We define a new propagating path Q. If $d_P(v_2) > 2$, then set $Q : l_2, v_2 \ldots, v_k$. Hence $|ME(Q)| = |ME(P)| - 1$ and $|V(Q)| = |V(P)|$. If $d_P(v_2) \leq 2$, then set $Q : v_2, v_3, \ldots, v_k$ and observe that $|ME(Q)| \leq |ME(P)|$ and $|V(Q)| = |V(P)| - 1$.

  Q forms a propagating path in H and $G_P = H_Q$ is a triangulation as required.

- $d(v_1)$ and $d(l_1)$ are both odd.
  We add another edge to the pair of vertices v_2, l_1 and subdivide both edges connecting v_2, l_1 by adding the vertices w_1 and w_2, respectively. Then we connect v_1, w_1, w_2, l_2 by a path of length 3. Obviously, the parity of the degree of vertices v_1, v_2, l_1 and l_2 changed in the new graph H and both w_1 and w_2 have even degree. Again define a new propagating path Q. If $d_P(v_2) > 2$ then let $Q : l_2, v_2, \ldots, v_k$. Hence $|ME(Q)| = |ME(P)| - 1$ and $|V(Q)| = |V(P)|$.

  If $d_P(v_2) \leq 1$, then set $Q : v_2, v_3, \ldots, v_k$ and oberve that $|ME(Q)| \leq |ME(P)|$ and $|V(Q)| = |V(P)| - 1$.

  The path Q forms a propagating path an $G_P = H_Q$ is as required.

- $d(v_1)$ is even.
  Let $Q : l_1, v_2, \ldots, v_k$. We observe that $|ME(Q)| = |ME(P)| - 1$ and $|V(Q)| = |V(P)|$.

Again Q is propagating and $G_P = G_Q$ is as required.

□

A *d-left-common-neighbor* of a propagating path $P : v_1, \ldots, v_k$ is a vertex v contained in a path $Q : v_i, v, v_j$ with $v \in \mathrm{LN}_P(v_i) \cap \mathrm{LN}_P(v_j)$, $1 < i < j < k$ and $(j - i) \geq d$. Observe here that $E(Q) \subseteq \mathrm{ME}(P)$. The propagating path P is called *left* if it contains no 2-left-common-neighbor.

Note that a directed shortest path does not contain any 3-left-common-neighbor.

Corollary 5.3. *Every triangulation G such that there is a left propagating path $P : v_1, \ldots, v_k$ with $\mathrm{Ext}(P) \supseteq Odd(G)$ has a vertex 3-coloring V_1, V_2 and V_3 such that each component C of $G[V_i]$ containing at least two vertices forms a star with center $v \in \{v_i \mid i \in \{2, \ldots, k-1\}\}$. In particular* $\mathrm{sbmc}_3(G) \leq \lceil \Delta(G)/2 \rceil - 1$.

Proof. We apply Lemma 5.3 to G and propagating path P, resulting in a triangulation G_P with $Odd(G_P) \subseteq T_P$. According to Lemma 5.2 and the fact that every graph, and in particular G, has got an even number of vertices of odd degree, $Odd(G_P) = \emptyset$. Due to Theorem 5.3 G_P has a proper 3-coloring χ. Let χ_G refer to χ restricted to the vertices of G.

Obviously every monochromatic of G in χ_G is contained in $\mathrm{ME}(P)$. Suppose χ_G contains a monochromatic path $R : u_1, u_2, u_3$ of length two with $u_2 \in \mathrm{Ext}(P) \setminus V(P)$. Both u_1 and u_3 are in $V(P)$ with $u_1 = v_i$ and $u_3 = v_j$, $1 < i < j < k$. Since P is left, u_2 is not a 2-left-common-neighbor of P and thus $j = i+1$, $v_i v_j \in E(G)$ and thus $\chi_G(v_i) \neq \chi_G(v_j)$. A contradiction.

Hence every monochromatic component in χ_G of size at least 2 forms a star S with center $v_i \in V(P)$ and leaves $l \in \mathrm{LN}_P(v_i)$. We conclude the proof since $d_P(v_i) \leq \Delta(G) - 2$ and all edges of consecutive pairs of vertices l_i, l_{i+1} in $\mathrm{LN}_P(v_i)$ exist in G_P. □

Lemma 5.4. *For every pair of vertices v_1 and v_k with $dist(v_1, v_k) \geq k - 1 \geq 3$ there is a left propagating path $P : v_1, \ldots, v_k$ in G. Moreover P can be chosen such that there is* no *vertex $v \in \mathrm{LN}_P(v_{k-2}) \cap N(v_k)$.*

Proof. For two adjacent vertices $\{u, v\} \in E(G)$ and a vertex w let $N^c(v, u) = (u_1 = u, u_2, \ldots, u_{d(v)})$ we define $\mathrm{lns}_w(v, u)$ to be the vertex $u_j, j \in \{1, \ldots, d(v)\}$ such that $d(u_j, w) \leq d(v, w) - 1$ (i.e., there is a shortest- path $v, u_j, \ldots, w$) and there is no vertex $u_{j'}$ with $1 < j' < j$ and $d(u_{j'}, w) = d(u_j, w)$.

We start with a shortest path Q between v_1 and v_k. Let v_2 be the neighbor of v_1 on that shortest path. Then we iteratively define $v_i = \text{lns}_{v_k}(v_{i-1}, v_{i-2})$, for $3 \leq i \leq k-1$, starting with $v_3 = \text{lns}(v_2, v_1)$.

Trivially by the choice of the initial shortest path Q and the definition of $\text{lns}_w(v, u)$, $P : v_1, \ldots . v_k$ is a shortest path in G connecting v_1 and v_k and thus P is propagating, contains no 3-left-common-neighbor and $d_P(v_{k-1}) > 0$ (refer to the observation made in the beginning of the proof). Suppose for a moment that P contains either a 2-left-common-neighbor or a path $R : v_{k-2}, w, v_k$, with $w \in \text{LN}_P(v_{k-2})$. We conclude that there is an $i \in \{2, \ldots k-1\}$ with $\text{lns}_{v_k}(v_i, v_{i-1}) \neq v_{i+1}$ (but possibly $\text{lns}_{v_k}(v_{i+1}, v_i) = w$), a contradiction. □

Corollary 5.4. *For every triangulation G with at most two vertices of odd degree,* $\text{sbmc}_3(G) \leq \lceil \Delta(G)/2 \rceil - 1$.

Proof. We can assume that G contains exactly two vertices of odd degree. Let u, v be the two vertices of odd degree. Note that $d(u, v) \geq 2$ according to Lemma 5.2. Choose a left propagating path $P : u, \ldots, v$. The existence of P follows from Lemma 5.4. Now we apply Corollary 5.3 with G and P. □

Bounded Number of Vertices with Odd Degree

Let $\mathcal{T} = \{P_1, \ldots, P_t\}$ be a family of propagating paths. We define $\text{Ext}(\mathcal{T}) = \bigcup_{i=1}^{t} \text{Ext}(P)$, and $E(\mathcal{T}) = \bigcup_{i=1}^{t} E(G[\text{Ext}(P_i)])$. Also define $\mathcal{T}_i$ to be the set of paths $\{P_1, \ldots, P_i\}$.

Definition 5.1. *A family $\mathcal{T} = \{P_1, \ldots, P_t\}$ of propagating paths is called a* propagating tree *if for every $i \in \{2, \ldots, t\}$ the following three conditions are fulfilled.*

(i) $T_{P_i} \subseteq \text{Ext}(\mathcal{T}_{i-1})$,

(ii) $(\text{Ext}(P_i) \cap \text{Ext}(\mathcal{T}_{i-1})) \subseteq (\text{LN}_{P_i}(w_{P_i}) \cup \text{LN}_{P_i}(x_{P_i}) \cup \{x_{P_i}, z_{P_i}\})$,

(iii) $(\text{ME}(P_i) \cap E(\mathcal{T}_{i-1})) \subseteq \{x_{P_i}, y_{P_i}\}$.

Let us call $\mathcal{T}$ a *left propagating tree* if every propagating path in $\mathcal{T}$ is left.

Lemma 5.5. *For every triangulation G there is a left propagating tree $\mathcal{T}$ such that* $\text{Ext}(\mathcal{T}) \supseteq$ *$Odd(G)$ and $|\mathcal{T}| \leq odd(G)$.*

Before we prove Lemma 5.5 let us see how it implies Theorem 5.4.

Proof of Theorem 5.4. Let $\mathcal{T}(G) = \{P_1, \ldots, P_t\}$ be a left propagating tree of G with $\mathrm{Ext}(\mathcal{T}(G)) \supseteq Odd(G)$ and $t \leq odd(G)$. In this proof we iteratively propagate the odd-degree vertices of G along the paths P_i (possibly with some minor modifications of P_i), starting with P_t and G, and resulting in an Eulerian triangulation H.

More formally we refer to $P(\mathcal{T}(G))$ as the propagating path P_t of $\mathcal{T}(G)$. Let $H_t = G$, and define H_i to be the triangulation obtained from an application of Lemma 5.3 with H_{i+1} and $P(\mathcal{T}(H_{i+1}))$. In the propagation from H_{i+1} to H_i with propagating path $P = P(\mathcal{T}(H_{i+1}))$ no edge from $E(\mathcal{T}(H_{i+1}) \setminus P)$ but possibly $\{x_P, y_P\}$ of H_{i+1} has been subdivided, see Lemma 5.3(*iii*) and Definition 5.1(*iii*). If $\{x_P, y_P\}$ has not been subdivided, then define $\mathcal{T}(H_i) = \mathcal{T}(H_{i+1}) \setminus \{P\}$. Otherwise if $\{x_P, y_P\}$ has been subdivided, then we let $\mathcal{Q}$ be the set of propagating paths Q in $\mathcal{T}(H_{i+1})$ with $\{x_P, y_P\} \subseteq E(Q)$. For every path $Q : u_1, \ldots, x_P, y_P, \ldots, u_{k'} \in \mathcal{Q}$ with $\{x_P, y_P\} \in E(Q)$, define Q^* of H_i as follows, $Q^* : u_1, \ldots, x_P, v_P, y_P, \ldots, u_k$ and let $\mathcal{Q}^*$ be the set of these paths Q^*. Set $\mathcal{T}(H_i) = (\mathcal{T}(H_{i+1}) \cup \mathcal{Q}^*) \setminus (\{P\} \cup \mathcal{Q})$. Note here that a path Q^* of H_i has no left-shortcut and no 3-left-common-neighbor but possibly contains the 2-left-common-neighbor v with $v \in N_{Q^*}(x_P)$ and $v \in N_{Q^*}(y_P)$.

Since $T_{P(\mathcal{T}(H_{i+1}))} \subseteq \mathrm{Ext}(\mathcal{T}(H_i))$ by the definition of a propagating tree and using Lemma 5.3, $Odd(H_i) \subseteq \mathrm{Ext}(\mathcal{T}(H_i))$.

Hence $H = H_1$ is an Eulerian triangulation. Let χ be a proper 3-coloring of H whose existence is assured by Theorem 5.3. Further we refer to χ_G as the coloring χ restricted to the vertices of G.

Let $R : u_1, u_2, u_3, u_4$ be a monochromatic path in χ_G of length three in G. Hence none of the edges $e_i = \{u_i, u_{i+1}\}$, for $1 \leq i \leq 3$, is contained in H and therefore $e_i \in \mathrm{ME}(P_l)$, for some $j \in [t]$. We claim that either u_2 or u_3 is contained in $\bigcup_{P \in \mathcal{T}(G)} \mathrm{LN}_P(w_P) \cup \mathrm{LN}_P(x_P) \cup \{x_P, z_P\}$. Suppose on contrary, that is, $\{u_2, u_3\} \subseteq \mathrm{Ext}(\mathcal{T}) \setminus (\bigcup_{P \in \mathcal{T}} \mathrm{LN}_P(w_P) \cup \mathrm{LN}_P(x_P) \cup \{x_P, z_P\})$. Thus according to Lemma 5.3(*ii*) the edge $\{u_2, u_3\}$ is such that without loss of generality $u_2 \in V(P_j)$ and $u_3 \in \mathrm{LN}_{P_j}(u_2)$. Since u_3 is not a 2-left-common-neighbor of P_j, all neighbors w of u_3 besides u_2 are colored with a distinct color than u_3 (because the edge $\{u_2, w\}$ or the edge $\{u_3, w\}$ or the edge $\{u_2, u_3\}$ exists in H). A contradiction.

Every monochromatic component of diameter at most 2 contains at most $\Delta(G)+1$ many vertices. For a monochromatic component C of diameter at least 3 we conclude that for every vertex v in C there is a vertex u in $\bigcup_{P\in\mathcal{T}} \mathrm{LN}_P(w_P)\cup \mathrm{LN}_P(x_P)\cup\{x_P,z_P\}$ with $d(v,u)\leq 2$. Since $|\bigcup_{P\in\mathcal{T}} \mathrm{LN}_P(w_P)\cup \mathrm{LN}_P(x_P)\cup\{x_P,z_P\}|\leq t2\Delta(G)\leq odd(G)2\Delta(G)$, every monochromatic component of G contains at most $odd(G)2\Delta(G)\cdot\Delta(G)^2 = 2odd(G)\Delta(G)^3$ many vertices. □

Proof of Lemma 5.5. We fix two vertices $u,v\in Odd(G)$ and let $P_1 : u = v_1, v_2,\ldots,v_k = v$ be a left propagating path connecting u and v, c.f. Lemma 5.4. Let $\mathcal{T}_1 = \{P_1\}$ and for $i>1$ let $\mathcal{T}_i$ be defined as follows. We choose a vertex $u'\in Odd(G)\setminus \mathrm{Ext}(\mathcal{T}_{i-1})$ and let x be a vertex in $\mathrm{Ext}(\mathcal{T}_{i-1})$ of shortest distance to u'. If $d(u',x)>1$ then we choose the left propagating path $Q : u' = u_1,u_2,\ldots,u_l = x$ with the additional property that there is no vertex u with $u\in N_Q(u_{l-2})$. The existence of Q is shown in Lemma 5.4. Otherwise (if $d(u',x)=1$) set $Q : u',x$. Let $N^c(x,u_{l-1}) = (u_{l-1},w_2,\ldots,w_{d(x)})$ and denote by $P_j, j<i$ the propagating path of $\mathcal{T}_{i-1}$ such that $x\in \mathrm{Ext}(P_j)$ and there are two vertices $w_k,w_{k+1}\in N(x)\cap \mathrm{Ext}(P_j)$ such that there is no $1<k'<k$ for which there is a propagating path P in $\mathcal{T}_{i-1}$ with $w_{k'},w_{k'+1}\in N(x)\cap \mathrm{Ext}(P)$ (w_k,w_{k+1} are the first two neighbors of x in $\mathrm{Ext}(\mathcal{T}_{i-1})$ in clockwise order that are incident to the same exterior of a propagating path), see Figure 5.2. Note here that the path P_i with $V(P_i) = V(Q)\cup\{z\}$ is a left propagating path with $w_{P_i} = u_{l-1}, x_{P_i} = x, y_{P_i} = w_k$ and $z_{P_i} = w_{k+1}$. We claim that $\mathcal{T}_i = \mathcal{T}_{i-1}\cup\{P_i\}$ is a left propagating tree. Obviously by the definition of P_i, $T_{P_i}\supseteq \mathrm{Ext}(\mathcal{T}_{i-1})$. By the fact that $u' = u_1,\ldots,u_l = x$ is a shortest path, no vertex of $\mathrm{Ext}(P_i)\setminus(\mathrm{LN}_{P_i}(w_{P_i})\cup \mathrm{LN}_{P_i}(x_{P_i})\cup\{x_{P_i},z_{P_i}\})$ is contained in $\mathrm{Ext}(\mathcal{T}_{i-1})$. Moreover by the definition of P_i, $\mathrm{ME}(P_i)\cap E(G[\mathrm{Ext}(\mathcal{T}_{i-1})]) = \{x_{P_i},z_{P_i}\}$.

We terminate this procedure if for the current left propagating tree $\mathcal{T}_t$ it holds that $\mathrm{Ext}(\mathcal{T}_t)\supseteq Odd(G)$. Since in every iteration of the procedure we add at least one vertex of G with odd degree to the current left propagating tree it holds that $t\leq odd(G)$

□

A Lower Bound

Proposition 5.3. *For the graph G_k in Figure 5.3 it holds that* $\mathrm{sbmc}_3(G_k) = \Theta(\sqrt{\Delta(G_k)})$.

A *double fan* F_k, see for instance the graph $F_k^{(1)}$ in Figure 5.3, con-

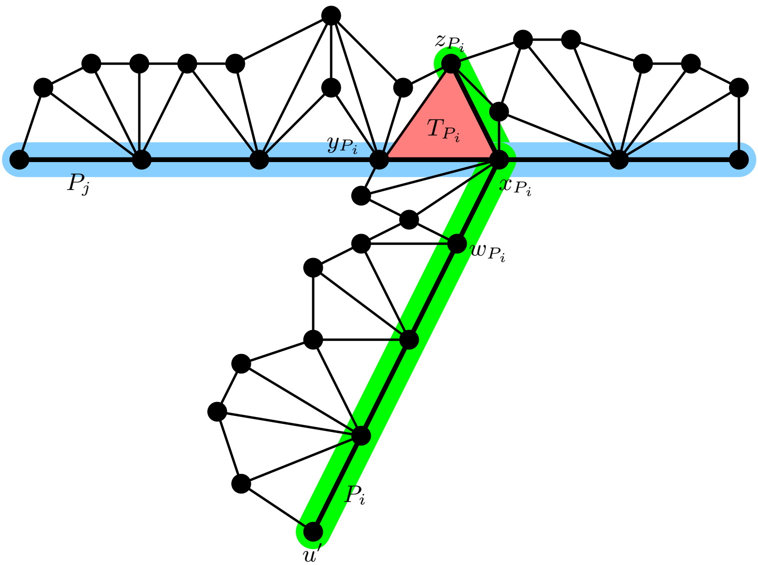


Figure 5.2: Construction of $\mathcal{T}_i$, with left propagating paths P_j and P_i.

sists of a path $P : u_1, \ldots, u_k$ and two non-adjacent vertices v_1 and v_2 connected to all vertices of P.

Lemma 5.6. *Let $k > 2c^2 + c - 2$, with $c \geq 1$. In every 3-coloring of a double fan F_k such that v_1 and v_2 are colored with distinct colors, there is a monochromatic component of order larger than c.*

Proof. Let χ be any such 3-coloring and suppose that there is no monochromatic component containing more than c many vertices. Without loss of generality $\chi(v_1) = 1$ and $\chi(v_2) = 2$. Hence there are at most $2(c-1)$ many vertices of color either 1 or 2 on the vertices $u_1, \ldots, u_k$. Since every set of $c+1$ many consecutive vertices of $u_1, \ldots, u_k$ has to contain at least one vertex of color either 1 or 2, the vertices $u_1, \ldots, u_k$ contain at most $(2(c-1)+1)c = 2c^2 - c$ vertices with color 3. We conclude that $k \geq 2(c-1) + 2c^2 - c = 2c^2 + c - 2$, a contradiction. □

Let us now prove Proposition 5.3 by constructing the graph G_k, see Figure 5.3.

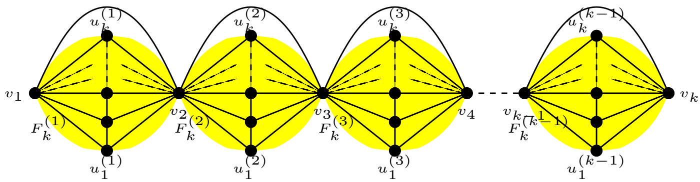


Figure 5.3: The graph G_k with double fans F_k.

Proof of Proposition 5.3. The graph G_k consists of a path $P : v_1, \ldots, v_k$ of length $k-1$ and $k-1$ copies of double fans, $F_k^{(1)}, \ldots, F_k^{(k)}$, where the vertices $v_1^{(i)}$ of the ith copy of F_k is identified with vertex v_i of P and the vertex $v_2^{(i)}$ is identified with vertex v_{i+1} of P.

We claim that in every vertex 3-coloring of G_k there is a monochromatic component of order larger than c, provided $k > 2c^2 + c - 2$.

Assume on the contrary that there is a 3-coloring χ with no monochromatic component of order larger than c. Hence there are two consecutive vertices of the path $P : v_1, \ldots, v_k$, say v_i and v_{i+1} that are colored with distinct colors. Applying Lemma 5.6 to $F_k^{(i)}$ yields a contradiction. On the other hand, if $k \leq 2c^2 + c - 2$, then we can define a c-sbmc 3-coloring χ as follows. For every double-fan $F_k^{(i)}$ we color the vertices $u_{c+1}^{(i)}, u_{2(c+1)}^{(i)}, u_{3(c+1)}^{(i)}, \ldots$ alternatingly again with colors 1 and 2. Finally we color all vertices that have not yet been colored with color 3. Let us check that every monochromatic componentcontains at most c vertices. This is trivially true for the vertices with color 3. The number of vertices $u_1^{(i)}, \ldots, u_k^{(i)}$ of the double-fan $F_k^{(i)}$ with color 1 is at most $k/(2(c+1)) \leq (c-1)$ (the same holds true for color 2). Thus we can conclude that $\text{sbmc}_3(G_k) = \Theta(\sqrt{\Delta(G_k)})$. □

Let us remark that the graph G_k can be slightly modified into a triangulation T_k with again $\text{sbmc}_3(T_k) = \Omega(\sqrt{\Delta(T_k)})$. Moreover T_k contains only $k-1$ vertices with odd degree.

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Maple Code

```
> with(Optimization);

> lps := proc(a1, b1, a2, b2, a3, b3)
    description "solve corresponding lp";
    local l1, l2, l3, dv4, dx12, dx13, dy12, dy13, dx21, dx23, dy21, dy23,
    dx31, dx32, x1, x2, x3, y1, y2, y3, x12, x13, x21, x23, x31, x32, y12,
    y13, y21, y23, y31, y32, a12, a13, a21, a23, a31, a32, b12, b13, b21,
    b23, b31, b32;
    unassign('l1, l2, l3, dv4, dx12, dx13, dy12, dy13, dx21, dx23, dy21,
    dy23, dx31, dx32, x1, x2, x3, y1, y2, y3, x12, x13, x21, x23, x31, x32,
    y12, y13, y21, y23, y31, y32, a12, a13, a21, a23, a31, a32, b12, b13,
    b21, b23, b31, b32');
    if a1 = 1 then assume(0 < x12, 0 < x13); a12 := 1; a13 := 1
    elif a1 = 2 then assume(0 < x12); x13 := 0; a12 := 1; a13 := 0;
    elif a1 = 3 then x12 := 0; assume(0 < x13); a12 := 0; a13 := 1
    end if;
    if b1 = 1 then assume(0 < y12, 0 < y13); b12 := 1; b13 := 1;
    elif b1 = 2 then assume(0 < y12); y13 := 0; b12 := 1; b13 := 0;
    elif b1 = 3 then y12 := 0; assume(0 < y13); b12 := 0; b13 := 1;
    end if;
    if a2 = 1 then assume(0 < x21, 0 < x23); a21 := 1; a23 := 1;
    elif a2 = 2 then assume(0 < x21); x23 := 0; a21 := 1; a23 := 0;
    elif a2 = 3 then x21 := 0; assume(0 < x23); a21 := 0; a23 := 1;
    end if;
    if b2 = 1 then assume(0 < y21, 0 < y23); b21 := 1; b23 := 1;
    elif b2 = 2 then assume(0 < y21); y23 := 0; b21 := 1; b23 := 0;
    elif b2 = 3 then y21 := 0; assume(0 < y23); b21 := 0; b23 := 1;
    end if;
```

```
if a3 = 1 then assume(0 < x31, 0 < x32); a31 := 1; a32 := 1;
elif a3 = 2 then assume(0 < x31); x32 := 0; a31 := 1; a32 := 0;
elif a3 = 3 then x31 := 0; assume(0 < x32); a31 := 0; a32 := 1;
endif;
if b3 = 1 then assume(0 < y31, 0 < y32); b31 := 1; b32 := 1;
elif b3 = 2 then assume(0 < y31); y32 := 0; b31 := 1; b32 := 0;
elif b3 = 3 then y12 := 0; assume(0 < y32); b31 := 0; b32 := 1;
end if;
l1 := {x1 + x2 + x3 <= dv4, 1 + y2 <= dx12, 1 + y3 <= dx13,
y2 + x21 + y31 + x31 <= dy12, y3 + x31 + y21 + x21 <= dy13,
1 + y1 <= dx21, 1 + y3 <= dx23, y1 + x12 + y32 + x32 <= dy21,
y3 + x32 + y12 + x12 <= dy23, 1 + y1 <= dx31, 1 + y2 <= dx32,
y1 + x13 + y23 + x23 <= dy31, y2 + x23 + y13 + x13 <= dy32};
l2 := {x1 <= x12 + x13, y1 <= y12 + y13, x1 + y1 = 1,
x2 <= x21 + x23, y2 <= y21 + y23, x2 + y2 = 1, x3 <= x31 + x32,
y3 <= y31 + y32, x3 + y3 = 1};
l3 := {dv4 <= d, a12 · dx12 <= d, a13 · dx13 <= d, b12 · dy12 <= d,
b13 · dy13 <= d, a21 · dx21 <= d, a23 · dx23 <= d, b21 · dy21 <= d,
b23 · dy23 <= d, a31 · dx31 <= d, a32 · dx32 <= d, b31 · dy31 <= d,
b32 · dy32 <= d};
return Minimize(d, l1 union l2 union l3, assume = nonnegative)[1];
end proc;

> Δ := 2;
  for i1 to 3 do
    for j1 to 3 do
      for i2 to 3 do
        for j2 to 3 do
          for i3 to 3 do
            for j3 to 3 do
              Δ := min(Δ, lps(i1, j1, i2, j2, i3, j3));
            end do
          end do
        end do
      end do
    end do
  end do
  print(Δ);
```

Curriculum Vitae

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1993-1998
Highschool
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1998-2003
Technical University
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Doccourse: Modern Methods in Ramsey Theory;
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