DISS. ETH NO. 16256, 2005

Games on Graphs

A dissertation submitted to the
SWISS FEDERAL INSTITUTE OF TECHNOLOGY, ZURICH

for the degree of
Doctor of Sciences

presented by
Miloš Stojaković
M.Sc. in Computer Science, University of Novi Sad, Serbia and Montenegro
born August 26, 1976
citizen of Serbia and Montenegro

accepted on the recommendation of
Prof. Dr. Emo Welzl, ETH Zurich, examiner
Dr. Tibor Szabó, ETH Zurich, co-examiner
Prof. Dr. József Beck, Rutgers Mathematics Department, New Brunswick, co-examiner

2005

Abstract

We introduce and study Maker-Breaker positional games on random graphs. Our goal is to determine the threshold probability $p_{\mathcal{F}}$ for the existence of Maker's strategy to claim a member of $\mathcal{F}$ in the unbiased (one-on-one) game played on the edges of the random graph $G(n,p)$, for various target families $\mathcal{F}$ of winning sets. More generally, for each probability above this threshold we study the smallest bias b such that Maker wins the $(1 : b)$ biased game. We investigate these functions for a number of basic games, like the connectivity game, the perfect matching game, the clique game, the Hamiltonian cycle game and the tree game. Particular attention is devoted to unbiased games, when $b = 1$.

Next, we consider the planarity game and the k-coloring game on the complete graph on n vertices. In the planarity game the winning sets are all non-planar subgraphs, and in the k-coloring game the winning sets are all non-k-colorable subgraphs. For both of the games we look at a $(1 : b)$ biased game. We are interested in determining the largest bias b such that Maker wins the Maker-Breaker version of the game. On the other hand, we want to find the largest bias b such that Forcer wins the Avoider-Forcer version of the game.

Finally, we deal with balanced online games on the random graph process. The game is played by a player called Painter. Edges in the random graph process are introduced two at a time. For each pair of edges Painter immediately and irrevocably chooses one of the two possibilities to color one of them red and the other one blue. His goal is to avoid creating a monochromatic copy of a prescribed fixed graph H, for as long as possible. We study the threshold m_H for the number of edges to be played to know that Painter almost surely will create a monochromatic copy of H, for H being a cycle, a path and a star.

Zusammenfassung

In dieser Arbeit führen wir sogenannte “Maker-Breaker” Spiele auf zufälligen Graphen $G(n,p)$ ein. Mit $p_{\mathcal{F}}$ bezeichnen wir den Schwellenwert für die Kantenwahrscheinlichkeit, ab der eine Strategie existiert, so dass “Maker” ein Element der Menge $\mathcal{F}$ für sich behaupten kann. Ein Ziel dieser Arbeit ist es, $p_{\mathcal{F}}$ für unterschiedliche Gewinnmengen $\mathcal{F}$ zu ermitteln. Gespielt wird zunächst die $(1:1)$ Variante, wobei in jeder Runde beide Spieler genau eine Kante des Graphen wählen. Darüber hinaus ermitteln wir für Wahrscheinlichkeiten über diesem Schwellenwert die kleinste zulässige Gewichtung b, welche es noch ermöglicht, dass Maker das ungleiche Spiel $(1:b)$ gewinnt. Wir untersuchen verschiedene Spiele, nämlich das Konnektivitätsspiel, das Perfekte-Matchingspiel, das Hamiltonsche-Kreisspiel und das Baumspiel. Besondere Beachtung schenken wir dabei immer der ausgeglichenen Variante der Spiele, d.h. dem Fall $b = 1$.

Desweiteren betrachten wir das Planaritätsspiel sowie das k-Färbungsspiel, beide auf dem vollständigen Graphen mit n Knoten. Die Gewinnmenge des Planaritätsspiels ist die Menge aller nicht planaren Subgraphen; die des k-Färbungsproblems die Menge aller Subgraphen, welche nicht k-färbbar sind. Für beide Spiele untersuchen wir abermals die $(1:b)$ Variante. Hier soll die grösstmögliche Gewichtung b ermittelt werden, sodass “Maker” die “Maker-Breaker” Varianten der Spiele gewinnt. Andererseits interessieren wir uns für die grösstmögliche Gewichtung b, sodass “Forcer” die “Avoider-Forcer” Varianten der Spiele gewinnt.

Zuletzt befassen wir uns mit balancierten Online-Spielen auf dem zufälligen Graph Prozess. Wir nennen den einzigen Spieler “Painter”. Pro Runde werden jeweils zwei zufällige Kanten gespielt. Für jedes dieser Kantenpaare entscheidet sich “Painter” unmittelbar und unwiderruflich für eine der beiden Kantenfärbungen, wobei eine der Kanten rot, die andere blau gefärbt wird. Dabei möchte “Painter” solange wie möglich die Entstehung einer monochromatischen Kopie des gegebenen Graphen verhindern. Wir betrachten den Grenzwert m_H der Anzahl gespielter Kanten, sodass “Painter” fast sicher eine monochromatische Kopie von H erzeugen wird, wobei hier m_H für Kreise, Pfade und Sterne untersucht wird.

Hobbes: *Well, summer is almost over, it sure went quick, didn't it?*
Calvin: *Yep. There's never enough time to do all the nothing you want.*

Calvin and Hobbes, by Bill Watterson

Acknowledgments

Writing a thesis is a non-trivial thing to do. Many have helped me on the way—both at work and off work—and I am grateful to them. My special thanks go to:

Emo, for giving me the opportunity to come here, be(come) a member of Gremo, and learn from him;

Tibor, for working with me, and making great bean soup (Go Buckeyes!);

József, for teaching me positional games, and playing tennis with me;

Danny, Dieter, Eva, Jirka, Jochen, József, Martin, Michael, Michael, Tibor and Udo, for collaborating with me during my stay in Zurich;

The European Graduate Program "Combinatorics, Geometry, and Computation", financed by ETH Zurich and the German Science Foundation (DFG), for providing me with a gratifying environment in which I completed my thesis;

Floris and Franziska, for resolving all the administrative issues that were in my way;

The PreDoc's, AleXX, Ingo, Robert, Philipp and Eva (in order of appearance), for being my office mates and liking the music I like;

Michael and Robert, for translating the abstract of this thesis to some obscure language;

All the co-presidents, members and associates of the Gremo Cinematic Group (GCG) who have watched movies with me, in particular to Ingo;

All the people who have played board games with me, in particular to the Game Master;

Bettina and Alex, for frequently laughing loud enough to be heard in the whole floor, thus breaking the silence;

All my basketball friends, for letting me elbow them when I needed to elbow someone;

Andreas, Branka and Davor, Eva and Ulli, Hannah and Martin, Ingo, Leon, Ljilja, Mambo, Paz, Péter, Robert, Shakhar, Shankar, Sonja, Uli, Ženjgi and all the others, for hanging out with me, going to concerts, watching sports, cooking, drinking, skiing, swimming, frisbeeing, and God knows what else;

Isabella, for being my girlfriend.

M.S., Zurich, September 2005

Contents

What-what-what?!
What do I want to ask?

Yoshio Okamoto

Chapter 1

Introduction

As the title suggests, this thesis is about games on graphs. At first sight, the topic may suggest lots of fun since games are widely recognized as amusing. But the reader should keep in mind that this is a collection of mathematical results and as such has a moderate fun impact.

What follows is a brief description of our results in a slightly imprecise manner, all for keeping it concise. We hope to make it shorter than the whole thesis (trivially true; by inclusion), but still longer than the abstract. Next to the results, one can find some information about their exact coordinates in the thesis.

1.1 A short version of the thesis

1.1.1 Chapter 2

In Chapter 2 we introduce the basic concepts that will be used throughout the thesis.

When we say games, we mostly mean positional games, and when we say graphs, we mostly mean random graphs. A positional game is a 4-tuple $(X, \mathcal{F}, a, b)$, where X is a finite set, the "board", $\mathcal{F} \subseteq 2^X$ is the set of winning sets, and a and b are positive integers. There are two basic variants of rules for playing a game.

In the first one, we call the players Maker and Breaker. They alternately claim unclaimed elements of X. Maker claims a of them per move, and Breaker b of them per move. The game ends when all elements of X are claimed. Maker wins the game if he claimed all elements of a set from $\mathcal{F}$, and otherwise Breaker wins.

In the other variant of the rules, we call the players Avoider and Forcer. The only difference to the Maker-Breaker variant is that Avoider *loses* if he claims all elements of a set from $\mathcal{F}$, and otherwise Forcer loses.

The random graph $G(n,p)$ is a graph on n vertices, such that each two vertices are connected by an edge with probability p, independently for every pair of vertices.

1.1.2 Chapter 3

In Chapter 3 we deal with Maker-Breaker games played on edges of the random graph $G(n,p)$. We consider winning sets $\mathcal{F}$ to be the set of all representatives of some well-known graph-theoretic structure: the set of all spanning trees, the set of all Hamiltonian cycles, the set of all perfect matchings, the set of all cliques of size k. Generally, we assume that n tends to ∞.

We first consider unbiased $(1:1)$ games. For each mentioned set of winning sets, it is easy to see that Maker can win the game when $p=1$. We are interested in the following. What is the smallest probability $p_{\mathcal{F}} = p_{\mathcal{F}}(n)$ for which Maker still can win the game a.s.?

More generally, for every $p > p_{\mathcal{F}}$ we would like to determine the largest bias $b^p_{\mathcal{F}}$ such that Maker still wins $(1:b^p_{\mathcal{F}})$ game on edges of $G(n,p)$.

In Section 3.3 our attempt is to determine the value $b^p_{\mathcal{F}}$ for the mentioned families of winning sets. We manage to find the order of $b^p_{\mathcal{F}}$ for all the games in question, except for the Hamiltonian cycle game for which we just give upper and lower bounds.

Next, we give particular attention to $(1:1)$ games in Section 3.4, trying to determine the threshold probability $p_{\mathcal{F}}$ more accurately. We deal with the connectivity game, the clique game, the Hamiltonian cycle game and the G-game for arbitrary fixed graph G. Unlike in the general case, we manage to give the exact threshold for the Hamiltonian cycle game.

The results presented in this chapter are joint work with Michael Krivelevich and Tibor Szabó [41, 30].

1.1.3 Chapter 4

In Chapter 4 we take a closer look at the planarity game and the k-coloring game, played on edges of the complete graph on n vertices. For each of them, we look at both the Maker-Breaker variant and the Avoider-Forcer variant of the game.

The planarity game is analyzed in Section 4.2. The winning sets are all non-planar subgraphs of the complete graph. Therefore, Maker wins the game if he claims a non-planar graph, and Breaker wins otherwise. On the other hand, Forcer wins if Avoider claims a non-planar graph, and Avoider wins otherwise.

In the biased $(1 : b)$ game, we would like to determine the smallest value of b such that Breaker (resp. Forcer) still can win the game. For the Maker-Breaker version we find an upper bound and a lower bound for this value that are both of order n. On the other hand, the bounds we get for the Avoider-Forcer version of the game are not of the same order.

In Section 4.3 we consider the k-coloring game. Similarly as in the planarity game, Maker wins the Maker-Breaker variant of the game if he claims a non-k-colorable graph, and Forcer wins the Avoider-Forcer variant of the game if Avoider claims a non-k-colorable graph.

Again, we would like to determine the smallest integer b such that Breaker (resp. Forcer) still can win $(1 : b)$ game. In the Maker-Breaker game, we exhibit an upper bound and a lower bound for this value that are both linear, whereas for the Avoider-Forcer version we just manage to give bounds that are not of the same order.

The results presented in this chapter are joint work with Dan Hefetz, Michael Krivelevich, and Tibor Szabó [27].

1.1.4 Chapter 5

In Chapter 5 we deal with games played by a single player (whom we call Painter) on the random graph process. We call these games balance avoidance games.

Edges of the complete graph on n vertices are introduced two at a time, in a random order. For each pair, Painter immediately and irrevocably chooses one of the two possibilities to color one of them red and the other one blue. His goal is to avoid creating a monochromatic copy of a prescribed fixed graph H.

We would like to determine the number of edges to be played such that Painter will lose a.s., independent of his strategy.

In Section 5.2.1 we give a generic theorem that gives this threshold for a class of graphs. As a consequence, we get the threshold for the game of avoiding cycles C_l, $l \geq 3$.

Next, in Section 5.2.2 we deal with stars. We estimate the number of stars of fixed size at any point of the game, and as a consequence give the number of moves that Painter can survive a.s. without creating a star S_k, $k \geq 2$. Finally, we deal with the game of avoiding paths in Section 5.2.3. We exhibit an upper bound for the number of moves when avoiding a path P_k, $k \geq 2$, but give the exact threshold for the number of moves only for $k = 2$ and $k = 3$.

The results presented in this chapter are joint work with Martin Marciniszyn and Dieter Mitsche [34].

1.2 Preliminaries

For a graph G, $e(G)$ and $v(G)$ denote the number of edges and vertices (respectively) of G, $\delta(G)$ denotes the minimum degree of G, $\Delta(G)$ denotes the maximum degree, and $E(G)$ and $V(G)$ denote the sets of edges and vertices (respectively). For $A, B \subseteq V(G)$, $A \cap B = \emptyset$, we define $E_G(A : B)$ to be the set of edges of G joining A and B, and $e_G(A : B) = |E_G(A : B)|$. If $C \subseteq V(G)$ and $v \in V(G)$, then $N_C(v)$ denotes the set of neighbors of v in C. The length of a path is equal to the number of its edges.

For every graph G we define $d(H) = \frac{e(H)}{v(H)}$, $m(G) = \max_{H\subseteq G} d(H)$, $d_2(H) = \frac{e(H)-1}{v(H)-2}$ and

$$m_2(G) = \max_{\substack{H\subseteq G\\ v(H)\geq 3}} d_2(H).$$

A graph G with $m(G) = d(G)$ is called balanced.

Let T be a rooted tree with root r. Then the *down-degree* of a vertex $v \in V(T)\backslash r$ is $\underline{d}(v) = d(v)-1$. The down-degree of the root r is $\underline{d}(r) = d(r)$. The depth $\nu(T)$ of the tree T is equal to the maximal length of a path in T with one endpoint in r. A vertex $v \in V(T)$ is said to be on the ith level, if there exists an $r - v$ path in T of length i. The root r is said to be on the 0th level.

The logarithm $\log n$ in this thesis is always of natural base.

For functions $f(n), g(n) \geq 0$, we say that $f = O(g)$ if there are constants C and K, such that $f(n) \leq Cg(n)$ for $n \geq K$; $f = \Omega(g)$ if $g = O(f)$; $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$; $f = o(g)$ if $f(n)/g(n) \to 0$ when $n \to \infty$; $f = \omega(g)$ if $g = o(f)$; $f \sim g$ if $f(n)/g(n) \to 1$ when $n \to \infty$.

Let X be a sum of independent indicator random variables, with $\mu := \mathbf{E}[X] > 0$. Throughout the thesis, we are going to use the following Chernoff bounds, see, e.g., [35, Theorem 4.1, Theorem 4.2].

For $0 < \delta < 1$ and $\rho > 1$, we have

$$\Pr[X \leq \delta\mu] < \left(\frac{e}{\delta}\right)^{\delta\mu} \cdot e^{-\mu} < e^{-\mu(1-\delta)^2/2},$$

and

$$\Pr[X \geq \rho\mu] < \left(\frac{e}{\rho}\right)^{\rho\mu} \cdot e^{-\mu}.$$

You can't win.
You can't break even.
You can't even quit the game.

Ginsberg's Thm, from Murphy's Laws

Chapter 2

Basics

2.1 Positional games

2.1.1 Combinatorial games

A natural and general way to classify games is the following: The games of pure chance, the games of mixed chance and skill, and the games of pure skill (sometimes also called the games of no chance).

Combinatorial game theory focuses on the games of pure skill and no chance. It studies strategies and outcomes of two-player games of perfect knowledge, like chess or Tic-Tac-Toe. An important distinction between this topic and classical game theory (a branch of economics) is that game players are assumed to move in sequence rather than simultaneously, so there is no point in randomization or other information-hiding strategies.

Typically, the payoff function has three values: win of the first player, win of the second player, and a draw. Since the games we are looking at are of perfect information and of pure skill, the outcome depends only on the skill of each of the players. If we assume that both players are playing according to the best possible strategy, it is known—in theory—who will win even before the game starts. In other words, depending on the outcome we can divide all such games into three classes. Either the first player has a strategy to win, or the second player has a strategy to win, or both players have a strategy to avoid losing.

Amongst all combinatorial games, we would like to devote particular attention to positional games.

2.1.2 Strong positional games

In the following three sections we largely rely on the preliminary version of the book [12] by Beck.

Before the formal description of the strong positional games, we give an example—a well-known positional game called Tic-Tac-Toe.

The board on which the game is played is a 3-by-3 square grid. Two players, Xena and Obelix, alternately put their marks into the squares. First Xena spots one of the nine squares and marks it with an "X", then Obelix spots one of the unmarked squares and marks it with an "O", and so on. The player who *first* marks three squares in a line wins. If all nine squares are marked and nobody has won, it is a draw.

It is well-known that this game is a draw if both players are playing as good as possible. To define the game more formally, we assign numbers to squares. Then the board on which the game is played can be seen as $X = \{1, 2, \ldots, 9\}$. In each move, Xena claims one unclaimed element of X, and then Obelix claims one unclaimed element of X. All the possible ways to win can be now expressed without the geometric interpretation—a player wins as soon as (s)he claims all elements of one of sets in

$$\begin{aligned}\mathcal{F} &= \{\{1,2,3\},\{4,5,6\},\{7,8,9\},\{1,4,7\},\\ &\quad\{2,5,8\},\{3,6,9\},\{1,5,9\},\{3,5,7\}\}.\end{aligned}$$

In the same fashion we can define a general notion of a strong positional game. Let X be a finite nonempty set and $\mathcal{F} \subseteq 2^X$. We refer to X as the board, and the set $\mathcal{F}$ as the set of winning sets. The players alternately claim elements of X and the winner is the one who first claims all elements of some winning set $F \in \mathcal{F}$. Often it is enough to fix the family of winning sets, as we can take the union $\cup_{F \in \mathcal{F}} F$ as the board.

Obviously, playing an extra move at any point of the game cannot hurt a player. In particular, if the second player would have a strategy to win a game, the first player could also apply the same strategy after his first move and win the game. That means that second player cannot win, if we assume that both players are playing the optimal strategies. Therefore, the only two possible outcomes of the game are a first player's win, and a draw. The argument that we informally described here appears in the literature as *strategy stealing* (see [9]).

Even though the outcome of a game is its property, no efficient way of determining it in general is known. One possibility is a brute force search through the whole game tree, which is of exponential size in the size of the board.

Another argument that can be applied to the strong positional games is a Ramsey type argument. Namely, we say that the board X has a Ramsey property with respect to the set of winning sets $\mathcal{F}$, if in any 2-coloring of X one of the elements of $\mathcal{F}$ is monochromatic. Then the draw is impossible, which implies that the first player has s strategy of winning the game.

A simple concept that helps us figure out that a game is a draw, is the existence of a *pairing strategy* for the second player. Suppose that for every $F \in \mathcal{F}$ there exist two elements $x_F^1, x_F^2 \in F$, such that for every two different $F, F' \in \mathcal{F}$ we have $\{x_F^1, x_F^2\} \cap \{x_{F'}^1, x_{F'}^2\} = \emptyset$, i.e., there exists a matching of the board elements, such that every $F \in \mathcal{F}$ contains both elements of a matched pair. Then the second player can avoid loosing. Indeed, he has a simple strategy that can prevent the first player from completely occupying a winning set—whenever the first player claims one element of such a two-set, the second player claims the other one. Then, since every winning set contains one such pair, at the end of the game each of the winning sets will have at least one element claimed by the second player.

On the topic of strong positional games many questions have been asked, but only few are answered. The concept of the game is rather simple and many of the well-known games fit into this framework. The problem is that we are still lacking the means to deal with strong positional games, and problems often seem hopelessly unsolvable. For most of the games, the only way to determine the outcome is brute force analysis of the game tree, for which we lack computational power even at very small instances. In the remaining we list some results on strong positional games.

One of the games that has been thoroughly studied is Tic-Tac-Toe itself, or more precisely, its generalizations. Here we look at the so called n^d-game.

The board is the d dimensional cube of edge size n, subdivided into n^d congruent subcubes. The original game of Tic-Tac-Toe is a 3^2 game according to this definition. The course of the game is also analog—the player who first occupies n cubes in a line wins.

Hales and Jewett [25] proved by a Ramsey type argument that for every n there exists $d(n)$ such that n^d game is a first player's win for $d \geq d(n)$. However, the bounds for the smallest such d are very far from being tight. The best bound from above given by Shelah in [40] is $d < \text{tower}_n(n)$, where $\text{tower}_k(n)$ denotes the k-fold iteration of the exponential function: $\text{tower}_1(n) = 2^n$, and $\text{tower}_k(n) = 2^{\text{tower}_{k-1}(n)}$. Hales and Jewett on the other hand proved that $d \geq n$, which is still tremendously far away from the upper bound.

Beck [11] resolved the outcome of several n^d games for concrete values of d and n. The smallest n^d game for which it is not known whether the first player wins is 5^3 game.

2.1.3 Weak positional games

The concept of weak positional games comes out as a simplification of the rules for the strong positional games, and we mainly deal with them in the thesis.

The setup remains the same—the game is played on a finite nonempty set X, and the set of winning sets is $\mathcal{F} \subseteq 2^X$. However, the way in which the game is played has one crucial difference. Only one player—whom we call Maker—has a goal of occupying a winning set, while the other one—Breaker—is just trying to stop him from doing so. More precisely, if at any point of the game Maker claims all elements of a winning set $F \in \mathcal{F}$, then he wins the game. If all the elements of the board are claimed and Maker did not win, Breaker wins. As we mentioned before, assuming that both players are playing best possible strategies, the winner is determined even before the game starts. Each game is either a Maker's win or a Breaker's win, and no draw is possible.

The pair $(X, \mathcal{F})$ we call a *weak positional game*, a *Maker-Breaker posi-*

tional game, or sometimes just a *positional game* if there is no confusion.

We come back to our first example—Tic-Tac-Toe, now played in a Maker-Breaker version. It turns out that the outcome of the game is different from the outcome of the strong version. It is easy to find a strategy for Maker to win, if he wants to claim three squares in a line, but not necessarily first.

Already this suggests that the concept of a weak game is substantially different from strong games. Before we get deeper into the topic of weak games, let us note which results on the strong games transfer to the weak games.

Firstly, we mention the obvious—if the first player has a winning strategy in a strong game, then Maker has a strategy in the corresponding weak game. Also, if Breaker can win a weak game, then the second player surely can force a draw in the strong game counterpart.

Since the roles of Maker and Breaker are crucially different, there is no point in looking at strategy stealing—Maker cannot win a game by adopting a strategy of Breaker in the same game, and vice versa. As we will see later, it may be that Maker can win a game by adopting the role of Breaker in a *different* game, but this concept is more subtle, it requires some analysis and in that case we cannot talk about strategy stealing anymore.

On the other hand, Ramsey type arguments can be applied to guarantee Maker's win in the weak game, since they give the existence of a winning strategy for the first player in the strong game.

As for the pairing strategy argument, we now see that this concept is actually tailored for the role of Breaker in a weak game, and its application for the second player's draw in the strong game is just an implication.

Unlike for strong games, the list of tools to tackle weak games extends beyond this point. In the following we present a few general criterions that can be applied on a wide spectrum of games, ensuring the win for one of the players.

The first to be mentioned is certainly the famous Erdős-Selfridge Theorem, that provides Breaker's win. It is surprisingly easy to state and can be applied on virtually every positional game that satisfies the single condition.

Theorem 1 *(Erdős-Selfridge [21]) If*

$$\sum_{A\in\mathcal{F}} 2^{-|A|} < 1/2,$$

then Breaker has a winning strategy in the $(X, \mathcal{F})$ *game.*

A particularly remarkable thing about this statement is that it depends neither on the size of the board, nor on the winning set structure—the only thing that matters is the size of the winning sets. One may expect that a statement of such simple form cannot give strong results. But that is not at all true—for several families of games its result is sharp, i.e., whenever Breaker can win, the theorem can be applied to confirm it (see, e.g., [10]). Another additional treat of the Erdős-Selfridge Theorem is that it not only ensures that Breaker can win, but its proof also provides a winning strategy for Breaker.

We switch sides now, and look at Maker's possibilities to win. The following theorem was proven by Beck in [4], giving a general criterion for Maker's win.

Theorem 2 *(Beck [4]) Let* $(X, \mathcal{F})$ *be a positional game. Assume that for any two points in* X *there are not more than* $\Delta_2 = \Delta_2(\mathcal{F})$ *winning sets* $A \in \mathcal{F}$ *that contain both of them. If*

$$\sum_{A\in\mathcal{F}} 2^{-|A|} > \frac{1}{8}\Delta_2|X|,$$

then Maker has a winning strategy in $(X, \mathcal{F})$.

We refer to $\Delta_2(\mathcal{F})$ as the pair-degree.

If X is a finite nonempty set, $\mathcal{F} \subseteq 2^X$ and a, b are positive integers, then the 4-tuple $(X, \mathcal{F}, a, b)$ is a *biased* $(a\colon b)$ *game.* In a biased $(a\colon b)$ game, Maker claims a elements (instead of 1) and Breaker claims b elements (instead of 1) in each move.

For an unbiased game $(X, \mathcal{F})$ that is a Maker's win, a natural question to ask is: What is the smallest integer $b_{\mathcal{F}}$ for which Breaker wins the $(1 : b_{\mathcal{F}})$ game (see Figure 2.1)? On the other hand, if an unbiased game

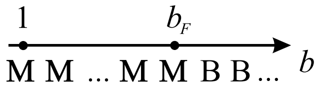

Figure 2.1: Threshold bias

is a Breaker's win, we can pose the analog question about the threshold for the $(a_{\mathcal{F}} : 1)$ game. The problem of finding this threshold for various positional games has been studied in [1, 2, 3, 13, 14, 15, 18].

Throughout the thesis we will assume that Maker starts the game, unless otherwise stated. But, we would also like to discuss the games in which Breaker starts. In order to avoid confusion, the biased game with board X and set of winning sets $\mathcal{F}$ in which Breaker starts is denoted by $(\widehat{X}, \mathcal{F}, a, b)$. Note that a is always the bias of Maker, independently from who is the first player to move.

A version of the Erdős-Selfridge Theorem for biased games was proved by Beck in [2]. We present here its version for the games started by Breaker.

Theorem 3 *(Beck, [2]) If*

$$\sum_{A \in \mathcal{F}} (1+b)^{-|A|/a} < 1,$$

then Breaker has a winning strategy in $(\widehat{X}, \mathcal{F}, a, b)$.

If Maker plays first, then 1 on the right hand side of the criterion is to be replaced by the fraction $\frac{1}{1+b}$.

A counterpart winning criterion for Maker for biased games was also proved by Beck.

Theorem 4 *(Beck, [2]) Let $(X, \mathcal{F}, a, b)$ be a biased positional game. Assume that for any two points in X there is not more than $\Delta_2 = \Delta_2(\mathcal{F})$ winning sets $A \in \mathcal{F}$ that contain both of them. If*

$$\sum_{A \in \mathcal{F}} \left(1 + \frac{b}{a}\right)^{-|A|} > \frac{a^2 b^2}{(a+b)^3} \Delta_2 |X|,$$

then Maker has a winning strategy in $(X, \mathcal{F}, a, b)$.

As a further generalization, we look at biased games in which the goal of Maker is to claim many (instead of one) winning sets. More precisely, Maker wins the game if he claims all elements of at least c winning sets from $\mathcal{F}$. Then we are actually talking about the game $(X, \{\cup_{B\in F} B : F \in \binom{\mathcal{F}}{c}\}, a, b)$, whose winning sets are all possible unions of c sets in $\mathcal{F}$. Later, we are going to make use of the following extension of the Biased Erdős-Selfridge Theorem.

Theorem 5 *([2, 13]) If for a positive integer c we have*

$$\sum_{A\in\mathcal{F}} (1+b)^{-|A|/a} < c\frac{1}{1+b},$$

then Breaker has a winning strategy in the $(X, \{\cup_{B\in F} B : F \in \binom{\mathcal{F}}{c}\}, a, b)$ *game.*

The games we are particularly interested in are the games on graphs. The board on which the game is played is the set of edges of a complete graph on n vertices, $X = E(K_n)$. The winning sets are usually representatives of some well-known graph-theoretic structure, like clique, path, cycle, spanning tree, Hamiltonian cycle, etc. These games are frequently studied in the literature in the last couple of years, and some results in connection with them can be found in [1, 5, 7, 9, 13, 14, 15, 18, 24, 31, 37, 41].

As a first example, we can look at the triangle Maker-Breaker game played on edges of K_5. For a precise definition, let $\mathcal{K}_3 = \mathcal{K}_3(K_5)$ be the set of all 3-sets of edges of K_5 which form a triangle. The game is formally described as $(E(K_5), \mathcal{K}_3)$.

Without going into details, we will show that Maker can win this game. He can start by claiming three (out of four) edges adjacent to a vertex. Then, after his third move, there are two unclaimed edges that potentially can close Maker's triangle. Breaker cannot take both of them, so Maker wins in his fourth move.

It is also not hard to check that the triangle game on edges of K_4 can be won by Breaker. On the other hand, $(E(K_n), \mathcal{K}_3)$ is a Maker's win for every fixed $n \geq 6$. That follows simply by a Ramsey type argument—it is well-known that every 2-coloring of K_n contains a monochromatic triangle, if $n \geq 6$.

We saw that on a large graph Maker can "easily" claim a triangle, meaning that he can do it in just four moves. This answers our initial question of who can win the game, but the proof and the result itself do not seem that breathtaking (in our opinion, at least). One obvious way to make it harder for Maker and spice up the game a little bit is to let Breaker claim more than one edge per move. The new, more interesting question is: Who wins the biased game $(E(K_n), \mathcal{K}_3, 1, b)$?! Depends on what is b, of course.

First thing to note is that if Breaker can win the $(1 : b)$ game for $b = b_0$, then he also can win it for every $b > b_0$. We already saw that for $b = 1$ Maker wins, and Breaker obviously can win when $b = \binom{n}{2} - 1$. Therefore, there has to exist the smallest integer $b_{\mathcal{K}_3} = b_{\mathcal{K}_3}(n)$ for which Breaker wins.

2.1.4 Avoider-Forcer and Picker-Chooser games

In the previous section we saw what a Maker-Breaker game is.

The so called Avoider-Forcer games are the misère version of Maker-Breaker games. There are two players, called Avoider and Forcer. The main difference—informally speaking—is that in Maker-Breaker games Maker's goal is to create, while in Avoider-Forcer games Avoider wants to avoid creating.

The setup stays the same, there is a board X and a set of winning sets $\mathcal{F} \subseteq 2^X$, and Avoider and Forcer alternately claim the unclaimed elements of X. Forcer wins as soon as Avoider claims all the elements of a winning set from $\mathcal{F}$. If the game ends and Forcer did not win, then Avoider wins (for more detail see, e.g., [9]).

Without going into details, we note that Beck gave the analogs of unbiased versions of Erdős-Selfridge Theorem (Theorem 1), now for Forcer's win, and Theorem 2, now for Avoider's win. Furthermore, Hefetz, Krivelevich and Szabó [26] gave the extension of the criterion for Avoider's win to all games with bias $(b : 1)$. To our knowledge there is still no general criterion of that kind for either Avoider's win or Forcer's win in $(a : b)$

game.

In so-called Picker-Chooser games we come across yet another way of playing on a board X. They are defined in [3] by Beck. The game is played by two players, Picker and Chooser. In each move, Picker picks two previously unpicked elements of the board, and then Chooser chooses which one of the two edges to claim, and the other one automatically goes back to Picker. Note that this way at any point of the game both players have the same number of elements claimed.

We still did not say how each of the players can win. Again we have the set of winning sets $\mathcal{F} \subseteq 2^X$, but there are still several versions of the game depending on how the winner is determined. Firstly, we can either look at the standard game in which the goal is to claim a winning set, or the misère version of it in which the goal is to avoid claiming a winning set.

Note that unlike in Maker-Breaker or Avoider-Forcer games, in the Picker-Chooser games we distinguish between the players already in the way the game is played. Therefore, there is the total of four different kinds of games.

If Chooser wants to occupy a winning set, then we call the game Chooser-Picker game. On the other hand, if the goal of Picker is to occupy a winning set, then we call the game Picker-Chooser game. For games of avoiding winning sets we also have two variants. If Chooser avoids, we call it Chooser-Picker Misère game, and if Picker avoids, we call it Picker-Chooser Misère game.

2.2 Random graphs

In this section we will introduce random graphs, and mention some of their basic properties. This topic is about 50 years old, but had its rapid expansion in the last two decades. One can get a better insight in the theory of random graphs by exploring books by Bollobás [17], and Janson, Łuczak, Ruciński [28].

2.2.1 Models of random graphs

Informally speaking, a random graph is a graph obtained as a result of a random procedure. This can be formalized by representing the "random procedure" by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is the set of all graphs with vertex set $[n]$, $\mathcal{F}$ is the family of all subsets of Ω, and $\mathbf{P}$ is a probability distribution on Ω. Depending on the definition of $\mathbf{P}$, there are two basic models of random graphs.

Given a real number p, with $0 \leq p \leq 1$, and an integer n, the *binomial random graph* is defined by setting

$$\mathbf{P}(G) = p^{e_G}(1-p)^{\binom{n}{2}-e_G}$$

for every graph G on vertex set $[n]$. The binomial random graph is denoted by $G(n,p)$.

One way to check that the function above is really a probability distribution is to calculate directly the sum of $\mathbf{P}(G)$ over all graphs G on vertex set $[n]$, using binomial formula. But it is also important to recognize the "random procedure" behind the definition. The distribution can be viewed as a result of $\binom{n}{2}$ independent coin flips, one for each couple of vertices, with probability of success equal to p. Then two vertices are connected if and only if the coin flip for this pair of vertices was successful.

The real number p should not necessarily be seen as a constant. Actually, large part of the random graph literature is devoted to cases in which $p = p(n) \to 0$ as $n \to \infty$.

Given an integer M, with $0 \leq M \leq \binom{n}{2}$, the *uniform random graph* is defined by setting

$$\mathbf{P}'(G) = \binom{\binom{n}{2}}{M}^{-1}$$

for every graph G with M edges on vertex set $[n]$, and

$$\mathbf{P}'(G) = 0$$

for all other graphs. The uniform random graph is denoted by $G(n,M)$.

The random graph process is a stochastic process that describes the random graph evolving in time. The process starts from an empty graph on vertex set $[n]$ and proceeds by adding one edge at a time, chosen uniformly at random from the set of all edges not yet chosen.

To be more formal, let $e_1, \ldots, e_m$ be the edges of K_n, where $m = \binom{n}{2}$. Choose a permutation $\pi \in S_m$ uniformly at random and define an increasing sequence of subgraphs (G_i) where $V(G_i) = [n]$ and $E(G_i) = \{e_{\pi(1)}, \ldots, e_{\pi(i)}\}$. It is clear that G_i is a graph with i edges, selected uniformly at random from all n-vertex graphs with i edges. In other words, $G_i = G(n, i)$.

2.2.2 Graph properties, thresholds and hitting time

A set $\mathcal{Q}$ of graphs with vertex set $[n]$ is called a *property of graphs of order* n if for every two graphs G and H on vertex set $[n]$, $G \in \mathcal{Q}$ and G being isomorphic to H implies that $H \in \mathcal{Q}$. If $G \in \mathcal{Q}$, we say that "graph G has property $\mathcal{Q}$".

A property is said to be *monotone increasing*, if for every two graphs G and H on vertex set $[n]$, $G \in \mathcal{Q}$ and $G \subseteq H$ implies that $H \in \mathcal{Q}$. That is, a property is called increasing if a graph that has this property cannot lose it when some edges are added to it. The complement of a monotone increasing property is called a *monotone decreasing* property.

For example, we can define the property "is connected" as the set $\mathcal{C}$ of all graphs on vertex set $[n]$ which are connected. This property is monotone increasing, since adding edges to a connected graph cannot make it disconnected. On the other hand, the complement property $\overline{\mathcal{C}}$ contains all graphs that are not connected, and it is decreasing. Obviously, taking away edges from a graph that has a decreasing property will not make it lose this property.

The following two propositions are crucial, as they provide a frequently exploited connection between the binomial and uniform random graphs. Let $m = \binom{n}{2}$.

Proposition 6 *[28, Proposition 1.12] Let $\mathcal{Q} = \mathcal{Q}(n)$ be an arbitrary (not necessarily monotone) property, $p = p(n) \in [0,1]$ and $0 \leq a \leq 1$. If for every sequence $M = M(n)$ such that*

$$M = mp + O(\sqrt{mp(1-p)})$$

it holds that $\Pr[G(n,M) \in \mathcal{Q}] \to a$ *as* $n \to \infty$*, then also* $\Pr[G(n,p) \in \mathcal{Q}] \to a$ *as* $n \to \infty$*.*

In the other direction we do not have asymptotic equivalence in such generality. But the analog statement holds if we assume that the property is monotone.

Proposition 7 *[28, Proposition 1.13] Let* $\mathcal{Q} = \mathcal{Q}(n)$ *be a monotone property,* $0 \leq M = M(n) \leq m$ *and* $0 \leq a \leq 1$*. If for every sequence* $p = p(n) \in [0,1]$ *such that*

$$p = \frac{M}{m} + O\left(\sqrt{\frac{M(m-M)}{m^3}}\right)$$

it holds that $\Pr[G(n,p) \in \mathcal{Q}] \to a$ *as* $n \to \infty$*, then also* $\Pr[G(n,M) \in \mathcal{Q}] \to a$ *as* $n \to \infty$*.*

For a monotone increasing property $\mathcal{Q}$, a sequence $\widehat{p} = \widehat{p}(n)$ is called a *threshold* if

$$\Pr[G(n,p) \in \mathcal{Q}] \to \begin{cases} 0 & \text{if } p = o(\widehat{p}) \\ 1 & \text{if } p = \omega(\widehat{p}) \end{cases} .$$

A threshold $\widehat{M} = \widehat{M}(n)$ for the uniform model of the random graphs is defined analogously by

$$\Pr[G(n,M) \in \mathcal{Q}] \to \begin{cases} 0 & \text{if } M = o(\widehat{M}) \\ 1 & \text{if } M = \omega(\widehat{M}) \end{cases} .$$

One surprising property of monotone thresholds is the so called threshold behavior. Ever since the following theorem was proved by Bollobás and Thomason, a lot of research has been directed towards finding thresholds for various properties of graphs.

Theorem 8 *[16] Every monotone property has a threshold.*

Knowing that every monotone property has this sudden jump at the threshold, a next question to ask is how sudden the jump really is? It turns out that it is not the same for all properties.

property	*threshold p*
G contains a (fixed) subgraph H	n^{-v_H/e_H}
G is connected	$\frac{\log n}{n}$
G has a perfect matching	$\frac{\log n}{n}$
G has a Hamiltonian cycle	$\frac{\log n}{n}$
G is non-planar	n^{-1}

Table 2.1: Thresholds for some graph properties in $G(n,p)$

If for a monotone property $\mathcal{Q}$ and sequence $\widehat{p} = \widehat{p}(n)$ we have

$$\Pr[G(n,p) \in \mathcal{Q}] \to \begin{cases} 0 & \text{if } p \leq (1-\varepsilon)\widehat{p} \\ 1 & \text{if } p \geq (1+\varepsilon)\widehat{p} \end{cases}$$

for every $\varepsilon > 0$, then we say that $\widehat{p}$ is a *sharp threshold* for property $\mathcal{Q}$.

In Table 2.1 we give some graph properties and thresholds for their appearance in $G(n,p)$ random graph model.

One additional property of the random graphs that we will use later is the following estimate for the number of copies of a fixed graph H appearing in $G(n,p)$. Let Y_H denote the number of copies of a fixed graph H appearing in the random graph $G(n,p)$ and $\mu = \mathbf{E}[Y_H]$.

Theorem 9 *[42, Theorem 2.1] If H is balanced and ε is a positive constant, then*

$$\Pr[Y_H \geq (1+\varepsilon)\mu] \leq \exp\left(-\Omega\left(\mu^{\frac{1}{v_H-1}}\right)\right).$$

We now come back to the random graph processes. Given a particular graph process (G_i) and a monotone increasing graph property $\mathcal{P}$ possessed by K_n, the *hitting time* $\tau(\mathcal{P}) = \tau(\mathcal{P}, (G_i))$ is the minimal i for which G_i has property $\mathcal{P}$.

It turns out that some pairs of properties have the same hitting time a.s., meaning that in almost all graph processes they come at the exactly same time. For example, it can be shown that the hitting time for "having a perfect matching" is a.s. the same as the hitting time for "having no isolated vertices" [28, Theorem 4.6].

2.3 Games on random graphs

We briefly describe two different games on random graphs that we are going to analyze in detail later.

For Maker-Breaker games, we introduce a new approach to even out a possible advantage Maker has in $(1\colon 1)$ game, by randomly reducing the board size and keeping only those winning sets which survive this thinning intact.

Definition 1 *Let* $(X, \mathcal{F}, a, b)$ *be a biased game. A* random game $(X_p, \mathcal{F}_p, a, b)$ *is a probability space of games where each* $x \in X$ *is independently included in* X_p *with probability* p*, and* $\mathcal{F}_p = \{W \in \mathcal{F} : W \subseteq X_p\}$*.*

Apart from the trivial case $\emptyset \in \mathcal{F}$, Breaker surely wins when $p = 0$. On the other hand, for $p = 1$ the winner of the random game is the same as in $(X, \mathcal{F}, a, b)$. For any other probability p, $0 < p < 1$, we cannot be sure who (Maker or Breaker) wins the random game $\mathcal{F}_p$. The best we can conclude is that Maker (or Breaker) wins a.s. (almost surely), i.e., the probability that Maker (Breaker) wins tends to 1 if the board size tends to infinity. (So we actually talk about an infinite family of probability spaces of games ...) We are particularly interested in random Maker-Breaker games on edges of a complete graph on n vertices. We consider this model in Chapter 3.

Now we switch to a different kind of games on random graphs, which we analyse in Chapter 5. They are played by a single player, Painter. He maintains a balanced 2-coloring in the random graph process, coloring two edges at a time in an online fashion. His goal is to avoid creating a monochromatic copy of a fixed graph F for as long as possible.

More precisely, if the edges in a graph process are coming in order $(e_{\pi(1)}, e_{\pi(2)}, \ldots, e_{\pi(m)})$, then in the ith move of the game Painter is presented with edges $e_{\pi(2i-1)}$ and $e_{\pi(2i)}$. He then immediately and irrevocably chooses one of the two possibilities to color one of them red and the other one blue. Therefore, after playing the first i moves, Painter has created a balanced 2-coloring of the graph G_{2i}. Note that at the move i he has no knowledge of the order in which the remaining edges will be presented to him in the moves to come.

Let F be a fixed graph. Painter loses the game as soon as he creates a monochromatic copy of F, i.e., Painter loses in the move $\min\{i : G_{2i}$ contains a monochromatic copy of $F\}$. His goal is to play as long as possible without losing, and we would like to determine how long that is.

Enemy spotted!
...
Hold this position!

radio messages from Counter Strike

Chapter 3

Positional games on random graphs

3.1 Introduction

Typical, well-studied examples of positional games are played on the edges of a complete graph, i.e., $X = E(K_n)$. Maker's goal usually is to build a graph theoretic structure – like a spanning tree, a perfect matching, a Hamiltonian cycle, or a clique of fixed size. It turns out that all these games are won easily by Maker if n is sufficiently large, so in order to make things more fair (if such thing exists; actually no game of perfect information is *fair* as the winner—in theory—is known in the beginning of the game) one could give Breaker extra power by allowing him to claim more than 1 edge in each move.

For a family $\mathcal{F}$ the smallest integer $b_{\mathcal{F}}$ is sought (and sometimes found; see [1, 2, 3, 13, 14, 15, 18]) for which Breaker wins the $(1 : b_{\mathcal{F}})$ game.

In the *connectivity game* Maker's goal is to build a connected spanning subgraph; i.e., in this game the family of winning sets is the family $\mathcal{T} = \mathcal{T}_n$ of all spanning trees on n vertices. Chvátal and Erdős proved [18] that $b_{\mathcal{T}} = \Theta(\frac{n}{\log n})$.

Beck [1] established $b_{\mathcal{H}} = \Theta(\frac{n}{\log n})$, where $\mathcal{H} = \mathcal{H}_n$ is the family of all

Hamiltonian cycles on n vertices.

For the family $\mathcal{K}_k = \mathcal{K}_{k,n}$ of all k-cliques on n vertices, Bednarska and Łuczak [13] showed that $b_{\mathcal{K}_k} = \Theta(n^{\frac{2}{k+1}})$. More generally, they proved that in the game in which Maker's goal is to claim an arbitrary fixed graph G, the threshold bias is $\Theta(n^{1/m_2(G)})$.

Playing on a random board. Let $(X, \mathcal{F})$ be a particular sequence of games, where $\emptyset \notin \mathcal{F}$, the board size tends to infinity, and $(X, \mathcal{F}, 1, 1)$ is won by Maker provided $|X|$ is big enough. The first natural question to ask is: What is the threshold probability $p_{\mathcal{F}}$ at which an almost sure Breaker's win turns into an almost sure Maker's win? More precisely, we would like to determine $p_{\mathcal{F}}$ for which

- $\Pr[(X_p, \mathcal{F}_p, 1, 1)$ is a Breaker's win$] \to 1$ for $p = o(p_{\mathcal{F}})$, and

- $\Pr[(X_p, \mathcal{F}_p, 1, 1)$ is a Maker's win$] \to 1$ for $p = \omega(p_{\mathcal{F}})$,

as in Figure 3.1.

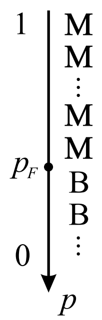

Figure 3.1: Threshold probability

Such a threshold $p_{\mathcal{F}}$ exists [16], since being a Maker's win is an *increasing property.*

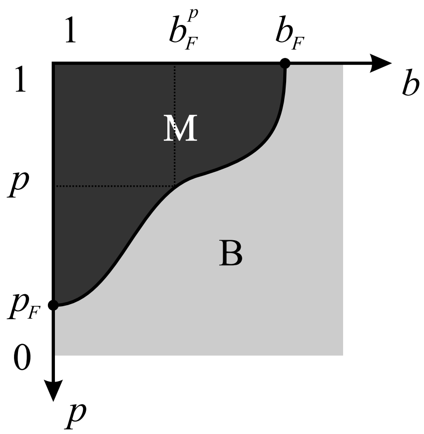

Figure 3.2: Dependency of bias and probability

Our main goal in this chapter is to establish a connection between the natural threshold values, $b_{\mathcal{F}}$ and $p_{\mathcal{F}}$, corresponding to the two different weakenings of Maker's power: bias and random thinning, respectively. We find that there is an intriguing reciprocal connection between these two thresholds in a number of well-studied games on graphs.

Recall the notations $\mathcal{T}$, $\mathcal{H}$, and $\mathcal{K}_k$, and let us denote by $\mathcal{M}$ the set of all perfect matchings on the graph K_n.

Theorem 10 *For positional games, played on $E(K_n)$, we have*

(i) $p_{\mathcal{T}} = \frac{\log n}{n}$,

(ii) $p_{\mathcal{M}} = \frac{\log n}{n}$,

(iii) $p_{\mathcal{H}} = \frac{\log n}{n}$,

(iv) $n^{-\frac{2}{k+1}-\varepsilon} \leq p_{\mathcal{K}_k} \leq n^{-\frac{2}{k+1}}$, *for every integer $k \geq 4$ and every constant*

$\varepsilon > 0$.

(v) $p_{\mathcal{K}_3} = n^{-\frac{5}{9}}$.

For the connectivity game $\mathcal{T}$ an even more precise statement is true. In Corollary 28 we observe that Maker starts to win a.s. at the very moment when the last vertex of the random graph process picks up its second incident edge.

Note that part (iii) of the last theorem implies that in the random graph with edge probability $p \geq C\frac{\log n}{n}$ Maker can build a Hamiltonian cycle in the one-on-one game. So Pósa's result [38] (which gives the existence of a Hamiltonian cycle) is true constructively even if an adversary is playing against us.

More generally, for every p we would like to find the smallest bias $b^p_{\mathcal{F}}$ such that Breaker wins the random game $(X_p, \mathcal{F}_p, 1, b^p_{\mathcal{F}})$ a.s.

Note that by definition $b_{\mathcal{F}} = b^1_{\mathcal{F}}$. Another trivial observation is that $b^p_{\mathcal{F}} = 0$ provided p is less than the threshold for the appearance of the first element of $\mathcal{F}$ in the random graph. Hence, we get a general dependency of bias and probability, as described in Figure 3.2.

We obtain the following.

Theorem 11 *There exist constants C_1, C_2, C_3, such that*

(i) $b^p_{\mathcal{T}} = \Theta\left(pb_{\mathcal{T}}\right) = \Theta\left(p\frac{n}{\log n}\right)$, *provided* $p \geq C_1\frac{1}{b_{\mathcal{T}}}$,

(ii) $b^p_{\mathcal{M}} = \Theta\left(pb_{\mathcal{M}}\right) = \Theta\left(p\frac{n}{\log n}\right)$, *provided* $p \geq C_2\frac{1}{b_{\mathcal{M}}}$,

(iii) $\Omega\left(p\frac{\sqrt{n}}{\log n}\right) \leq b^p_{\mathcal{H}} \leq O\left(p\frac{n}{\log n}\right)$, *provided* $p \geq C_3\frac{\log n}{\sqrt{n}}$,

(iv) There exists $c_k > 0$, *such that* $b^p_{\mathcal{K}_k} = \Theta\left(pb_{\mathcal{K}_k}\right) = \Theta\left(pn^{\frac{2}{k+1}}\right)$, *provided* $p = \Omega\left(\frac{\log^{c_k} n}{b_{\mathcal{K}_k}}\right)$.

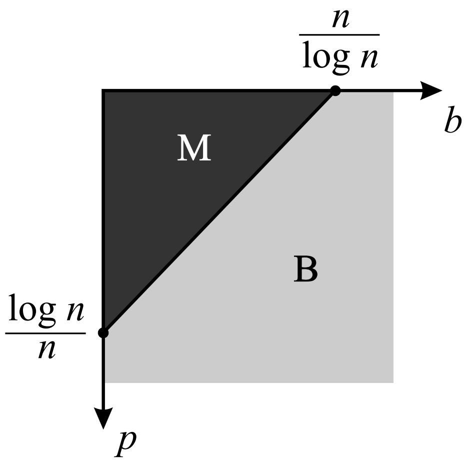

Figure 3.3: Connectivity game and matching game

One can see that $b^p_{\mathcal{F}}$ is of order $p/p_{\mathcal{F}} = pb_{\mathcal{F}}$ for the connectivity game and the perfect matching game (Figure 3.3), provided $p \geq Cp_{\mathcal{F}}$ for some constant C. In particular for these games $p_{\mathcal{F}} = \Theta(1/b_{\mathcal{F}})$.

In the Hamiltonian cycle game, we can obtain the exact value only for (1 : 1) games, for other values of bias we just give some upper and lower bounds (see Figure 3.4). Nevertheless we think that our arguments for finding $p_{\mathcal{H}}$ can be extended for general bias, to prove that the Hamiltonian cycle game behaves "nicely", i.e., the same way as the connectivity game and the perfect matching game.

Conjecture 1 *Let $\mathcal{H}$ be the set of Hamiltonian cycles in K_n. There exists a constant C such that*

$$b^p_{\mathcal{H}} = \Theta\left(p\frac{n}{\log n}\right), \text{ provided } p \geq C\tfrac{\log n}{n}.$$

Note that the Theorem 10 (iii) implies that the conjecture is valid when p is of order $\frac{\log n}{n}$.

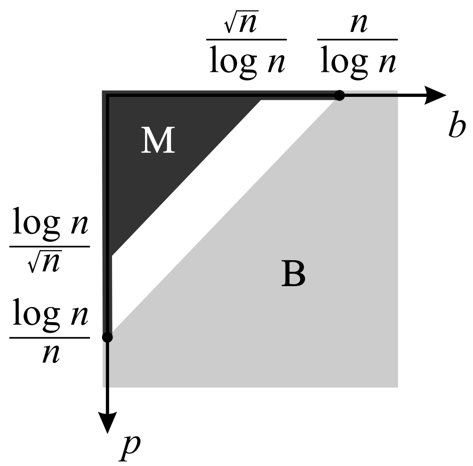

Figure 3.4: Hamiltonian cycle game

In part (iv) of Theorem 11, generalizing the arguments of Bednarska and Łuczak [13] we show that one can estimate $b^p_{\mathcal{K}_k}$ up to a constant factor, for all probabilities down to a polylogarithmic factor away from the critical probability $1/b_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$, see Figure 3.5 (left).

On the other hand Theorem 10 part (v) shows that in the case $k = 3$ we cannot get arbitrarily close to probability $1/b_{\mathcal{K}_k}$, since Maker *can win* even for probabilities below $1/b_{\mathcal{K}_3} = n^{-1/2}$, as in Figure 3.5 (right).

3.2 A criterion

As we saw, one of few general, but still very applicable results to decide the winner of biased positional games is Theorem 3, the biased version of the Erdős–Selfridge Theorem. It provides a criterion for Breaker to win, applicable on any game.

In this section we give an adaptation of this criterion which proves to be very useful in dealing with positional games on a random board. We

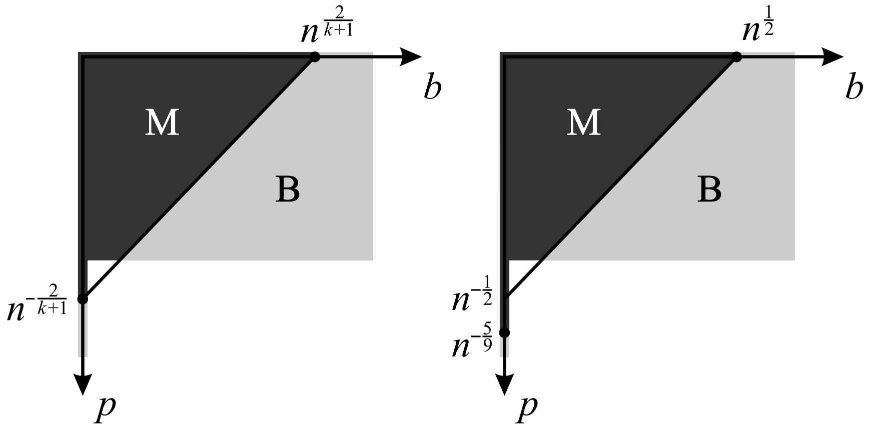

Figure 3.5: k-clique game, $k \geq 4$ (left), and triangle game (right)

need the following technical definition.

Definition 2 *Let* $(X, \mathcal{F}, a, b)$ *be a biased game.* Random game $(X_p, \mathcal{F}_p^{\cap}, a, b)$ with induced set of winning sets *is a probability space of games, where* X_p *is defined as in Definition 1 and*

$$\mathcal{F}_p^{\cap} = \{W : \exists F \in \mathcal{F},\, W = F \cap X_p\}.$$

The following statement is the randomized version of the Erdős-Selfridge Theorem. It is stated for the biased $(b\colon 1)$ game in which Breaker is the first player, because this is the version we will need in our applications.

Theorem 12 *Let* $\mathcal{F}$ *be a set of winning sets on* X *with*

$$\sum_{A \in \mathcal{F}} 2^{-\frac{|A|}{b}} < 1 \tag{3.1}$$

(i.e., the condition of the Erdős–Selfridge Theorem holds for the $(\widehat{X}, \mathcal{F}, b, 1)$ *game), and*

$$\lim_{n \to \infty} \min_{A \in \mathcal{F}} \frac{|A|}{b} = \infty. \tag{3.2}$$

If p and $\delta > 0$ are chosen so that

$$p > \frac{4 \log 2}{\delta^2 b}$$

holds, then the game $(\widehat{X}_p, \mathcal{F}_p^{\cap}, (1-\delta)pb, 1)$ is a Breaker's win a.s.

Proof. For each $A \in \mathcal{F}$ and its corresponding set $A' \in \mathcal{F}_p^{\cap}$ we have $\mathbf{E}[|A'|] = p|A|$. If all winning sets $A' \in \mathcal{F}_p^{\cap}$ have size at least $(1-\delta)p|A|$, then

$$\sum_{A' \in \mathcal{F}_p^{\cap}} 2^{-\frac{|A'|}{(1-\delta)pb}} \le \sum_{A \in \mathcal{F}} 2^{-\frac{(1-\delta)p|A|}{(1-\delta)pb}} = \sum_{A \in \mathcal{F}} 2^{-\frac{|A|}{b}} < 1.$$

Using the Erdős–Selfridge theorem we obtain that Breaker wins the $(\widehat{X}_p, \mathcal{F}_p^{\cap}, (1-\delta)pb, 1)$ game, provided $|A'| \ge (1-\delta)p|A|$ for all $A' \in \mathcal{F}_p^{\cap}$.

Next we check that this condition holds almost surely. Using a Chernoff bound, we obtain that

$$\Pr[\exists A \in \mathcal{F} : |A'| \le (1-\delta)p|A|] \le \sum_{A \in \mathcal{F}} e^{-\frac{\delta^2 p|A|}{2}}.$$

If we denote $\min_{A \in \mathcal{F}} \frac{|A|}{b}$ by m_n, then we have

$$\sum_{A \in \mathcal{F}} e^{-\frac{\delta^2 p|A|}{2}} \le \sum_{A \in \mathcal{F}} 2^{-2\frac{|A|}{b}} \le \sum_{A \in \mathcal{F}} 2^{-m_n} 2^{-\frac{|A|}{b}} < 2^{-m_n} \to 0,$$

and therefore all winning sets $A' \in \mathcal{F}_p^{\cap}$ have size at least $(1-\delta)p|A|$ a.s. □

3.3 Games

3.3.1 Connectivity game

The first game we study is a random version of the biased connectivity game $(E(K_n), \mathcal{T}, 1, b)$ on a complete graph on n vertices K_n. Maker's goal is to build a spanning, connected subgraph, i.e., $\mathcal{T}$ is the set of all spanning trees on n vertices.

It is obvious that $p_{\mathcal{T}} = \Omega(\frac{\log n}{n})$, since for lower probabilities the random graph is a.s. not connected, and Breaker wins even if he does not claim any edges.

First we generalize this for arbitrary probability p by providing Breaker with a strategy to isolate a vertex. One of our main tools is the following winning criterion of Chvátal and Erdős on games with disjoint winning sets.

Theorem 13 *[18] In a biased* $(b\colon 1)$ *game with* k *disjoint winning sets of size* s *in which Breaker makes the first move, Maker wins if*

$$s \leq (b-1)\sum_{i=1}^{k-1}\frac{1}{i}. \tag{3.3}$$

Corollary 14 *In a biased* $(b\colon 2)$ *game with* k *disjoint winning sets of size at most* s *Maker wins if*

$$s \leq \left(\left\lfloor\frac{b}{2}\right\rfloor - 1\right)\sum_{i=1}^{k-1}\frac{1}{i}.$$

Proof of Corollary. Recall that as a default Maker starts the game. We will prove that when Breaker starts, the bias is $(2b : 2)$, there are k winning sets and (3.3) holds, then Maker still wins. Indeed, since the winning sets are disjoint, after Breaker's move Maker can just pretend to play a $(b : 1)$ game and answer with his first b moves to one of the two selections of Breaker, and answer with his second b moves to the other move of Breaker, both according to the $(b : 1)$ strategy. Now the Corollary follows, since starting instead of being second player cannot hurt Maker. □

Theorem 15 *There exists* $K_0 > 0$ *so that for arbitrary* $p \in [0,1]$ *and* $b \geq K_0 p\frac{n}{\log n}$ *Breaker, playing the* $(1 : b)$ *game on the edges of the random graph* $G(n,p)$*, can achieve that Maker's graph has an isolated vertex a.s.*

Proof. Let us fix

$$b = \left\lfloor \frac{K_0 pn}{\log n} \right\rfloor,$$

where K_0 is a constant to be determined later. Note that we can assume $p > \log n/2n$, since otherwise the random graph has an isolated vertex a.s., thus Breaker achieves his goal without having to play any moves.

We present a strategy for Breaker to claim all the edges incident to some vertex of $G(n,p)$. If successful, this strategy prevents Maker from building a connected spanning subgraph. A similar strategy was introduced by Chvátal and Erdős [18] for solving the problem on the complete graph.

Let C be an arbitrary subset of the vertex set of cardinality $\lfloor n/\log n \rfloor$. Breaker will claim all the edges incident to some vertex $v \in C$ (thus preventing Maker from claiming any edge incident to v). We would like to use the game from Corollary 14, with the winning sets being the $\lfloor n/\log n \rfloor$ stars of size at most $n-1$ whose center is in C. Since these stars are not necessarily disjoint, formally we will talk about ordered pairs of vertices: the winning sets are denoted by $W_v = \{(v,u) : u \in V\}$, $v \in C$. We call this game *Box*. To avoid confusion with Maker and Breaker of the game from Theorem 15, the players from Corollary 14 will be called BoxMaker and BoxBreaker. Recall that in Box the bias is $(b:2)$.

Breaker will utilize the strategy of BoxMaker from Corollary 14 to achieve his goal. How? He will play a game of Box in such a way that a win for BoxMaker automatically implies a win for Breaker. When Maker selects an edge uv, Breaker interprets it as BoxBreaker claimed the elements (u,v) and (v,u) in Box. Whenever Breaker would like to make a move, he looks at the current move of BoxMaker in Box, and takes those edges which correspond to the b ordered pairs BoxMaker selected. If he is supposed to select an edge which has already been selected by him, he selects an arbitrary unoccupied edge. Note that the above strategy never calls for Breaker to select an edge which has already been selected by Maker.

It is also obvious, that if BoxMaker wins Box, then Breaker occupied all incident edges of a vertex from C.

In order to apply Corollary 14 it is enough then to show that the size $d(v)$ of each winning set is appropriately bounded from above, i.e., for each

$v \in C$ we have

$$d(v) \leq \frac{K_0}{8} pn \leq \left(\left\lfloor \frac{b}{2} \right\rfloor - 1 \right) \sum_{i=1}^{k-1} \frac{1}{i}$$

a.s.

Indeed, using a Chernoff bound and K_0 large enough, we obtain that for every $v \in C$

$$\Pr\left[d(v) > \frac{K_0}{8} pn \right] \leq e^{-\frac{K_0 pn}{8}} \leq n^{-\frac{K_0}{16}}.$$

Therefore we have

$$\Pr\left[\exists v \in C : \, d(v) > \frac{K_0}{8} pn \right] \leq n \cdot n^{-\frac{K_0}{16}} \to 0,$$

provided K_0 is large enough. Then Corollary 14 guarantees BoxMaker's win, thus Breaker's win a.s., and the proof of Theorem 15 is complete. □

Next we give a winning strategy for Maker in the connectivity game, thus determining the threshold bias $b^p_{\mathcal{T}}$ up to a constant factor.

Obviously, Breaker wins if and only if he claims all the edges of a cut, i.e., all the edges connecting some set of vertices with its complement. In order to win Maker has to claim one edge in each of the cuts. This observation enables us to formulate the connectivity game in a different way, where winning sets are cuts and roles of players are exchanged – Breaker wants to occupy a cut and Maker wants to prevent Breaker from doing so. To avoid confusion we refer to the players of this "cut-game" by CutMaker and CutBreaker.

This new point of view enables us to give Maker a winning strategy using Theorem 12, which is a criterion for CutBreaker's win. Observe, that in this "cut-game" CutBreaker (alias Maker) only cares about occupying the existing edges of a cut, that's why we are going to look at the family $\mathcal{F}_p^{\cap}$ instead of $\mathcal{F}_p$.

Theorem 16 *There exists $k_0 > 0$, so that for $p > \frac{32 \log n}{n}$ and $b \leq k_0 p \frac{n}{\log n}$ Maker wins the random connectivity game $(E(K_n)_p, \mathcal{T}_p, 1, b)$ a.s.*

Proof. For
$$b_0 = \frac{\log 2}{2} \cdot \frac{n}{\log n}$$
we are going to prove that the conditions of Theorem 12 are satisfied if $\mathcal{F}$ is the set of all cuts in a complete graph with n vertices.

On one hand, Beck [2] showed
$$\sum_{k=1}^{n/2} \binom{n}{k} 2^{-\frac{k(n-k)}{b_0}} \to 0,$$
which means that condition (3.1) holds in this setting.

For a cut $A \in \mathcal{F}$ we have $|A| \geq n-1$ which implies condition (3.2). If we set $\delta = 1/2$ we can apply Theorem 12 which gives that
$$(\widehat{E(K_n)}_p, \mathcal{F}_p^{\cap}, \frac{\log 2}{4} p \frac{n}{\log n}, 1)$$
is a CutBreaker's win a.s. The statement of the theorem immediately follows. □

Theorem 15 and Theorem 16 together imply part (i) of both Theorem 10 and 11.

3.3.2 Hamiltonian cycle game

Here we investigate the random version of the $(1\colon b)$ biased game $(E(K_n), \mathcal{H}, 1, b)$ on the complete graph K_n, where $\mathcal{H}$ is the set of all Hamiltonian cycles. Maker's goal is to occupy all edges of a Hamiltonian cycle, while Breaker wants to prevent that. Breaker can obviously win when Maker is not able to claim a connected graph and thus from Theorem 15 we obtain the following corollary.

Corollary 17 *There exists $H_0 > 0$ so that for every $p \in [0,1]$ and $b \geq H_0 p \frac{n}{\log n}$ Breaker wins the random Hamiltonian cycle game $(E(K_n)_p, \mathcal{H}_p, 1, b)$ a.s.*

In the proof of the following theorem, we give a Maker's winning strategy for the Hamiltonian cycle game.

Theorem 18 *There exists $h_0 > 0$, so that for $p > \frac{32\log n}{\sqrt{n}}$ and $b \leq h_0 p \frac{\sqrt{n}}{\log n}$ Maker wins the random Hamiltonian cycle game $(E(K_n)_p, \mathcal{H}_p, 1, b)$ a.s.*

Proof. Maker wins, if at the end of the game the subgraph G_M (containing the edges claimed by Maker) has connectivity $\kappa(G_M)$ greater or equal than independence number $\alpha(G_M)$. Indeed, from the criterion of Chvátal and Erdős for Hamiltonicity [19], we obtain that G_M then contains a Hamiltonian cycle.

We show that Maker, using only his odd moves, can ensure that the connectivity of his graph at the end of the game is greater than $k = \sqrt{n}/2$ and, using his even moves, can make the independence number at the end of the game smaller than $k = \sqrt{n}/2$. In other words we will look at two separate games where in each of them Maker plays one move against Breaker's $2b$ moves. This is a correct strategy, because moves of Maker made in one of these games cannot hurt him in the other.

We first look at the odd Maker's moves. To ensure that $\kappa(G_M) \geq k$, Maker has to claim one edge in every cut of a graph obtained from the initial graph by removing some k vertices. More precisely, we are going to prove the conditions of Theorem 12 for the biased $(b' : 1)$ game, where

$$b' = \frac{\log 2}{2} \cdot \frac{\sqrt{n}}{\log n}$$

and

$$\begin{aligned} \mathcal{F} \ = \ \Big\{ & \{v_1 v_2 : v_1 \in V_1, v_2 \in V_2\} : \\ & V(K_n) = V_0 \dot\cup V_1 \dot\cup V_2,\ |V_0| = k,\ V_1, V_2 \neq \emptyset \Big\}. \end{aligned}$$

That is, Maker plays the role of "CutBreaker" by trying to break all the cuts in $\mathcal{F}$.

Since the size of each of the sets in $\mathcal{F}$ is at least $n - k - 1$ we have

$$\lim_{n\to\infty} \min_{A\in\mathcal{F}} \frac{|A|}{b'} = \lim_{n\to\infty} \frac{2\log n(n - \sqrt{n}/2 - 1)}{\log 2\sqrt{n}} = \infty,$$

and the condition (3.2) holds. Next, we have

$$\begin{aligned}\sum_{A\in\mathcal{F}} 2^{-\frac{|A|}{b'}} &= \sum_{i=1}^{\frac{n-k}{2}} \binom{n}{i}\binom{n-i}{k} 2^{-\frac{i(n-i-k)}{b'}} \\ &< \sum_{i=1}^{k} n^{2k} 2^{-\frac{n-k-1}{b'}} + \sum_{i=k+1}^{\frac{n-k}{2}} 2^{2n-\frac{k(n-2k)}{b'}} \\ &< k\cdot n^{-\sqrt{n}} + n\cdot n^{-n} \to 0,\end{aligned}$$

which gives the condition (3.1). Therefore, CutBreaker (alias Maker) wins the game

$$\left(\widehat{E(K_n)}_p, \mathcal{F}_p^{\cap}, \frac{\log 2}{4} p \frac{\sqrt{n}}{\log n}, 1\right)$$

a.s., provided $p \geq \frac{32\log n}{\sqrt{n}}$.

In the other part of the game using even moves Maker has to ensure that $\alpha(G_M) \leq k = \sqrt{n}/2$. That is going to be true if Maker manages to claim at least one edge in every clique of k elements. To prove that it is possible we again use Theorem 12 for a biased (b': 1) game with the same value of

$$b' = \frac{\log 2}{2} \cdot \frac{\sqrt{n}}{\log n}.$$

But now $\mathcal{F}$ is the family of the edge-sets of all cliques of size k and Maker will play the role of "CliqueBreaker" in this game.

We have

$$\lim_{n\to\infty} \min_{A\in\mathcal{F}} \frac{|A|}{b'} = \lim_{n\to\infty} \frac{2\log n \binom{\frac{\sqrt{n}}{2}}{2}}{\log 2\sqrt{n}} = \infty,$$

and the condition (3.2) is satisfied. It remains to prove that the condition (3.1) holds.

$$\begin{aligned}\sum_{A\in\mathcal{F}} 2^{-\frac{|A|}{b'}} &= \binom{n}{k} 2^{-\frac{\binom{k}{2}}{b'}} \\ &< \left(\frac{ne}{k} 2^{-\frac{k-1}{2b'}}\right)^k \\ &< 2^{-\sqrt{n}} \to 0.\end{aligned}$$

Therefore, CliqueBreaker wins the game

$$\left(\widehat{E(K_n)}_p, \mathcal{F}_p^{\cap}, \frac{\log 2}{4} p \frac{\sqrt{n}}{\log n}, 1\right)$$

a.s., provided $p \geq \frac{32 \log n}{\sqrt{n}}$.

Putting the two parts of the game together we have that Maker wins

$$\left(E(K_n)_p, \mathcal{H}_p, 1, \frac{1}{16} p \frac{\sqrt{n}}{\log n}\right)$$

a.s. □

Combining the statements of Corollary 17 and Theorem 18 we obtain part (*iii*) of both Theorems 10 and 11.

3.3.3 Perfect matching game

The upper and lower bounds obtained in the previous subsection for the threshold bias of the random Hamiltonian cycle game are not tight. We firmly believe that our strategy for Maker in that game is not optimal. The game we consider next is simpler for Maker, and for that we are able to obtain bounds optimal up to a constant factor.

Recall that $\mathcal{M}$ is the set of all perfect matchings on K_n. We will assume that n is even. In the game $(E(K_n), \mathcal{M}, 1, b)$ Maker's goal is to occupy all edges of a perfect matching, while Breaker wants to prevent that.

The following theorem provides the winning strategy in the random perfect matching game for Maker.

Theorem 19 *There exists $m_0 > 0$, so that for $p > 64\frac{\log n}{n}$ and $b \leq m_0 p \frac{n}{\log n}$ Maker wins the random perfect matching game $(E(K_n)_p, \mathcal{M}_p, 1, b)$ a.s.*

Proof. We can show that Maker can win in a slightly harder game. More precisely, if the set of vertices of K_n is partitioned into two sets A and B of equal size before the game starts, we are going to show that Maker can claim a perfect matching with edges going only between A and B.

For disjoint sets $X, Y \subset V(K_n)$, we define $E(X,Y)$ to be the set of edges between X and Y. Let $\mathcal{F}$ be a family of sets of edges,

$$\mathcal{F} = \{E(X,Y) : \emptyset \neq X \subset A, \emptyset \neq Y \subset B, |X| + |Y| = \frac{n}{2} + 1\}.$$

Suppose that at the end of the game Maker has not claimed all edges of any perfect matching between A and B. Hall's necessary and sufficient condition for existence of a perfect matching implies that there exist sets $X_0 \subset A$ and $Y_0 \subset B$ such that $|X_0| > |Y_0|$ and all edges in $E(K_n)_p \cap E(X_0, B \setminus Y_0)$ were claimed by Breaker.

Therefore, in order to win, Maker has to claim at least one edge in each of the sets from $\mathcal{F}$, i.e., the game $(\widehat{E(K_n)}_p, \mathcal{F}_p^{\cap}, b, 1)$, which we call *Hall*, should be a HallBreaker's win.

To prove that HallBreaker wins we are going to use Theorem 12. We set $\delta = 1/2$ and

$$b_0 = \frac{\log 2}{4} \cdot \frac{n}{\log n}.$$

First we show that condition (3.1) holds. We have

$$\begin{aligned}
\sum_{k=1}^{n/2} \binom{n/2}{k} \binom{n/2}{n/2 - k + 1} 2^{-\frac{k(n/2-k+1)}{b_0}} &< 2 \sum_{k=1}^{\lfloor n/4 \rfloor} \binom{n/2}{k}^2 2^{-\frac{k(n/2-k+1)}{b_0}} \\
&< 2 \sum_{k=1}^{\lfloor n/4 \rfloor} \left(e^{2\log(n/2) - 2\log n}\right)^k \\
&= 2 \sum_{k=1}^{\lfloor n/4 \rfloor} \left(\frac{1}{4}\right)^k < 1.
\end{aligned}$$

Since

$$\lim_{n\to\infty} \min_{A \in \mathcal{F}} \frac{|A|}{b_0} > \lim_{n\to\infty} \log n = \infty,$$

the condition (3.2) is also satisfied and we can apply Theorem 12 proving that HallBreaker wins the random game

$$\left(\widehat{E(K_n)}_p, \mathcal{F}_p^{\cap}, \frac{\log 2}{8} p \frac{n}{\log n}, 1\right)$$

a.s., provided $p > 64 \log n/n$.

This immediately implies that Maker wins $(E(K_n)_p, \mathcal{M}_p, 1, b)$ a.s. □

Theorem 15 ensures a win for Breaker in the perfect matching game, if

$$b > \frac{K_0 pn}{\log n}.$$

This, together with the above Theorem 19 proves part (ii) of Theorems 10 and 11.

3.3.4 Clique game

Here we look at the random version of the $(1\colon b)$ biased clique game $(E(K_n), \mathcal{K}_k, 1, b)$ on a complete graph K_n, where $\mathcal{K}_k$ is the set of all cliques of constant size k. Maker's goal is to occupy all edges of a clique of size k while Breaker wants to prevent that.

The deterministic clique game was extensively studied by Bednarska and Łuczak in [13]. They proved a more general result by determining the order of the threshold bias for the whole family of games in which Maker's goal is to claim an arbitrary fixed graph H. In this section, we will largely rely on the constructions and ideas from their paper.

If $\{F_1, \ldots, F_t\}$ is a family of k-cliques having two common vertices, and $e_i \in E(F_i)$, $i = 1, \ldots, t$ are distinct edges, then we call the graph $\cup_{i=1}^t F_i$ a *t-2-cluster* and the graph $\cup_{i=1}^t (F_i - e_i)$ a *t-fan.* If furthermore the k-cliques have three vertices in common, then a t-2-cluster is called a *t-3-cluster* and a t-fan is called a *t-flower*. A t-fan or a t-2-cluster is said to be *simple*, if the pairwise intersections (of any two k-cliques) have size exactly 2.

In order to prevent Maker to occupy a clique K_k, Breaker will play two auxiliary games. In the first one he prevents Maker from occupying a 3-cluster of constant size.

Lemma 20 *There exists $t = t(k)$, so that for $\varepsilon = \frac{1}{2(k+2)}$, $p = \omega(n^{-\frac{2}{k+1}})$ and $b > pn^{\frac{2(1-\varepsilon)}{k+1}}$ Breaker wins the game $(E(K_n)_p, t\text{-}3\text{-}clusters, 1, b)$ a.s.*

Proof. To apply Theorem 3, it is enough to check that there exists t

such that for the random variable

$$Y := \sum_{t\text{-3-cluster } C \text{ in } G(n,p)} (1+b)^{-e(C)},$$

$Y < \frac{1}{b+1}$ holds a.s.

We have

$$\mathbf{E}[Y] = \sum_{t\text{-3-cluster } C \text{ in } K_n} \left(\frac{p}{1+b}\right)^{e(C)}.$$

Let $b_1 = \frac{b+1}{p} - 1$. In [13], it is shown that there exists t for which

$$\sum_{t\text{-3-cluster } C \text{ in } K_n} \left(\frac{p}{1+b}\right)^{e(C)} \leq K_0 \frac{1}{b_1^{1+k_0}},$$

where $k_0, K_0 > 0$ are constants depending on k. This implies $\mathbf{E}[Y] = o\left(\frac{1}{b+1}\right)$, and by Markov inequality we get that $Y < \frac{1}{b+1}$ a.s. □

During a game, a t-fan (or t-flower) is said to be *dangerous* if all the t edges missing from the cliques that make up the t-fan are present in the graph on which the game is played, but not yet claimed by any of the players. Note that if at any moment of the game $(E(K_n)_p, t\text{-3-clusters}, 1, b)$ Maker claimed a dangerous $(b+1)t$-flower, then he could win since he could claim a t-3-cluster in his next t moves by simply claiming missing edges, one by one. Hence, Lemma 20 implies the following.

Corollary 21 *There exists $t = t(k)$ so that for $\varepsilon = \frac{1}{2(k+2)}$ and $p = \omega(n^{-\frac{2}{k+1}})$, Breaker playing a $(1\colon pn^{\frac{2(1-\varepsilon)}{k+1}})$ game on edges of the random graph $E(K_n)_p$ can make sure that Maker does not claim a dangerous $\left(pn^{\frac{2(1-\varepsilon)}{k+1}} t\right)$-flower at any moment of the game.*

Next we deal with the second auxiliary game of Breaker; in this game he prevents the appearance of too many simple b^{ε}-fans.

Lemma 22 *There exists $C_0 > 0$, such that for $\varepsilon_1 = \frac{1}{6(k+2)}$,*

$$p \geq \frac{\log^{1/\varepsilon_1} n}{n^{\frac{2}{k+1}}},$$

$b > C_0 p n^{\frac{2}{k+1}}$ and $s = b^{\varepsilon_1}$ Breaker wins the game

$$\left(E(K_n)_p, \textit{unions of } \frac{1}{2}\binom{b}{s} \textit{ simple } s\textit{-fans}, 1, b/2 \right)$$

a.s.

Proof. Let $c_s(n)$ be the number of simple s-2-clusters contained in K_n, and let X_s be the random variable counting the number of simple s-2-clusters contained in $G(n,p)$. Using the first moment method we get

$$\Pr[X_s \geq \mathbf{E}[X_s] \log n] \leq \frac{1}{\log n} \longrightarrow 0,$$

and using this, a.s. we have that

$$\begin{aligned}
& \sum_{\substack{\text{dangerous simple} \\ s\text{-fan } C \text{ in } G(n,p)}} (1+b/2)^{-e(C)} \\
\leq & \sum_{\substack{\text{simple } s\text{-2-cluster } K \\ \text{in } G(n,p)}} \binom{k}{2}^s (1+b/2)^{-s(\binom{k}{2}-2)-1} \\
\leq & \binom{k}{2}^s \log n \cdot c_s(n) p^{s(\binom{k}{2}-1)+1} 2^{sk^2} b^{-s(\binom{k}{2}-2)-1} \\
\leq & \log n \cdot C_1^s \binom{n}{2} \frac{\binom{n}{k-2}^s}{s!} \left(\frac{p}{b}\right)^{s(\binom{k}{2}-1)+1} b^s \\
\leq & n^3 \cdot C_1^s n^{(k-2)s} \left(\frac{1}{C_0 n^{\frac{2}{k+1}}}\right)^{s(k+1)(k-2)/2+1} \frac{b^s}{s!} \\
\leq & n^3 \cdot \left(\frac{C_1}{C_0^{\binom{k}{2}-1}}\right)^s \left(\frac{1}{C_0 n^{\frac{2}{k+1}}}\right) \frac{b^s}{s!} \\
< & \frac{1}{2}\binom{b}{s} \frac{1}{b+1},
\end{aligned}$$

where $C_1 = C_1(k)$ is a constant. The last inequality is valid since $p \geq n^{-\frac{2}{k+1}} \log^{1/\varepsilon_1} n$, and for C_0 large enough

$$\left(\frac{C_1}{C_0^{\binom{k}{2}-1}} \right)^s \leq n^{-5}.$$

This enables us to apply Theorem 5, and the statement of the lemma is proved. □

Now we are ready to state and prove the theorem ensuring Breaker's win in the clique game on the random graph. In the proof, we are going to use this result of Bednarska and Łuczak.

Lemma 23 *[13] For every $0 < \varepsilon < 1$ there exists b_0 so that every graph with $b > b_0$ vertices and at most $b^{2-\varepsilon}$ edges has at least $\frac{1}{2}\binom{b}{b^{\varepsilon/3}}$ independent sets of size $b^{\varepsilon/3}$.*

Theorem 24 *There exists $C_0 > 0$ so that for $p \geq n^{-\frac{2}{k+1}} \log^{6k+12} n$ and $b \geq C_0 p n^{\frac{2}{k+1}}$ Breaker wins the random clique game $(E(K_n)_p, (\mathcal{K}_k)_p, 1, b)$ a.s.*

Proof. Breaker will use $b/2$ of his moves to defend "immediate threats", i.e., to claim the remaining edge in all k-cliques in which Maker occupied all but one edge. In order to be able to do this Breaker must ensure that he never has to block more than $b/2$ immediate threats, that is, there is no dangerous $b/2$-fan.

He will use his other $b/2$ moves to prevent Maker from creating a dangerous $(b/2)$-fan.

From Corollary 21 we get that Breaker can prevent Maker from claiming a dangerous f-flower (where $f = tpn^{\frac{2(1-\varepsilon)}{k+1}}$, $\varepsilon = \frac{1}{2(k+2)}$ and t is a positive constant) using less than $b/4$ edges per move. On the other hand, from Lemma 22 we have that if C_0 is large enough Breaker can prevent Maker from claiming $\frac{1}{2}\binom{b/2}{s}$ simple s-fans using $b/4$ edges per move, where $s = (b/2)^{\varepsilon/3}$.

Suppose that Maker managed to claim a dangerous $(b/2)$-fan. We define an auxiliary graph G' with the vertex set being the set of all $b/2$ k-cliques of this dangerous fan, and two k-cliques being connected with an edge if they have at least 3 vertices in common. Since there is no dangerous f-flower in Maker's graph, the degree of each of the vertices of the graph G' is at most fk and therefore

$$e(G') < \frac{bfk}{2} \leq \left(\frac{b}{2}\right)^{2-\varepsilon}.$$

On the other hand, the number of independent sets in G' of size s cannot be more than $\frac{1}{2}\binom{b/2}{s}$, since each of the independent sets in G' corresponds to a simple s-fan in Maker's graph.

Since the last two facts are obviously in contradiction with Lemma 23, Maker cannot claim a dangerous $b/2$-fan and the statement of the theorem is proved. □

To prove the theorem for Maker's win, we need the following lemma which is a slight modification of a result from [13, Lemma 4]. It is stated in more general form—for each graph H containing a cycle, not only for cliques.

Lemma 25 *Let H be a graph containing a cycle. There exists $0 < \delta_k < 1$, such that a.s. for $M = 2\lfloor n^{2-1/m_2(H)} \rfloor$ each subgraph of the random graph $G(n, M)$ with $\lfloor (1-\delta_k)M \rfloor$ edges contains a copy of H.*

Proof. For $0 < \delta_k < 1$, we call a subgraph F of K_n bad, if F has M edges and it contains a subgraph F' with $\lfloor (1-\delta_k)M \rfloor$ edges that does not contain a copy of H. In [13], it is proved that there exist constants $0 < \delta_k < 1$ and $c_1' > 0$ such that the number of bad subgraphs of K_n is bounded from above by

$$e^{-c_1' M/6} \binom{\binom{n}{2}}{M} = o(1) \binom{\binom{n}{2}}{M}.$$

Hence, the probability that $G(n, M)$ is bad tends to 0. □

Using the last lemma we can prove a theorem for Maker's win in the H-game, where H is arbitrary fixed graph containing a cycle.

Theorem 26 *Let H be a graph containing a cycle, and let $\mathcal{F}_H$ be the set of all copies of H in K_n. There exists $c_0 > 0$ so that for $p > \frac{1}{c_0} n^{-\frac{1}{m_2(H)}}$*

and $b \leq c_0 p n^{\frac{1}{m_2(H)}}$ *Maker wins the random* H*-game* $(E(K_n)_p, (\mathcal{F}_H)_p, 1, b)$ *a.s.*

Proof. We will follow the analysis of the random Maker's strategy proposed in [13], looking at $G(n, M')$, where $M' = p\binom{n}{2}$. We will prove that H-game on $G(n, M')$ is a Maker's win a.s., which implies that the same is true on $G(n,p)$, as being a Maker's win is a monotone property [17, Chapter 2].

In each of his moves Maker chooses one of the edges of $G(n, M')$ that was not previously claimed by him, uniformly at random. If the edge is free he claims it and we call that a successful Maker's move. If the edge was already claimed by Breaker, then Maker skips his move (e.g., claims an arbitrary free edge, and that edge we will not encounter for the future analysis).

Let $0 < \delta_k < 1$ be chosen so that the conditions of Lemma 25 are satisfied. We look at the course of game after $M = 2\lfloor n^{2-1/m_2(H)} \rfloor$ moves.

By choosing $c_0 \leq \delta_k/12$, we have

$$\begin{aligned} M &\leq \frac{\delta_k}{6c_0} \lfloor n^{2-1/m_2(H)} \rfloor \\ &\leq \frac{\delta_k}{2} \frac{1}{b+1} p \binom{n}{2}. \end{aligned}$$

That means that only at most $\delta_k/2$ fraction of the total number of elements of the board $E(G(n, M'))$ is claimed (by both players) after move M. Therefore, the probability that the edge randomly chosen in Maker's mth move, $m \leq M$, is already claimed by Breaker is bounded from above by $\delta_k/2$. That means that Maker has at least $(1-\delta_k)M$ successful moves a.s.

Since in each of his moves Maker has chosen edges uniformly at random (without repetition) from $E(G(n, M'))$, the graph containing edges chosen by Maker in his first M moves (both successful and unsuccessful) actually is the random graph $G(n, M)$. Applying Lemma 25, we get that the graph containing edges claimed by Maker in his successful moves contains graph H a.s., which means that there exists a non-randomized winning strategy for Maker a.s. □

Corollary 27 *There exists* $c_0 > 0$ *so that for* $p > \frac{1}{c_0} n^{-\frac{2}{k+1}}$ *and* $b \leq c_0 p n^{\frac{2}{k+1}}$ *Maker wins the random clique game* $(E(K_n)_p, (\mathcal{K}_k)_p, 1, b)$ *a.s.*

Combining the statements of Theorem 24 and Corollary 27 we obtain part (*iv*) of Theorem 11.

3.4 Unbiased games

3.4.1 Connectivity one-on-one

A theorem of Lehman enables us to determine the threshold probability $p_{\mathcal{T}}$ with extraordinary precision. Namely, Lehman [31] proved that the unbiased connectivity game is won by Maker (now as the second player!) if and only if the underlying graph contains two edge-disjoint spanning trees. The threshold for the appearance of two edge-disjoint spanning trees was determined exactly by Palmer and Spencer [36].

The consequence of the theorems of Lehman, and Palmer and Spencer is that the very moment the last vertex receives its second adjacent edge, the unbiased connectivity game is won by Maker a.s. More precisely, the following is true.

Corollary 28 *For the unbiased connectivity game we have that a.s.*

$$\tau(\textit{Maker wins } \mathcal{T}) = \tau(\exists \textit{ two edge-disjoint spanning trees}) = \tau(\delta(G) \geq 2).$$

In particular, for edge-probability

$$p = \frac{\log n + \log\log n + g(n)}{n},$$

where $g(n)$ tends to infinity arbitrarily slowly, Maker wins the unbiased connectivity game a.s., while if $g(n) \to -\infty$, then Breaker wins a.s.

Remark. The assumption that Maker is the second player is just technical, for the sake of smooth applicability of Lehman's Theorem. If Maker is the first player, then from the proof of Lehman's Theorem one can infer that Maker wins if and only if the base graph contains a spanning tree and a spanning forest of two components, which are edge-disjoint. This property has the same sharp threshold as the presence of two edge-disjoint spanning trees, and the hitting time should be the same when the next to last vertex receives its second incident edge.

3.4.2 Hamiltonian cycles one-on-one

Let $H = ([n], E)$ be a graph, and let

$$\text{LARGE} = \text{LARGE}(H) = \{v \in [n] \colon d(v) \geq \frac{\log n}{10}\},$$

and

$$\text{SMALL} = \text{SMALL}(H) = [n] \setminus \text{LARGE}.$$

Frieze and Krivelevich proved in [23] that if a graph H satisfies the following seven properties, than it is Hamiltonian.

P1 $\delta(H) \geq 2$,

P2 SMALL contains no edges,

P3 No $v \in [n]$ is within distance 2 of more than one vertex from SMALL,

P4 $S \subseteq \text{LARGE}$ and

$$|S| \leq n \frac{\log \log n}{\log n}$$

implies that

$$|N(S)| \geq |S| \frac{\log n}{10 \log \log n},$$

P5 $A, B \subseteq [n]$, $A \cap B = \emptyset$,

$$|A|, |B| \geq 20n \frac{\log \log n}{\log n}$$

implies that there are at least

$$|A||B| \frac{\log n}{2n}$$

edges joining A and B,

P6 $A, B \subseteq [n]$, $A \cap B = \emptyset$, $|A| \leq |B| \leq 4|A|$ and

$$|B| \leq 200n\frac{\log\log n}{\log n}$$

implies that there are at most $2400|A|\log\log n$ edges joining A and B,

P7 If

$$|A| \leq 30n\frac{\log\log n}{\log n},$$

then A contains at most $100|A|\log\log n$ edges.

Lemma 29 *[23] If $G = ([n], E)$ satisfies P1–P7 above, then G is Hamiltonian.*

We will use this result to give a strategy for Maker in the Hamiltonian cycle game.

Theorem 30 *There exists a constant C, such that for $p \geq C\frac{\log n}{n}$ Maker can win the Hamiltonian cycle game $(E(K_n)_p, \mathcal{H}_p)$ a.s.*

Proof. We are going to show that there exists a Maker's wining strategy a.s., for $p = 5.4\frac{\log n}{n}$. Since "being a Maker's win" is an increasing graph property [16], Maker can win a.s. for any $p \geq 5.4\frac{\log n}{n}$.

From now on, we set $p = 5.4\frac{\log n}{n}$. We will prove that Maker can a.s. play the game in a way that in the end of the game the graph containing edges claimed by Maker satisfies properties P1-P7, and thus is Hamiltonian.

Before we discuss the strategy, we need to verify several properties of $G = G(n, p)$. First, we show that

$$\delta(G) > 2.2\log n \tag{3.4}$$

holds a.s. Indeed, using Chernoff bounds we get

$$\begin{aligned}\Pr[\exists v \in [n] \colon d(v) \leq 2.2\log n] &\leq n\left(\frac{5.4 \cdot e}{2.2}\right)^{2.2\log n} e^{-5.4\log n}\\ &\leq n^{-4.4+2.2\cdot(1-\log 0.41)} = o(1).\end{aligned}$$

It follows that a.s. for every $v \in [n]$ we can select a set $S(v)$ of exactly $2.2 \log n$ edges incident to v.

Next, we have that a.s. for all $A \subseteq [n]$,

$$\text{if } a = |A| < \frac{n}{\log^2 n}, \text{ then } e_G(A) \leq 3a, \tag{3.5}$$

since

$$\begin{aligned} & \Pr\left[\exists A \subseteq [n] \colon a = |A| < \frac{n}{\log^2 n}, e_G(A) > 3a\right] \\ \leq & \sum_{a=1}^{n/\log^2 n} \binom{n}{a}\binom{\binom{a}{2}}{3a} p^{3a} \\ \leq & \sum_{a=1}^{n/\log^2 n} \left(\frac{ne}{a}\left(\frac{5.4 \cdot ae \log n}{6n}\right)^3\right)^a \\ \leq & \sum_{a=1}^{n/\log^2 n} \left(\frac{5.4^3 \cdot e^4}{6^3} \log^{-1} n\right)^a = o(1). \end{aligned}$$

We also get that a.s. for every two sets $A, B \subseteq [n]$ with $A \cap B = \emptyset$,

$$\frac{n}{\log^3 n} \leq a = |A| \leq n \frac{\log \log n}{\log n},$$

and

$$b = |B| = n - a\left(1 + \frac{\log n}{10 \log \log n}\right),$$

we have

$$e_G(A : B) \geq 0.7 \cdot abp, \tag{3.6}$$

following from

$$
\begin{aligned}
& \Pr[\exists A, B \subseteq [n] \colon e_G(A:B) < 0.7 \cdot abp] \\
\leq & \sum_{a=\frac{n}{\log^3 n}}^{n\frac{\log\log n}{\log n}} \binom{n}{a} \binom{n}{a\frac{\log n}{10\log\log n}} e^{-\frac{0.3^2 \cdot abp}{2}} \\
\leq & \sum_{a=\frac{n}{\log^3 n}}^{n\frac{\log\log n}{\log n}} \left(\frac{ne}{a} \cdot \left(\frac{10en\log\log n}{a\log n} \right)^{\frac{\log n}{10\log\log n}} \cdot e^{-\frac{9\cdot 0.3^2 \cdot 5.4(1-o(1))}{20}\log n} \right)^a \\
\leq & \sum_{a=\frac{n}{\log^3 n}}^{n\frac{\log\log n}{\log n}} \left(\log^3 n \cdot \left(10e\log^2 n\log\log n\right)^{\frac{\log n}{10\log\log n}} \cdot e^{1-0.218\cdot\log n} \right)^a \\
\leq & \sum_{a=\frac{n}{\log^3 n}}^{n\frac{\log\log n}{\log n}} e^{\frac{a\log n}{10}(-0.18+o(1))} \\
= & \ o(1).
\end{aligned}
$$

In the last derivation we used that $b > n\left(\frac{9}{10} - o(1)\right)$.

Finally, we have that a.s. for every two sets $A, B \subseteq [n]$ with $A \cap B = \emptyset$ holds that

$$
\text{if } a = |A| = |B| = 20n\frac{\log\log n}{\log n}, \text{ then } e(A:B) > 0.8a^2 p, \tag{3.7}
$$

since

$$
\begin{aligned}
\Pr\left[\exists A, B \subseteq [n] \colon e(A:B) \leq 0.8a^2 p\right] &\leq \binom{n}{a}^2 e^{-0.02\cdot a^2 p} \\
&\leq \left(\frac{ne}{a} \cdot e^{-0.01\cdot ap}\right)^{2a} \\
&\leq e^{2a(-0.08\cdot\log\log n + o(\log\log n))} \\
&= o(1).
\end{aligned}
$$

This implies that a.s. for every two sets A and B satisfying the conditions above there exists a set $T(A,B) \subseteq E(A:B)$ of cardinality $8.6a^2\frac{\log n}{2n}$.

Now we can define the set of winning sets for an auxiliary game AUX. We will show that on a graph satisfying properties (3.4), (3.5), (3.6) and (3.7), AUXBreaker can win the game. In particular, since the graph $G(n,p)$ satisfies these properties a.s., this will mean that AUXBreaker can win on $G(n,p)$ a.s. Knowing this, we will show that Maker can claim a graph that satisfies properties **P1**-**P5** by taking the role of AUXBreaker.

We define

$$\begin{aligned}
\mathcal{F}_1 &:= \{K \colon \exists v \in [n], K \subseteq S(v), |K| = 2.1 \log n\},\\
\mathcal{F}_2 &:= \{E(A:B) \colon A, B \subseteq [n], A \cap B = \emptyset, \frac{n}{\log^3 n} \leq |A| \leq n \frac{\log\log n}{\log n},\\
&\qquad |B| = n - a\left(1 + \frac{\log n}{10 \log\log n}\right)\},\\
\mathcal{F}_3 &:= \{C \colon \exists A, B \subseteq [n], A \cap B = \emptyset, a = |A| = |B| = 20n\frac{\log\log n}{\log n},\\
&\qquad C \subseteq T(A,B), |C| = 7.6a^2 \frac{\log n}{2n}\},\\
\mathcal{F} &= \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3.
\end{aligned}$$

Family $\mathcal{F}$ is the set of winning sets for the game AUX. To prove that AUXBreaker has a winning strategy, it remains to verify the conditions of Theorem 3 (Erdős-Selfridge Theorem). We do the calculation separately for $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$.

First, we have

$$\begin{aligned}
\sum_{A \in \mathcal{F}_1} 2^{-|A|} &\leq n \binom{2.2 \log n}{0.1 \log n} 2^{-2.1 \cdot \log n}\\
&\leq n (22e)^{0.1 \cdot \log n} n^{-2.1 \cdot \log 2}\\
&\leq n^{\frac{11 + \log 22}{10} - 2.1 \cdot \log 2}\\
&= o(1).
\end{aligned}$$

Since $0.7 \cdot \log 2 > \frac{0.3^2}{2}$, from the calculation used to prove (3.6) we get

$$\begin{aligned}\sum_{A\in\mathcal{F}_2} 2^{-|A|} &\leq \sum_{a=\frac{n}{\log^3 n}}^{n\frac{\log\log n}{\log n}} \binom{n}{a}\binom{n}{a\frac{\log n}{10\log\log n}} e^{-0.7\cdot\log 2\cdot abp} \\ &= o(1).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\sum_{A\in\mathcal{F}_3} 2^{-|A|} &\leq \binom{n}{a}^2 \binom{8.6a^2\frac{\log n}{2n}}{a^2\frac{\log n}{2n}} 2^{-7.6a^2\frac{\log n}{2n}} \\ &\leq \left(\frac{ne}{a}\right)^{2a} (8.6\cdot e)^{a^2\frac{\log n}{2n}} e^{-2a(1.9\cdot\log 2\cdot a\frac{\log n}{n})} \\ &\leq e^{2a(\log\log n(6+5\log 8.6-38\log 2)+o(\log\log n))} \\ &= o(1).\end{aligned}$$

and the condition of the Erdős-Selfridge Theorem is satisfied a.s., since we have

$$\begin{aligned}\sum_{A\in\mathcal{F}} 2^{-|A|} &= \sum_{A\in\mathcal{F}_1} 2^{-|A|} + \sum_{A\in\mathcal{F}_2} 2^{-|A|} + \sum_{A\in\mathcal{F}_3} 2^{-|A|} \\ &= o(1).\end{aligned}$$

Therefore, Maker can a.s. win the auxiliary game as AUXBreaker, i.e., he can claim an edge in each of the sets from $\mathcal{F}$. Let E_M be the set of all edges of $E(K_n)_p$ claimed by Maker during the game, and let $M = ([n], E_M)$. Assuming that AUXBreaker wins the game, we will prove that M satisfies properties **P1**-**P5** a.s.

Knowing that each set in $\mathcal{F}_1$ has at least one edge from E_M, the minimal degree of M must be at least $\frac{\log n}{10}$. That means that SMALL$(M) = \emptyset$, satisfying properties **P1**, **P2** and **P3**.

We verify that M satisfies **P4**. If $s = |S| \leq \frac{n}{\log^3 n}$ and

$$|N_M(S)| < s\frac{\log n}{10\log\log n},$$

then
$$|S \cup N_M(S)| \le s\left(1 + \frac{\log n}{10 \log\log n}\right) < s\frac{\log n}{60} \le \frac{n}{\log^2 n}.$$
Applying (3.5) on $S \cup N_M(S)$ we have
$$e_M(S \cup N_M(S)) \le 3|S \cup N_M(S)| < \frac{s \log n}{20}.$$
But we know that this cannot hold, since from (3.4) we get
$$e_M(S \cup N_M(S)) \ge \frac{1}{2}|S|\frac{\log n}{10} = \frac{s \log n}{20},$$
implying that this will not happen a.s.

Suppose that there is a set $S \subseteq [n]$ satisfying
$$n\frac{\log\log n}{\log n} > s = |S| > \frac{n}{\log^3 n}$$
and
$$|N_M(S)| \le s\frac{\log n}{10 \log\log n}.$$
Then, none of the edges of $E(S : [n] \setminus (S \cup N_M(S)))$ are in M. That is a contradiction, since $E(S : [n] \setminus (S \cup N_M(S)))$ contains a set from $\mathcal{F}_2$.

Now we prove that M satisfies **P5**. If
$$a = |A| = |B| = 20n\frac{\log\log n}{\log n}$$
and
$$e_M(A : B) < a^2\frac{\log n}{2n},$$
(3.7) implies that a.s. there is a set of at least $7.6a^2\frac{\log n}{2n}$ edges in $T(A, B)$ that are not claimed by Maker. This set contains a set from $\mathcal{F}_3$, which is in contradiction with the Maker's win in the game AUX on $\mathcal{F}_3$ (playing as AUXBreaker).

Let both A and B have size at least $a = 20n\frac{\log\log n}{\log n}$. The expectation of $\frac{e_M(A':B')}{a^2}$, where A' and B' are random a-subsets of A and B, respectively, is $\frac{e_M(A:B)}{|A||B|}$. We obtained in the previous case that $\frac{e_M(A':B')}{a^2}$ is larger than $\frac{\log n}{2n}$ for all A', B' a.s., and therefore $e_M(A:B) \geq |A||B|\frac{\log n}{2n}$.

It remains to check that the Maker's graph satisfies properties **P6** and **P7**. To do that, we will prove that $G(n,p)$ satisfies the properties. Since both properties are decreasing, this implies that the Maker's graph satisfies them as well.

First, we look at property **P6**. If $|A| \leq \frac{n}{\log^3 n}$, then $|A \cup B| \leq \frac{n}{\log^2 n}$. From (3.5) we have

$$e(A:B) \leq e(A \cup B) \leq 3|A \cup B| \leq 15|A| \leq 2400|A| \log\log n.$$

Next, we look at the case $|A| > \frac{n}{\log^3 n}$. We define

$$K = \left\{ \frac{a}{b} : a, b \in \mathbb{N},\ a, b \leq n,\ 1 \leq \frac{b}{a} \leq 4 \right\}.$$

Obviously, $|K|$ is bounded from above by n^2. Using Chernoff bounds, we get

$$\begin{aligned}
&\Pr[M \text{ does not satisfy } \mathbf{P6}] \\
\leq\ & n^2 \max_{k \in K} \sum_{a=\frac{n}{\log^3 n}}^{\frac{200n \log\log n}{k \log n}} \binom{n}{a}\binom{n}{ka} \cdot \\
&\cdot \left(\frac{5.4eka\log n}{2400n\log\log n}\right)^{2400a\log\log n} e^{-5.4\cdot ka^2\frac{\log n}{n}} \\
\leq\ & n^2 \max_{k \in K} \sum_{a=\frac{n}{\log^3 n}}^{\frac{200n \log\log n}{k \log n}} \left(e\log^3 n \cdot e^4 \log^{12} n\right)^a \cdot \\
&\cdot \left(\frac{5.4eka\log n}{2400n\log\log n}\right)^{2400a\log\log n} e^{-5.4\cdot ka^2\frac{\log n}{n}} \\
\leq\ & n^2 \max_{k \in K} \sum_{a=\frac{n}{\log^3 n}}^{\frac{200n \log\log n}{k \log n}} \exp\left\{ a\log\log n \left(15 + 2400\log\frac{5.4 \cdot ek}{2400} + \right.\right. \\
&\left.\left. + 2400\log a - 2400\log\frac{n\log\log n}{\log n} - 5.4\cdot ka\frac{\log n}{n\log\log n} + o(1)\right)\right\}.
\end{aligned}$$

Note that we can apply Chernoff bounds here, as

$$\rho = \frac{2400n \log\log n}{5.4ka \log n} > \frac{2400}{5.4 \cdot 200} > 1.$$

Since

$$2400 \log a - 5.4ka \frac{\log n}{n \log\log n}$$

as function of a over the interval of the summation is a growing function, we can substitute

$$a = \frac{200n \log\log n}{k \log n}$$

to get an upper bound, and thus

$$\begin{aligned}
& \Pr[M \text{ does not satisfy } \mathbf{P6}] \\
\leq \quad & n^2 \max_{k \in K} \sum_{a=\frac{n}{\log^3 n}}^{\frac{200n \log\log n}{k \log n}} \exp\Bigg\{ a \log\log n \cdot \\
& \cdot \left(15 + 2400 \log \frac{5.4 \cdot e}{12} - 1080 + o(1) \right) \Bigg\} \\
\leq \quad & n^2 \sum_{a=\frac{n}{\log^3 n}}^{\frac{200n \log\log n}{\log n}} \exp\{-a \log\log n\} = o(1).
\end{aligned}$$

It remains to check that **P7** holds for $G(n,p)$, and thus also for M. If $|A| < \frac{n}{\log^2 n}$ then from (3.5) we get that a.s. $e(A) \leq 3|A| \leq 100|A| \log\log n$.

For

$$\frac{n}{\log^2 n} \leq a = |A| < \frac{30n \log\log n}{\log n},$$

using Chernoff bounds we have

$$\begin{aligned}
&\Pr[M \text{ does not satisfy } \mathbf{P7}]\\
\leq &\sum_{a=\frac{n}{\log^2 n}}^{\frac{30n\log\log n}{\log n}} \binom{n}{a}\left(\frac{5.4ea\log n}{200n\log\log n}\right)^{100a\log\log n} e^{-\frac{5.4\cdot a^2\log n}{n}\binom{a}{2}}\\
\leq &\sum_{a=\frac{n}{\log^2 n}}^{\frac{30n\log\log n}{\log n}} \left(e\log^2 n\right)^a\left(\frac{5.4ea\log n}{200n\log\log n}\right)^{100a\log\log n} e^{-\frac{5.4\cdot a^2\log n}{n}\binom{a}{2}}\\
\leq &\sum_{a=\frac{n}{\log^2 n}}^{\frac{30n\log\log n}{\log n}} \exp\Bigg\{a\log\log n\Bigg(2+100\log a+100\log\frac{5.4\cdot e}{200}\\
&-100\log\frac{n\log\log n}{\log n}-\frac{2.7\cdot a\log n}{n\log\log n}+o(1)\Bigg)\Bigg\}.
\end{aligned}$$

Note that we can apply Chernoff bounds here, as

$$\rho=\frac{200n\log\log n}{5.4a\log n}\geq\frac{200}{5.4\cdot 30}>1.$$

Since

$$100\log a-\frac{2.7\cdot a\log n}{n\log\log n}$$

as function of a over the interval of the summation is a growing function, we can substitute

$$a=\frac{30n\log\log n}{\log n}$$

to get an upper bound, and thus

$$\begin{aligned}
&\Pr[M \text{ does not satisfy } \mathbf{P7}]\\
\leq &\sum_{a=\frac{n}{\log^2 n}}^{\frac{30n\log\log n}{\log n}} \exp\left\{a\log\log n\left(2+100\log\frac{3\cdot 5.4\cdot e}{20}-15\cdot 5.4+o(1)\right)\right\}\\
\leq &\sum_{a=\frac{n}{\log^2 n}}^{\frac{30n\log\log n}{\log n}} \exp\{-a\log\log n\}=o(1).
\end{aligned}$$

Therefore, M satisfies all properties **P1**-**P7** a.s., and from Lemma 29 it follows that M is Hamiltonian a.s. □

3.4.3 k-cliques one-on-one

Let us fix k and let $(F_1, \ldots, F_s)$ be a sequence of k-cliques. Then $F = \cup_{i=1}^{s} F_i$ is called an *s-bunch* if $V(F_i) \setminus (\cup_{j=1}^{i-1} V(F_j)) \neq \emptyset$ and $|V(F_i) \cap (\cup_{j<i} V(F_j))| \geq 2$, for each $i = 2, \ldots, s$. Recall that an s-bunch in which the pairwise intersection of any two cliques is the same two vertices, was called a *simple s-2-cluster*. Let us denote the simple s-2-cluster by C_s.

Recall that the *density* of G is defined as $d(G) = \frac{e(G)}{v(G)}$, and the *maximum density* of G is defined as $m(G) = \max_{H \subseteq G} d(H)$. A graph G with $m(G) = d(G)$ is called *balanced*. The maximum density of a graph G determines the threshold probability for the appearance of G in the random graph. More precisely, (i) if $p = o(n^{-1/m(G)})$, then $G(n,p)$ does not contain G a.s., and (ii) if $p = \omega(n^{-1/m(G)})$, then $G(n,p)$ does contain G a.s.

We need two properties of simple s-2-clusters and s-bunches.

Lemma 31 *For every positive integer s, C_s is balanced and has maximum density*

$$m(C_s) = d(C_s) = \frac{k+1}{2} - \frac{k}{sk - 2s + 2}.$$

Proof. It is easy to check that $v(C_s) = s(k-2)+2$, $e(C_s) = s\binom{k}{2} - s + 1$, and thus

$$d(C_s) = \frac{e(C_s)}{v(C_s)} = \frac{k+1}{2} - \frac{k}{sk - 2s + 2}.$$

Let T be a subgraph of C_s. We want to prove $d(T) \leq d(C_s)$. Since C_s is the union of k-cliques, $C_s = \cup_{i=1}^{s} F_i$, if we set $E_i = F_i \cap T$ we have that $T = \cup_{i=1}^{s} E_i$, and we can assume that each E_i is a clique of order $k_i \leq k$.

We can also assume that the two vertices in $\cap_{i=1}^{s} V(F_i)$ are in T, since otherwise their inclusion would increase the density. This implies $k_i \geq 2$ for $i = 1, \ldots, s$.

Let us relabel the cliques in such a way that $E_i \neq F_i$ if and only if $i = 1, \ldots, s_1$. Then

$$\frac{e(C_s)}{v(C_s)} \geq \frac{e(T)}{v(T)} = \frac{e(C_s) - \sum_{i=1}^{s_1}\left(\binom{k}{2} - \binom{k_i}{2}\right)}{v(C_s) - \sum_{i=1}^{s_1}(k - k_i)},$$

since

$$\frac{e(C_s)}{v(C_s)} < \frac{k+1}{2} \leq \frac{\sum_{i=1}^{s_1}(k-k_i)\frac{k+k_i-1}{2}}{\sum_{i=1}^{s_1}(k-k_i)}.$$

The last inequality is true since the last fraction is the weighted average of the numbers $(k + k_i - 1)/2$, each of them being at least $(k + 1)/2$. □

Lemma 32 *Let $s \geq 3$ be a positive integer. No s-bunch has smaller maximum density than the simple s-2-cluster.*

Proof. When $k = 3$, the s bunch is a union of triangles. Then any s-bunch has the same number of vertices as the simple s-2-cluster, while the number of edges, and thus the density is minimized for the simple s-2-cluster.

From now on let us assume that $k \geq 4$. Let $s \geq 3$, and let $(F_1, F_2, \ldots, F_s)$ be the sequence of k-cliques of an arbitrary s-bunch $B_s = \cup_{i=1}^{s} F_i$. For every $i \in \{2, 3, \ldots, s\}$, let $F_i' = \left(\cup_{j=1}^{i-1} F_j\right) \cap F_i$. Then, we have

$$\begin{aligned}
d(B_s) &= \frac{s\binom{k}{2} - \sum_{i=2}^{s} e(F_i')}{sk - \sum_{i=2}^{s} v(F_i')} \\
&= \frac{e(C_s) - \sum_{i=2}^{s}(e(F_i') - 1)}{v(C_s) - \sum_{i=2}^{s}(v(F_i') - 2)} \\
&\geq \frac{e(C_s) - \sum_{i=2}^{s}(\binom{v(F_i')}{2} - 1)}{v(C_s) - \sum_{i=2}^{s}(v(F_i') - 2)} \\
&\geq \frac{e(C_s)}{v(C_s)}.
\end{aligned}$$

In the last inequality the terms with $v(F_i') = 2$ disappear, and otherwise we use that $v(F_i') \leq k - 1$ for every i, so

$$\frac{\binom{v(F_i')}{2} - 1}{v(F_i') - 2} \leq \frac{k}{2} \leq \frac{e(C_s)}{v(C_s)}.$$

Hence, simple s-2-clusters have the smallest density among all s-bunches. For any s-bunch B_s and the simple s-2-cluster C_s we immediately obtain

$$m(B_s) \geq d(B_s) \geq d(C_s) = m(C_s),$$

and the lemma is proved. □

Remark. The previous lemma is of course true for $s = 1$, but not for $s = 2$.

As a consequence of the last two lemmas we get a strategy for Breaker in the $(1:1)$ clique game.

Let H be a graph and consider the auxiliary graph G_H with vertices corresponding to the k-cliques of H, two vertices being adjacent if the corresponding cliques have at least two vertices in common. Let $F_1, \ldots, F_s$ be the cliques corresponding to a connected component of G_H. Then the graph $\cup_{i=1}^{s} F_i$ is called an *s-collection* or just a *collection* of H. Note that the edge-set of any H is uniquely partitioned into sets N and $E(A_i)$, where N contains the edges which do not participate in a k-clique, while the A_i are the collections of H.

Theorem 33 *For every $k \geq 4$ and $\varepsilon > 0$, $p_{\mathcal{K}_k} \geq n^{-\frac{2}{k+1}-\varepsilon}$. For $k = 3$, we have that $p_{\mathcal{K}_3} \geq n^{-\frac{5}{9}}$.*

Proof. First we give a strategy for Breaker to win $\mathcal{K}_k$ if the game is played on the edge-set of a $(2k-4)$-degenerate graph L. Consider the ordering $v_1, \ldots, v_{v(L)}$ of $V(L)$, such that $|N_{V_j}(v_{j+1})| \leq 2k-4$ for $j = 1, \ldots, v(L)-1$, where $V_j = \{v_1, \ldots, v_j\}$. Then Breaker's strategy is the following: if Maker takes an edge connecting v_{j+1} to V_j, then Breaker takes another one also connecting v_{j+1} to V_j. If there is no such edge available, then Breaker takes an arbitrary edge. Suppose for a contradiction that Maker managed to occupy a k-clique $v_{i_1}, \ldots, v_{i_k}$ against this strategy, where $i_1 < \cdots < i_k$. This is impossible, since Maker could have never claimed $k-1$ of the edges $v_j v_{i_k}$, $j < i_k$.

Let $E(K_n)_p = N \dot{\cup} E(A_1) \dot{\cup} \ldots \dot{\cup} E(A_h)$ be the partition of the edges, such that N contains all edges that do not participate in any k-clique, and each A_i is a collection of k-cliques. (Corresponding to the connected components of the auxiliary graph $G_{G(n,p)}$ defined on the set of k-cliques of $G(n,p)$.)

Breaker can play the game $(E(K_n)_p, (\mathcal{K}_k)_p, 1, 1)$ by playing separately on each of the sets $E(A_i)$. More precisely, whenever Maker claims an edge which is in some $E(A_i)$, Breaker can play according to a strategy restricted just to $E(A_i)$. Since, crucially, the edge-set of each k-clique is completely contained in exactly one of the $E(A_i)$, Maker can only win the game on $E(K_n)_p$ if he wins on one of the $E(A_i)$.

Now we are going to show that every collection A on $v(A) = v$ vertices contains a $\lceil \frac{v-2}{k-2} \rceil$-bunch. We take an arbitrary k-clique F_1 from A, and build a bunch recursively as follows. If we picked k-cliques $F_1, \ldots, F_i$, then we choose F_{i+1} such that $|V(F_{i+1}) \cap (\cup_{j=1}^{i} V(F_j))| \geq 2$ and $V(F_{i+1}) \setminus (\cup_{j=1}^{i} V(F_j)) \neq \emptyset$. Note that this means that $\cup_{j=1}^{i+1} F_j$ is an $(i+1)$-bunch. Since the auxiliary graph G_A of the collection is connected we can keep doing this until $V(A) = \cup_{j=1}^{i_0} V(F_j)$ for some i_0. Knowing that $v(F_i) = k$ for all $i \leq i_0$, we have

$$i_0 \geq 1 + \frac{v-k}{k-2} = \frac{v-2}{k-2}.$$

So there exists an $\lceil \frac{v-2}{k-2} \rceil$-bunch which is a subgraph of A.

We first look at the case $k \geq 4$. Let $\varepsilon > 0$ be a constant. From Lemma 31 it follows that there exists an integer v such that for $s_0 = \lceil \frac{v-2}{k-2} \rceil$ we have

$$m(C_{s_0}) \geq \frac{k+1}{2} - \frac{k}{v} > \left(\frac{2}{k+1} + \varepsilon \right)^{-1}.$$

Then for $p = O(n^{-\frac{2}{k+1} - \varepsilon})$ it follows that there is no s_0-bunch in $G(n,p)$ a.s., since we have that the first s_0-bunch that appears in the random graph is the one of the minimum maximum density, which, by Lemma 32, is the simple s_0-2-cluster. Note here that there is a constant (depending on k and ε) number of non-isomorphic s_0-bunches.

Since in $G(n,p)$ there are no s_0-bunches a.s., there are also no collections on v vertices a.s.

Finally, all the collections A_i are $(2k-4)$-degenerate a.s., since graphs which are not $(2k-4)$-degenerate have maximum density at least $\frac{2k-3}{2} \geq$

$\frac{k+1}{2}$, provided $k \geq 4$. Note that we know already that a.s. all collections have order at most v and thus there are at most a constant (depending on k and ϵ) number of non-isomorphic non-$(2k-4)$-degenerate graphs.

This proves that Breaker has a winning strategy a.s., if $k \geq 4$ and $p = O(n^{-\frac{2}{k+1}-\varepsilon})$.

Next, we look at the case $k = 3$. As we saw, any collection of triangles on v vertices contains a $(v-2)$-bunch. Thus for $p = o(n^{-5/9})$, no v-collection with $v \geq 15$ will appear in $G(n,p)$ a.s., since it would contain a 13-bunch, whose maximum density is at least $m(C_{13}) = 2 - \frac{3}{15}$. This observation makes the problem finite: one has to check who wins on collections up to 14 vertices.

Suppose that Maker can win the triangle game on some collection of triangles on $v \leq 14$ vertices and with maximum density less than $9/5$. Let A be a minimal such collection (Maker cannot win on any proper subcollection of A).

If there was a vertex $w \in V(A)$ with $d_A(w) \leq 2$, the minimality of A would imply that Breaker has a winning strategy on A. Indeed, Breaker plays according to his strategy on $A - w$, and as soon as Maker claims one edge adjacent to w Breaker claims the other edge adjacent to w (if that edge exists otherwise he does not move). This would mean that Breaker can win on A, a contradiction. Thus, $\delta_A \geq 3$.

Let B be a $(v-2)$-bunch contained in A, with $V(A) = V(B)$. Since $\delta_B = 2$, we have $e(A) \neq e(B)$. Then

$$2 - \frac{3}{v} = m(C_{v-2}) \leq \frac{e(B)}{v} < \frac{e(A)}{v} < \frac{9}{5},$$

and

$$2v - 3 = e(C_{v-2}) \leq e(B) < e(A) < \frac{9v}{5}.$$

It is easy to check that Maker cannot win the game on a graph with less than 5 vertices, thus $v > 4$, so $e(B) = e(C_{v-2})$ and $e(A) - e(B) = 1$.

Let $\{e\} = E(A) \setminus E(B)$, and let $T_1, \ldots, T_{v-2}$ be the sequence of triangles whose union is the $(v-2)$-bunch B. Since $e(B) = e(C_{v-2})$, for every $i = 2, \ldots, v-2$ we have that T_i has a common edge with $\cup_{j=1}^{i-1} T_j$. Then B must have at least 2 vertices of degree 2. From $\delta_{B \cup \{e\}} = \delta_A = 3$ we obtain that B has exactly two vertices b_1, b_2 with $d_B(b_1) = d_B(b_2) = 2$, and moreover $e = \{b_1, b_2\}$. Since e has to participate in at least one triangle of the collection A, b_1 and b_2 have to be connected with a 2-path in B, which is possible only if all $T_1, \dots, T_{v-2}$ share a vertex. That means that A is a $(v-1)$-wheel and it is easy to see that Breaker can win the triangle game on a wheel of arbitrary size by a simple pairing strategy.

This contradiction proves that for $p = o(n^{-5/9})$, a.s. there is no triangle collection in $G(n,p)$ on which Maker can win, which means that Breaker a.s. wins the game on the whole graph. □

From Corollary 27 we get that Maker can win the game $(E(K_n)_p, (\mathcal{K}_k)_p, 1, 1)$ for $p = \Theta(n^{-\frac{2}{k+1}})$ and thus we immediately obtain $p_{\mathcal{K}_k} = O(n^{-\frac{2}{k+1}})$. For the triangle game $\mathcal{K}_3$ a stronger upper bound can be found.

Proposition 34 *The game* $(E(K_n)_p, (\mathcal{K}_3)_p, 1, 1)$ *is a Maker's win a.s., provided* $p = \omega(n^{-\frac{5}{9}})$.

Proof. It is easy to check that Maker can claim a triangle in the $(1\colon 1)$ game if the board on which the game is played is on edges of $K_5 - e$, where $e \in E(K_5)$, Figure 3.6. Indeed, in the first two moves Maker can claim two out of three edges of the triangle in the middle. Denote those edges by e_1 and e_2. Note that Breaker has to claim the third edge of that triangle in one of the first two moves, otherwise he loses after the third move of Maker.

Hence, after two moves Breaker has claimed one edge of the triangle, and one more edge, say adjacent to the left-most vertex. Then in the third move Maker claims the edge adjacent to e_1, e_2, and the right-most vertex. Note that the remaining two edges adjacent to the right-most vertex are unclaimed and both potentially close Maker's triangle. Breaker cannot claim both of them in the third move, and thus he loses.

Therefore, as soon as the graph $G(n,p)$ contains $K_5 - e$ a.s., the initial game can be won by Maker a.s. □

Corollary 27, Theorem 33 and Proposition 34 imply parts (iv) and (v) of Theorem 10.

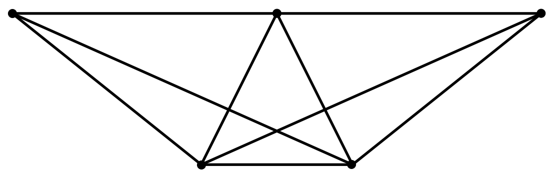

Figure 3.6: Maker can win the triangle game on K_5 minus an edge

3.4.4 G-game one-on-one

Here we give several results for the $(1:1)$ game of making an arbitrary fixed graph G. Let $\mathcal{F}_G$ be a family of subgraphs of K_n isomorphic to G. First, we apply similar argument to the one used in the clique game to obtain the following general criterion. We define the notions of bunch of G and collection of G, analogously as in the clique case.

Let $(F_1, \ldots, F_s)$ be a sequence of copies of G. Then $F = \cup_{i=1}^{s} F_i$ is called an *s-bunch of G*, or just a *bunch of G*, if $V(F_i) \setminus (\cup_{j=1}^{i-1} V(F_j)) \neq \emptyset$ and $|E(F_i) \cap (\cup_{j<i} E(F_j))| \neq \emptyset$, for each $i = 2, \ldots, s$.

Let L be a graph and consider the auxiliary graph L_G with vertices corresponding to the subgraphs of L isomorphic to G, two vertices being adjacent if the corresponding graphs have at least one edge in common. Let $F_1, \ldots, F_s$ be the copies of G corresponding to a connected component of G_H. Then the graph $\cup_{i=1}^{s} F_i$ is called an *s-collection of G* or just a *collection* of G.

Theorem 35 *Let G be a graph s.t. there exists an integer $k \geq 2$ with $m_2(G) \in (k-1, k-\frac{1}{2}]$. For arbitrarily small $\varepsilon > 0$ and for $p = n^{-m_2(G)^{-1}-\varepsilon}$, Breaker can win the game $(E(K_n)_p, (\mathcal{F}_G)_p, 1, 1)$ a.s.*

Proof. There exists $H \subseteq G$ with $d_2(H) = m_2(H) = m_2(G)$. First, note that $m_2(H) = d_2(H) > k - 1$ implies $\delta(H) \geq k$. Indeed, if we assume that H has a vertex of degree less than k, then removal of that vertex would increase the 2-density.

In every collection C of H on a vertices there is also a bunch B of H on the same number of vertices. That bunch contains a sequence of at least $b = a/v(H)$ copies of H.

In general, if a copy H' of H is added to a bunch B', we define $v_{\text{old}} = \#(V(H') \cap V(B'))$, $v_{\text{new}} = \#(V(H') \setminus V(B'))$, $e_{\text{old}} = \#(E(H') \cap E(B'))$, $e_{\text{new}} = \#(E(H') \setminus E(B'))$.

Then,

$$d_2(H) = \frac{e(H') - 1}{v(H') - 2} = \frac{e_{\text{new}} + e_{\text{old}} - 1}{v_{\text{new}} + v_{\text{old}} - 2}.$$

Since $\frac{e_{\text{old}} - 1}{v_{\text{old}} - 2} \leq d_2(H)$, we have $\frac{e_{\text{new}}}{v_{\text{new}}} \geq d_2(H)$. If we imagine that the bunch B is rebuilt from its sequence of copies of H, by starting from a single copy and adding one copy at a time, then we get

$$d(B) = \frac{e(H) + e_{\text{new}}^{(2)} + \cdots + e_{\text{new}}^{(b)}}{v(H) + v_{\text{new}}^{(2)} + \cdots + v_{\text{new}}^{(b)}} \geq d_2(H) - \varepsilon'(b),$$

where $\varepsilon'(b)$ is a positive function tending to 0 when $b \to \infty$.

Let b_0 be an integer, such that for every $b > b_0$ we have $\varepsilon'(b) < \varepsilon$. Then for $p = n^{-m_2(G)^{-1} - \varepsilon}$, $G(n,p)$ a.s. contains no bunch of order greater then $a_0 := v(H) \cdot b_0$, and thus it also contains only collections of order at most a_0.

For every collection C we have $d(C) < m_2(G) < k - \frac{1}{2}$ a.s., thus C is $2(k-1)$-degenerate. Then there is an obvious pairing strategy for Breaker on C to keep Maker's graph $(k-1)$-degenerate.

But $\delta(H) \geq k$ implies that H cannot be a subgraph of a $(k-1)$-degenerate graph, and Breaker wins the game on each of the collections a.s. □

Theorem 26 gives the upper bound for $p_{\mathcal{F}_G}$ if G contains a cycle, and thus we have that all graphs with $m_2(G) \in (k-1, k-\frac{1}{2}]$ for some $k \geq 2$ have the threshold $p_{\mathcal{F}_G}$ between $n^{-m_2(G)^{-1} - \varepsilon}$ and $n^{-m_2(G)^{-1}}$.

One class of graphs for which the last theorem does not give a bound on the threshold is trees, since $m_2(T) = 1$ for every tree T. The following statement allows us to observe the tree game on the random graph locally.

Lemma 36 *Let T be a tree.*

(i) There exists a tree T' such that Maker can win the T-game $(E(T'), \mathcal{T}'_T, 1, 1)$, where $\mathcal{T}'_T$ is the set of all copies of T in T'.

(ii) Let $\overline{T}$ be a tree of minimal size such that Maker can win the T-game $(E(\overline{T}), \overline{\mathcal{T}_T}, 1, 1)$, where $\overline{\mathcal{T}_T}$ is the set of all copies of T in $\overline{T}$.

Then we have $p_{\mathcal{F}_T} = n^{-\frac{e(\overline{T})+1}{e(\overline{T})}}$.

Proof. (i) We assume that T is rooted arbitrarily, with depth $\nu(T)$. Let T' be the rooted tree of depth $\nu(T)$, such that for every $i = 0, 1, \ldots, \nu(T)$ the down-degree of every vertex $v \in V(T')$ on level i is

$$\underline{d}(v) = 2 \cdot max_{\substack{u \in V(T) \\ u \text{ on } i\text{th level of } T}} \underline{d}(u),$$

i.e., doubled maximal down-degree of ith level of T. In the following, we exhibit a winning strategy for Maker (as the second player) in the game $(\widehat{E(T')}, \mathcal{T}'_T, 1, 1)$, which will obviously imply the statement of the lemma.

For every vertex $v \in V(T')$ the down-degree $\underline{d}(v)$ is even, and therefore we can arbitrarily pair up the edges going downwards from v. Maker now simply applies a pairing strategy, thus claiming half of the edges going down from each of the vertices of T'. At the end of the game, the graph claimed by Maker is a tree of depth $\nu(T)$, and every vertex on ith level has down-degree $max_{\substack{u \in V(T), \\ u \text{ on } i\text{th level of } T}} \underline{d}(u)$. This tree obviously contains T as a subgraph, and thus Maker wins the game.

(ii) If $p = o\left(n^{-\frac{e(\overline{T})+1}{e(\overline{T})}}\right)$, then the graph $G(n,p)$ a.s. contains no connected component of order $e(\overline{T})$ or larger, and every connected component is a tree. Since we assumed that $\overline{T}$ is the minimal tree on which Maker can win the T-game, he cannot win the game on any of the connected components of $G(n,p)$, and thus he also cannot win the game on the whole edge-set of $G(n,p)$.

On the other hand, if $p = \omega\left(n^{-\frac{e(\overline{T})+1}{e(\overline{T})}}\right)$, then $G(n,p)$ contains a copy of $\overline{T}$ a.s. In that case Maker can win the game on the whole $G(n,p)$ by simply restricting his play to edges of $\overline{T}$. □

In the following three propositions, we give some bounds on the threshold for some special classes of trees, with the help of the previous lemma.

Proposition 37 *The threshold probability for the l-path game is*

$$p_{\mathcal{F}_{P_l}} = n^{-\frac{e_l+1}{e_l}}, \text{ where } e_l = \Theta\left(2^{l/2}\right).$$

Proof. Using Lemma 36, we only need to show that for the minimal size e_l of a tree on which Maker can win the path game we have $e_l = \Theta\left(2^{l/2}\right)$.

It is easy to check that Maker can win the game on the tree T_0 that has a root of degree 3, and below it three binary trees of depth $\lceil l/2 \rceil - 1$. Indeed, if he starts by claiming an edge adjacent to the root, and then proceeds by a pairing strategy described in the proof of Lemma 36 (i), he will claim two edges adjacent to the root, and one edge going down from every other vertex (which is not a leaf). Therefore, at the end of the game there will be two disjoint paths of length $\lceil l/2 \rceil$ adjacent to the root that are claimed by Maker, and thus also an l-path claimed by Maker. The size of the tree T_0 is $1 + 3 \cdot \left(2^{\lceil l/2 \rceil} - 2\right)$.

On the other hand, Breaker can win the game on any tree of size less than $2^{\lceil l/2 \rceil}$. To prove that, we root the tree arbitrarily and we apply Theorem 3 on the set L of all paths of length $\lceil l/2 \rceil$ that are contained in a path connecting the root and a leaf. Each such path is uniquely determined by the one of its endpoints that is further away from the root ("lowest point"), and thus the number of elements of L is less than the number of vertices of the tree. Therefore, if the game is played on the edges of a tree with less than $2^{\lceil l/2 \rceil}$ vertices, then Breaker can prevent Maker from claiming a path from L. Since every l-path in a rooted tree must contain one of the paths from L, Breaker can also win the l-path game. □

Proposition 38 *The threshold probability for the d-star game is*

$$p_{\mathcal{F}_{S_d}} = n^{-\frac{2d}{2d-1}}.$$

Proof. Maker will win the d-star game playing on edges of a star of size $2d - 1$, independent of his strategy.

On the other hand, on any tree of size less than $2d - 1$ Breaker wins the d-star game independent of his strategy, since throughout the game Maker claims at most $d - 1$ edges.

Therefore, using Lemma 36 we immediately obtain the statement of the proposition. □

As we saw in the last two propositions, the size of the smallest tree on which Maker can win the game is linear in terms of the size of the winning set for the star game, but exponential for the path game. Even though we cannot determine the threshold for an arbitrary tree, in the next proposition we analyse another, more general class of trees.

Proposition 39 *Let F be a rooted tree with depth l and let d_i, $i = 1, \dots, l$ be integers, such that the degree of every vertex $v \in V(F)$ on the ith level of tree F has down-degree d_i.*

Then, the threshold probability for the F-game is $p_{\mathcal{F}_F} = n^{-\frac{t+1}{t}}$, where

$$2^{l-1} \cdot (d_0 - 1) \cdot d_1 \cdots d_{l-1} \leq t \leq \sum_{i=1}^{l} 2^i \cdot d_0 \cdot d_1 \cdots d_{i-1}.$$

Proof. Similarly as in the proof of Lemma 36 (i), Maker can win using a pairing strategy on a tree obtained from F by "doubling" down-degrees on every level. That tree has size $\sum_{i=1}^{l}(2d_0) \cdot (2d_1) \cdots (2d_{i-1}) = \sum_{i=1}^{l} 2^i \cdot d_0 \cdot d_1 \cdots d_{i-1}$.

On the other hand, Breaker can win on a rooted tree if he prevents Maker from claiming $(d_1 - 1) \cdot d_2 \cdots d_l$ different paths of length l that are contained in a path connecting the root and a leaf. By Theorem 5, Breaker can do that if the total number of such paths is less than $2^{l-1} \cdot (d_0 - 1) \cdot d_1 \cdots d_{l-1}$. This is true for every tree with the number of vertices less than that, by the same argument as in Proposition 37. □

Note that both stars and paths of even length satisfy the conditions of the last proposition. For $(2l)$-paths we have

$$2^{l-1} \leq t \leq \sum_{i=1}^{l} 2^i \cdot 2 \leq 2^{l+2},$$

and for d-stars we get $d-1 \leq t \leq 2d$. So, even though it is not as precise as the previous two propositions, it still gives the correct order of magnitude.

The most important question that remains unanswered is what can we say for the probability threshold for graphs whose m_2 density is outside the intervals mentioned in Theorem 35. We know only that for some games—like the triangle game and the tree games—the probability threshold is not close to the inverse of the bias threshold for the game on the complete graph. For example, for the triangle game we have $n^{-\frac{5}{9}} = p_{\mathcal{K}_3} \neq (b_{\mathcal{K}_3})^{-1} = n^{-\frac{1}{2}}$.

3.5 Open questions

More sharp thresholds? We saw that the connectivity game has a sharp threshold, and even more. We think that both the perfect matching game and the Hamiltonian cycle game have the same sharp threshold $\frac{\log n}{n}$, and maybe even more... It would be very interesting to decide whether the following conjectures are true.

Conjecture 2 *We have*

$$\begin{aligned} &(i) \quad \tau(\textit{Maker wins } \mathcal{M}) = \tau(\delta(G) \geq 2) \textit{ a.s., and} \\ &(ii) \quad \tau(\textit{Maker wins } \mathcal{H}) = \tau(\delta(G) \geq 4) \textit{ a.s.} \end{aligned}$$

Clique game/H-game. The exact determination of the threshold $p_{\mathcal{K}_k}$ for the k-clique game remains outstanding.

Problem 1 *Decide whether* $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$ *for* $k \geq 4$.

The arguments of Bednarska and Łuczak [13] could be extended to full generality to positional games on random graphs along the lines of Section 3.3.4. More precisely, the following is true. Let $\mathcal{K}_H$ be the family of subgraphs of K_n, isomorphic to H. Then for any fixed graph H there is a constant $c(H)$, such that

$$b^p_{\mathcal{K}_H} = \Theta\left(pb_{\mathcal{K}_H}\right) = \Theta\left(pn^{-1/m_2(H)}\right),$$

provided

$$p \geq \Omega\left(\frac{\log^{c(H)} n}{n^{1/m_2(H)}}\right).$$

Concerning the one-on-one game, it would be desirable to extend Theorem 35 to determine all fixed graphs for which an extension of the low-density Maker's win, à la Proposition 34, exists.

Problem 2 *Characterize those graphs H for which there exists a constant $\epsilon(H) > 0$, such that the unbiased game $\mathcal{K}_H$ is a.s. a Maker's win if $p = n^{-1/m_2(H)-\epsilon(H)}$.*

For such graphs the determination of the threshold $p_{\mathcal{K}_H}$ is a finite problem, in a way similar to the case $H = K_3$. We saw from Theorem 35 that these graphs must have the 2-density out of intervals $(k-1, k-\frac{1}{2}]$, $k \in \mathbb{N}$.

Relationships between thresholds. It is an intriguing task to understand under what circumstances the following is true.

Problem 3 *Characterize those games $(X, \mathcal{F})$ for which*

$$p_{\mathcal{F}} = \frac{1}{b_{\mathcal{F}}}.$$

More generally, characterize the games for which

$$b^p_{\mathcal{F}} = \Theta\left(pb_{\mathcal{F}}\right),$$

for every $p = \omega\left(\frac{1}{b_{\mathcal{F}}}\right)$.

This is not true in general as the triangle game shows. What is the reason it is true for the connectivity game and the perfect matching game? Is it because the appearance of these properties has a sharp threshold in $G(n,p)$? Or because the winning sets are not of constant size?

Problem 4 *Suppose $p_{\mathcal{F}} = 1/b_{\mathcal{F}}$. Is it true that for every $p \geq p_{\mathcal{F}}$, $b^p_{\mathcal{F}} = \Theta\left(pb_{\mathcal{F}}\right)$?*

It would be very interesting to relate the thresholds $b_{\mathcal{F}}$ and $p_{\mathcal{F}}$ to some thresholds of the family $\mathcal{F}$ in the random graph $G(n,p)$ (or, more generally, in the random set X_p). It seems to us that if the family $\mathcal{F}_p$ is quite dense and well-distributed in X, then Maker still wins the $(1\colon 1)$ game.

Problem 5 *Characterize those games* $(X, \mathcal{F})$ *for which there exists a constant* K*, such that for any probability* p *with*

$$\Pr\left[\min_{x \in X_p} |\{F \in \mathcal{F}_p : x \in F\}| > K\right] \longrightarrow 1,$$

we have $p_{\mathcal{F}} = O(p)$ *and/or* $b_{\mathcal{F}} = \Omega(1/p)$.

Homer: *God, if you really are God, you'll get me tickets to that game!*

The Simpsons, by Matt Groening

Chapter 4

Planarity game and k-coloring game

4.1 Introduction

In this chapter, we will take a closer look at the planarity game and the k-coloring game. For both of them we consider two versions of the game—the Maker-Breaker version and the Avoider-Forcer version.

Before we describe the games in more detail, we would like to say a few words about the monotonicity of positional games in general. Namely, if Maker can win the $(a_1 : b)$ biased game $\mathcal{F}$, it is easy to see that he can also win the $(a_2 : b)$ biased game $\mathcal{F}$, for any $a_2 \geq a_1$. The analog statement holds also for Breaker's bias—if Maker can win the $(a : b_1)$ biased game $\mathcal{F}$, he can also win the $(a : b_2)$ biased game $\mathcal{F}$, for any $b_2 \leq b_1$. We call this property *bias-monotonicity.*

Surprisingly, Avoider-Forcer games are not bias-monotone. An easy example from [26] shows this. Let n be an integer. The board on which we play is $X = \{1, 2, \ldots, 2n-1, 2n\}$, and the winning sets are $\mathcal{F} = \{\{2i-1, 2i\} : i = 1, 2, \ldots, n\}$. Now, if Forcer starts and the game is played with bias $(1 : b)$, then it is easy to see that for n large enough Forcer wins if and only if b is even. Indeed, if b is odd, then Avoider can always claim an unclaimed element of the board whose neighbor was already claimed by Forcer. Therefore, he will never claim a winning set. On the other hand, if b is even, then Forcer can claim $b/2$ whole winning sets in every move, and every time Avoider is forced to claim an element whose neighbor is not yet claimed. Thus, if the board is large enough, he will lose as soon as there is no completely unclaimed winning set.

But, there is a way to resolve this problem. If in a $(a : b)$ biased game we allow Avoider to claim at least a elements per move and we allow Forcer to claim at least b elements, then Avoider-Forcer games become monotone. Suppose that in this new setting Avoider can win $(a_1 : b_1)$ game $\mathcal{F}$. That means that he has a winning strategy, a rule book that tells him which $\geq a_1$ edges to claim in every move, knowing the current board situation. But he can then use the exact same strategy for $(a_2 : b_1)$ game $\mathcal{F}$ for any $a_2 \leq a_1$, and win. Similarly, the same strategy would also ensure a win for Avoider in $(a_1 : b_2)$ game $\mathcal{F}$ for any $b_2 \geq b_1$. Note that in this case divisibility arguments do not decide games anymore. If the game from the previous example is played with these rules, Forcer always wins. In this chapter we are going to look at the bias-monotone version of the rules for the Avoider-Forcer game.

In the Maker-Breaker case, according to the definition given before, in $(a : b)$ biased games Maker claims exactly a elements of the board in each move, and Breaker claims exactly b. If we also change the rules and allow Maker to claim *at most* a and Breaker *at most* b elements in each move, the winner stays the same—it is not hard to show that claiming less elements per move than the bias does not improve players' playing power, since claiming extra elements of the board in a move does not hurt a player. Hence, we can change the rules in this fashion, and everything stays the same—Maker-Breaker games remain bias-monotone.

We come back to the concrete games we want to analyze. Firstly, we consider the planarity game. The game is played on the edges of complete graph K_n on n vertices, and the set of winning sets contains edge-sets of all non-planar subgraphs of K_n. In the Maker-Breaker version, Maker's goal is to claim a non-planar graph. We will show that Maker can do that in the $(1 : b)$ biased game if $b < (1/2 - \varepsilon)n$, for any fixed $\varepsilon > 0$. On the other hand, Breaker can win the game if $b \geq n/2$.

In the Avoider-Forcer version of the game, Avoider wants to avoid claiming a non-planar graph. In other words, he would like to keep his graph planar until the end of the game. We can show that he can do that in the $(1:b)$ biased game if $b \geq 2n^{5/4}$, and that Forcer wins if $b < (1/2-\varepsilon)n$, for any fixed $\varepsilon > 0$.

The other game we consider is the k-coloring game. It is played on the edges of the complete graph K_n on n vertices and the set of winning sets contains edge-sets of all non-k-colorable subgraphs of K_n. Maker wins in the Maker-Breaker version of the game if he claims a non-k-colorable graph. We show that Maker can win the $(1:b)$ game if $b \leq \frac{n}{3k\log k}$, and Breaker can prevent him from winning if $b \geq s_k n$, where s_k is a constant depending on k, satisfying $s_k \sim \frac{3}{k\log k}$ as $k \to \infty$.

In the Avoider-Forcer version, Avoider would like to keep his graph k-colorable until the end of the game. We show that he can do this for $b \geq 2kn^{1+\frac{1}{2k-3}}$, whereas Forcer wins for $b \leq \frac{n}{3k\log k}$.

4.2 Planarity game

As we mentioned already, the set of winning sets $\mathcal{F}$ contains edge-sets of all non-planar subgraphs of K_n.

4.2.1 Maker-Breaker planarity game

The following theorem shows that the threshold bias at which Maker's win changes to Breaker's win in the planarity game is "around" $n/2$.

Theorem 40 *If $b \geq n/2$, then Breaker wins the $(1:b)$ planarity game and, for any fixed $\varepsilon > 0$, if $b \leq (\frac{1}{2}-\varepsilon)n$ then Maker wins the $(1:b)$ planarity game.*

Proof. Let $b \geq n/2$. We will use the following result of Bednarska and Pikhurko.

Theorem 41 *[15, Corollary 10] Suppose that Maker and Breaker select respectively 1 and b edges of K_n and Maker wants to build a cycle. Then Maker wins the game (no matter who starts) if and only if $b < \lceil n/2 \rceil - 1$.*

The last theorem implies that with the bias $b \geq n/2$ Breaker can prevent Maker from building a cycle. That implies that Maker's graph is a forest at the end of the game, which is obviously planar.

Let $\varepsilon > 0$ and let $b \leq (\frac{1}{2} - \varepsilon)n$. We will assume that $b = \lfloor(\frac{1}{2} - \varepsilon)n\rfloor$ which is legitimate since Maker-Breaker games are monotone in the bias. Let $\alpha > 0$ be the real number satisfying the equation

$$(1+\alpha)n = \frac{\binom{n}{2}}{b+1}$$

and let k be the smallest positive integer such that

$$\left(1+\frac{\alpha}{2}\right) > \frac{k}{k-2}$$

holds.

Maker's goal is to avoid cycles of length smaller than k, which we will call "short cycles", during the first $(1+\frac{\alpha}{2})n$ moves. If he succeeds, Maker's graph will at that point of the game have

$$(1+\frac{\alpha}{2})n > \frac{k}{k-2}n$$

edges and girth at least k. But, it is well-known that a planar graph with girth at least k cannot have more than $\frac{k}{k-2}(n-2)$ edges. Hence, Maker's graph is non-planar, and he won.

It remains to show that Maker can avoid claiming a short cycle during the first $(1+\frac{\alpha}{2})n$ moves. His strategy is the following. For as long as possible he claims edges (u,v) that satisfy the following two properties:

(a) (u,v) does not close a short cycle,

(b) the degrees of u and v in Maker's graph are less than $n^{1/(k+1)}$.

It suffices to prove that when this is no longer possible, that is, every remaining unclaimed edge violates either (a) or (b), Maker has claimed at least $(1+\frac{\alpha}{2})n$ edges.

Every edge that violates property (b) must have at least one endpoint of degree $n^{1/(k+1)}$ in Maker's graph. Since Maker's graph at any point of the game contains at most $(1+\alpha)n$ edges, there are at most $2(1+\alpha)n^{1-1/(k+1)}$ vertices of degree $n^{1/(k+1)}$. Therefore, the number of edges that violate property (b) is at most

$$n \cdot 2(1+\alpha)n^{1-1/(k+1)} = o(n^2).$$

For any fixed $s < k$, the number of edges adjacent to a vertex v that close a cycle of length s is at most Δ^s, where Δ is the maximum degree in Maker's graph. If we assume that property (b) has not been violated, then $\Delta \leq n^{1/(k+1)}$. Therefore, there are at most

$$n \cdot \sum_{s=3}^{k-1} n^{s/(k+1)} = o(n^2)$$

edges that close a short cycle.

The total number of edges that violate (a) or (b) if claimed by Maker, is $o(n^2)$. On the other hand, after $(1+\frac{\alpha}{2})n$ moves have been played, the number of unplayed edges is $\frac{\alpha}{2}n(b+1) = \Theta(n^2)$. Hence, in the first $(1+\frac{\alpha}{2})n$ moves Maker can play edges that satisfy properties (a) and (b), which means that he does not claim a short cycle. This completes the proof of the theorem. □

Even though the larger bias of Breaker generally makes the game for Maker harder to win, Maker's strategy in the proof of Theorem 40 is about avoiding and thus works as good for the larger Breaker's bias. Of course, the successful avoiding of small cycles alone is not enough to prove Maker's win in the planarity game. We would also need that Maker claims more than n edges during the game, and this does not hold if Breaker's bias is larger.

4.2.2 Avoider-Forcer planarity game

In the following theorem we give an upper bound and a lower bound for the threshold bias at which Forcer's win changes into Avoider's win in the planarity game.

Theorem 42 *If $b \geq 2n^{5/4}$, then Avoider wins the $(1:b)$ planarity game, and for any fixed $1 > \varepsilon > 0$, if $b \leq \frac{n}{2}(1-\varepsilon)$, then Forcer wins the $(1:b)$ planarity game.*

Proof. Assume that $b \geq 2n^{5/4}$. We divide the game in four stages, and Avoider's strategy is the following.

In the first stage, he builds a matching by repeatedly claiming an edge that connects two vertices such that neither of them is adjacent to any other edge previously claimed by Avoider. The first stage ends when no such edge is free and Avoider cannot further extend his matching. We denote the set of vertices that are covered by Avoider's matching by M.

Next, in the second stage, Avoider claims edges with one endpoint in M and the other one in $V \setminus M$ such that every vertex of $V \setminus M$ throughout the second stage has degree at most one in Avoider's graph. When no such edge is available, the second stage ends.

In the third stage Avoider builds another matching on M. More precisely, he is repeatedly claiming edges that connect two vertices such that neither of them is adjacent to any other edge previously claimed *in the third stage* by Avoider. The third stage ends when no such edge is unclaimed and Avoider cannot further extend the matching.

In the final stage, Avoider plays arbitrarily to the end of the game. If we prove that in the final stage Avoider will play at most one edge, the theorem is proved. Indeed, the graph that contains Avoider's edges from the first and the third stage is a union of two matchings, i.e., a union of disjoint paths and cycles. Further more, if we add Avoider's edges from the second stage to this graph, we actually add a number of hanging edges (edges with one vertex having degree 1). Obviously, if we now add the only edge from the fourth stage to that graph, it remains planar.

Let e be the number of edges that Avoider claims in the whole game. After the first stage, Forcer must have claimed all the edges on the vertices of $V \setminus M$. Since Avoider's matching on M has size at most e, we have $|V \setminus M| \geq n - 2e$ and therefore Forcer has already claimed at least $\binom{n-2e}{2}$ edges. It follows that there are at most

$$\binom{n}{2} - \binom{n-2e}{2} \leq 2en$$

unclaimed edges left in the graph. Avoider will claim at most $\frac{e}{\sqrt[4]{n}}$ of these edges, as $b \geq 2n^{5/4}$.

In the second stage, Avoider claims edges between M and $V \setminus M$. When this is no longer possible, every unclaimed edge between M and $V \setminus M$ is adjacent to a vertex of $V \setminus M$ which has degree one in Avoider's graph. It follows that, at this point, the number of unclaimed edges between M and $V \setminus M$ is at most

$$2e \cdot \frac{e}{\sqrt[4]{n}} = \frac{2e^2}{\sqrt[4]{n}}.$$

In the third stage, Avoider builds his second matching on M and when this is no longer possible the number of unclaimed edges with both endpoints in M is at most

$$\binom{2e}{2} - \binom{2e - 2e/\sqrt[4]{n}}{2} \leq \frac{4e^2}{\sqrt[4]{n}}.$$

To see this, it is enough to observe that the order of the second matching is at most $2e/\sqrt[4]{n}$, and that all edges with endpoints in M that are not adjacent to the second matching must be claimed by Forcer after the third stage.

Putting everything together, the total number of unclaimed edges after the third stage is at most

$$\frac{2e^2}{\sqrt[4]{n}} + \frac{4e^2}{\sqrt[4]{n}} = \frac{6e^2}{\sqrt[4]{n}}.$$

Since $e < \binom{n}{2}/b$, we have that the number of edges to be played in the fourth stage is $\frac{6e^2}{\sqrt[4]{n}} \leq b$, which means that in the fourth stage Avoider plays at most one move.

Next, assume that $b \leq \frac{n}{2}(1 - \varepsilon)$. Let k be the smallest positive integer such that $\frac{1}{1-\varepsilon} > \frac{k}{k-2}$. Forcer's goal is to prevent Avoider from claiming a cycle of length smaller than k, which we will call "a short cycle". That way, Avoider's graph at the end of the game will have

$$\frac{\binom{n}{2}}{b+1} > \frac{k}{k-2}n$$

edges, and girth at least k. As we mentioned before, graph with such properties cannot be planar.

It remains to show that Forcer can prevent Avoider from claiming a short cycle. To do that we will use the following theorem of Bednarska and Łuczak.

Theorem 43 *[13, Theorem 1] For every graph G which contains at least three non-isolated vertices there exist positive constants c and n_0 such that, playing the $(1 : q)$ game on K_n, Breaker can prevent Maker from building a copy of G provided that $n > n_0$ and $q > cn^{1/m_2(G)}$.*

For a cycle C_i, we have $m_2(C_i) = \frac{i-1}{i-2}$. Therefore, there exist constants c_i, $i = 3, \ldots, k-1$ such that Forcer can prevent Avoider to claim a copy of C_i, if the number of edges he is allowed to play in a move is $c_i n^{\frac{i-2}{i-1}}$. Now, since

$$b = \omega\left(\sum_{i=3}^{k-1} c_i n^{\frac{i-2}{i-1}}\right)$$

Forcer can at the same time prevent Avoider from claiming any short cycle C_i, $3 \leq i < k$, by simply playing all $k-3$ games in parallel. This concludes the proof of the theorem. □

4.3 k-coloring game

The k-coloring game is played on edge-set of K_n and the set of winning sets $\mathcal{F}$ contains edge-sets of all non-k-colorable subgraphs of K_n. Throughout this section we assume that k is a constant.

4.3.1 Maker-Breaker k-coloring game

The following theorem shows that the threshold bias at which Maker's win changes to Breaker's win in the k-coloring game is of order n.

Theorem 44 *For every k there exists a constant s_k such that $s_k \sim \frac{3}{k \log k}$ as $k \to \infty$, and if $b \geq s_k n$ then Breaker wins the $(1 : b)$ k-coloring game.*

If $b \le \frac{n}{3k\log k}$ then Maker wins the $(1:b)$ k-coloring game.

Proof. Assume first that $b \le \frac{n}{3k\log k}$. Maker's goal is to prevent Breaker from building a clique of size $\lceil n/k \rceil$, and this is enough to ensure his win. Indeed, Maker's graph is surely not k-colorable if there is no independent set of size at least $\lceil n/k \rceil$ in it.

Let $\mathcal{F}$ be the hypergraph whose vertices are the edges of K_n and whose hyperedges are the $\lceil n/k \rceil$-cliques of K_n. We name the players of the $(b:1)$ game $\mathcal{F}$ CliqueMaker and CliqueBreaker. As we saw, Maker will win the k-coloring game if he claims an edge in every clique in $\mathcal{F}$, and therefore he can win by adopting a winning strategy of CliqueBreaker. Such strategy is provided by Theorem 3, since

$$\begin{aligned}\sum_{D\in\mathcal{F}} 2^{-|D|/b} &\le \binom{n}{\lceil n/k\rceil} 2^{-\binom{\lceil n/k\rceil}{2}/b} \le (ek)^{\lceil n/k\rceil} 2^{-\binom{\lceil n/k\rceil}{2}/b} \\ &\le 2^{\frac{n\log_2 e}{k} + \frac{n\log_2 k}{k} - \frac{3n^2 k\log k}{2k^2 n} + \frac{n3k\log k}{2kn}} = o(1).\end{aligned}$$

Hence, Maker wins the game.

Assume now that $b \ge s_k n$, where s_k is a constant depending on k that will be specified later. We will make use of the following theorem of Kim.

Theorem 45 *[29, Corollary 1.2] If G is a graph with maximum degree Δ and girth at least 5, then*

$$\chi(G) \le (1+\nu(\Delta))\frac{\Delta}{\log\Delta},$$

where $\nu(\Delta) \to 0$ as $\Delta \to \infty$.

Let Δ_0 be the maximal value of Δ for which

$$(1+\nu(\Delta))\frac{\Delta}{\log\Delta} \le k.$$

Since $\nu(\Delta) \to 0$ as $\Delta \to \infty$, we have that $\Delta_0 \sim k\log k$ as $k \to \infty$. Breaker's goal is to force Maker to build a graph with maximum degree at most Δ_0 and girth at least 5. By Theorem 45 Maker's graph will then be k-colorable.

In each move, Breaker will play $c_3 n^{1/2}$ of his edges to prevent Maker from building a triangle ($m_2(C_3) = 2$), and $c_4 n^{2/3}$ of his edges to prevent Maker from building a cycle of length 4 ($m_2(C_4) = 3/2$), where c_3 and c_4 are the constants whose existence is guaranteed by Theorem 43. All the remaining edges, so $b' := b - c_3 n^{1/2} - c_4 n^{2/3} = (1 - o(1))b$ edges, Breaker will use to make sure that the maximum degree in Maker's graph does not surpass Δ_0. Hence, if uv is the last edge played by Maker, Breaker will claim $\frac{1}{2}b'$ edges incident with u and $\frac{1}{2}b'$ edges incident with v. It follows that the maximum degree in Maker's graph will be at most

$$1 + \frac{n-1}{b'/2} \leq \frac{2n}{\frac{2}{3}b} \leq \frac{3}{s_k}.$$

Therefore, if $s_k = \lceil \frac{3}{\Delta_0} \rceil$, then Breaker can force Maker to build a graph with maximum degree at most Δ_0 and girth at least 5, and thus he wins. Note that s_k defined in this way satisfies $s_k \sim \frac{3}{k \log k}$ as $k \to \infty$. This concludes the proof. □

4.3.2 Avoider-Forcer k-coloring game

In the following theorem we give an upper bound and a lower bound for the threshold bias at which the Forcer's win changes into the Avoider's win in the k-coloring game.

Theorem 46 *If $b \leq \frac{n}{3k \log k}$ then Forcer wins $(1:b)$ k-coloring game, and if $b > 2kn^{1+\frac{1}{2k-3}}$ then Avoider wins $(1:b)$ k-coloring game.*

Proof. Assume first that $b \leq \frac{n}{3k \log k}$. Forcer's goal is to avoid building a clique of size $\lceil n/k \rceil$. If he achieves this goal, Avoider's graph will not contain an independent set of size $\lceil n/k \rceil$ and for that reason will not be k-colorable, thus he will win. Let $\mathcal{F}$ be the hypergraph whose vertices are the edges of K_n and whose hyperedges are the $\lceil n/k \rceil$-cliques of K_n. We name the players of the $(b:1)$ game $\mathcal{F}$ CliqueAvoider and CliqueForcer. As we saw, Forcer will win the k-coloring game if he does not claim all edges in any clique in $\mathcal{F}$, and therefore he can win by adopting a winning strategy of CliqueAvoider.

We will use the following general criterion of Hefetz, Krivelevich and Szabó for Avoider's win.

Theorem 47 *[26, Theorem 1.1] If*

$$\sum_{D\in\mathcal{H}}\left(1+\frac{1}{b}\right)^{-|D|}<\left(1+\frac{1}{b}\right)^{-b},$$

then Avoider wins the $(1:b)$ *game* $\mathcal{H}$.

We have

$$\begin{aligned}\sum_{D\in\mathcal{F}}\left(1+\frac{1}{b}\right)^{-|D|} &\leq \binom{n}{\lceil n/k\rceil}\left(1+\frac{1}{b}\right)^{-\binom{\lceil n/k\rceil}{2}}\\ &\leq (ek)^{\lceil n/k\rceil}2^{-\binom{\lceil n/k\rceil}{2}/b}\\ &\leq 2^{\frac{n\log_2 e}{k}+\frac{n\log_2 k}{k}-\frac{3n^2k\log k}{2k^2n}+\frac{n3k\log k}{2kn}}=o(1),\end{aligned}$$

applying the last theorem we conclude that there exists a winning strategy for CliqueAvoider, and thus Forcer wins the game.

Next, let $b>2kn^{1+\frac{1}{2k-3}}$. We will present a strategy for Avoider to claim a $(k-1)$-degenerate graph. Clearly, that would entail Avoider's win in the k-coloring game as every $(k-1)$-degenerate graph is k-colorable.

Avoider will play several auxiliary minigames one after the other, until all edges are claimed and the game is over. Before we describe his strategy in detail, let us define two basic types of minigames.

Let A be a set of vertices, and let us assume that the game is in progress, meaning that some edges may have already been claimed. When we say that Avoider is playing the (A)-*minigame*, we mean that Avoider repeatedly claims independent edges with both endpoints in A for as long as possible, i.e., he extends a matching on A while he can. When Avoider cannot further extend his matching, we denote the set of vertices in A incident to the matching by A_1, and let $A_2=A\setminus A_1$. Note that all the edges with both endpoints in A_2 are already claimed by one of the players.

Let A and B be two disjoint sets of vertices. Again, we assume that the game is in progress, meaning that some edges may have already been claimed. When we say that Avoider is playing the $(A : B)$-*minigame*, we mean that Avoider repeatedly claims edges in the $A : B$ cut such that no vertex in B is incident with more than one of Avoider's edges claimed in *this* minigame. When this is no longer possible, the minigame is over. At this point, let B_1 denote the set of vertices of B that are incident with an edge claimed by Avoider in *this* minigame, and let $B_2 = B \setminus B_1$. Note that all edges in the cut $A : B_2$ have already been claimed by one of the players.

We say that the size of an (A)-minigame is $\frac{1}{2}|A|^2$, and the size of an $(A : B)$-minigame is $|A| \cdot |B|$. Note that the size of a minigame is an upper bound for the number of edges it contains.

Now we can describe the way Avoider plays the game. We introduce *a minigame pool* $\mathcal{P}$, which is a dynamic collection of minigames that will be updated during the game – it will contain minigames waiting to be played by Avoider.

To each minigame in the pool, we assign an integer parameter, that will help us keep track of the degeneracy of Avoider's graph throughout the game. So, instead of the (A)-minigame (or the $(A : B)$-minigame), we will consider the $(A)_l$-minigame (or the $(A : B)_l$-minigame) for an appropriate integer l. In the beginning of the game $\mathcal{P}$ contains only one minigame – the $(V(K_n))_{k-1}$-minigame.

When the game is played, Avoider repeatedly chooses a minigame of the largest size in the pool $\mathcal{P}$, removes it from the pool, plays it to its end, and then updates $\mathcal{P}$ as follows. If the minigame played was an $(A)_l$-minigame, then he places two new minigames into $\mathcal{P}$, the $(A_1)_{l-1}$-minigame and the $(A_1 : A_2)_l$-minigame. On the other hand, if the minigame played was an $(A : B)_l$-minigame, then he places only the $(A : B_1)_{l-1}$-minigame back into $\mathcal{P}$.

Note that at any point of the game, every unclaimed edge is in exactly one of the minigames in $\mathcal{P}$. It is easy to see that as long as the parameters of all the minigames in $\mathcal{P}$ are non-negative, Avoider's graph is $(k-1)$-degenerate. Therefore, it suffices to prove that after the first minigame with parameter 1 is taken out of the pool to be played, Avoider plays at most one more move before the k-coloring game ends.

As long as all the minigame parameters are non-negative, the number of minigames in the pool $\mathcal{P}$ is at most k. This follows from the initial state of the pool and the way it is being updated.

We will prove by induction on l that any minigame in the pool which has parameter $0 \leq l \leq k-1$ is of size at most $n^2\left(\frac{2k^2n^2}{b^2}\right)^{k-1-l}$. First, for the base step, note that the size of the (unique) minigame with parameter $l = k-1$ is less than n^2. Now let us assume that the induction hypotheses holds for all games with parameter l, where $l > l_0 \geq 0$. For a minigame M in the pool with parameter l_0 we consider the following three cases.

Case 1. M is an $(A_1)_{l_0}$-minigame that was put into the pool after the $(A)_{l_0+1}$-minigame has ended. Just before Avoider started playing the $(A)_{l_0+1}$-minigame there was no minigame in the pool of a larger size. Since the total number of games in the pool was at most k, the total number of unplayed edges at that point was at most $k \cdot |A|^2 \leq kn^2\left(\frac{2k^2n^2}{b^2}\right)^{k-1-l_0-1}$. The number of edges Avoider will play during the $(A)_{l_0+1}$-minigame is certainly bounded from above by the total number of edges that Avoider will claim until the end of the k-coloring game, which is at most $\frac{kn^2}{b}\left(\frac{2k^2n^2}{b^2}\right)^{k-1-l_0-1}$. Avoider's strategy for the $(A)_{l_0+1}$-minigame guarantees that the set A_1 will be of size at most twice this much, and hence the $(A_1)_{l_0}$-minigame will be of size at most

$$\frac{1}{2}|A_1|^2 \leq \frac{1}{2}\left(\frac{2kn^2}{b}\left(\frac{2k^2n^2}{b^2}\right)^{k-1-l_0-1}\right)^2 \leq n^2 \cdot \left(\frac{2k^2 \cdot n^2}{b^2}\right)^{k-1-l_0}.$$

Case 2. M is an $(A_1 : A_2)_{l_0}$-minigame that was put into the pool after the $(A)_{l_0}$-minigame has ended. The size of the $(A_1 : A_2)_{l_0}$-minigame is obviously less than the size of the $(A)_{l_0}$-minigame, which can be upper-bounded as in Case 1.

Case 3. M is an $(A : B_1)_{l_0}$-minigame that was put into the pool after the $(A : B)_{l_0+1}$-minigame has ended. As in Case 1, we can bound the number of edges Avoider will play during the $(A : B)_{l_0+1}$-minigame from above, by the total number of edges that Avoider will claim until the end of the k-coloring game. Thus, knowing that the $(A : B)_{l_0+1}$-minigame was of maximal size before it was played, we get

$$\frac{k|A||B|}{b} \leq \frac{kn^2}{b}\left(\frac{2k^2n^2}{b^2}\right)^{k-1-l_0-1}.$$

Therefore, the size of B_1 is also at most that much. Since the size of A is at most n^2/b, the total size of the $(A : B_1)_{l_0}$-minigame is at most $n^2 \left(\frac{2k^2n^2}{b^2}\right)^{k-1-l_0}$. This concludes the induction step.

If a minigame with parameter 1 is taken from the pool, then the total number of edges to be played in the remainder of the game is at most $kn^2 \left(\frac{2k^2n^2}{b^2}\right)^{k-2}$ which is less than b, meaning that Avoider will play at most one move before the game ends. □

...već pobeže, oj nesrećo,
na kraj sveta, oj nevoljo!
Plavi zec,
čudni zec,
jedini na svetu!

Plavi zec, by Duško Radović

Chapter 5

Balanced avoidance games on random graphs

5.1 Introduction

The games we study in this chapter are played by a single player, whom we call Painter. He maintains a balanced 2-coloring in the random graph process, coloring two edges at a time in an online fashion. His goal is to avoid creating a monochromatic copy of a fixed graph F for as long as possible.

We now give a more precise description of the game's setup and rules, and of Painter's goals. Let $e_1, e_2, \ldots, e_M$ be the edges of K_n where $M = \binom{n}{2}$, and let $\pi \in S_M$ be a permutation of the set $[M]$, chosen uniformly at random. By G_i, $i = 1, \ldots, M$, we denote the graph on n vertices with the edge-set $E(G_i) = \{e_{\pi(1)}, e_{\pi(2)}, \ldots, e_{\pi(i)}\}$.

In the ith move of the game, Painter is presented with edges $e_{\pi(2i-1)}$ and $e_{\pi(2i)}$. He then immediately and irrevocably chooses one of the two possibilities to color one of them red and the other one blue. Therefore, after playing the first i moves, Painter has created a balanced 2-coloring of the graph G_{2i}. Note that at the move i he has no knowledge of the order in which the remaining edges will be presented to him in the moves to come.

Let F be a fixed graph. Painter loses the game as soon as he creates a monochromatic copy of F, i.e., Painter loses in the move $\min\{i : G_{2i}$ contains a monochromatic copy of $F\}$. His goal is to play as long as possible without losing. It is well-known that for n large enough every 2-coloring of edges of K_n contains a monochromatic copy of F. Therefore, Painter cannot survive to the end of the game. Assuming that his strategy is fixed, for every graph process there is an integer i such that Painter loses in his ith move playing on that particular graph process. Since the graph process on which the game is played is chosen uniformly at random, for fixed n and i we can reason about the probability that Painter loses before his ith move. Note that, generally speaking, Painter can lose the game in two ways. If one of the two edges to be colored closes both red and blue copy of F, then he obviously cannot color it. We call this a bichromatic threat. Also, if both edges to be colored close a monochromatic copy of F of the same color, the game is over. We refer to this as monochromatic threat. It is easy to see that these are the only two possibilities.

Our results. In this chapter, we attempt to determine the maximal number of moves which Painter can a.s. play without losing. More precisely, we would like to find a threshold function $m_F = m_F(n)$ for which

- there exists a strategy for Painter, such that for $m = o(m_F)$ we have $\Pr[\text{Painter loses in the first } m \text{ moves}] \to 0$,
- regardless of the strategy of Painter, for $m = \omega(m_F)$ we have $\Pr[\text{Painter loses in the first } m \text{ moves}] \to 1$,

when $n \to \infty$. Our interests lie in determining this threshold for a number of graph-theoretic structures. Observe that the existence of this threshold is not guaranteed—there may exist a game for which there is no such threshold.

We give a generic theorem which gives upper and lower bounds, and in some cases the exact threshold. As a consequence of this result, the exact threshold for cycles is determined, $m_{C_l} = n^{\frac{2l}{2l-1}}$. For the game in which Painter's goal is to avoid k-stars S_k we can prove that $m_{S_k} = n^{\frac{2k-2}{2k-1}}$. In the game of avoiding k-paths P_k, we give the exact value of m_{P_k} only for $k \leq 3$. For greater k we just exhibit some bounds.

Motivation and known results. In [22] Friedgut et al. introduce the concept of an online game played on the random graph process. In this game the player is 2-coloring edges, one at a time in an online fashion, maintaining the balancedness of the coloring. His goal is to avoid a monochromatic copy of a triangle for as long as possible.

Extending this result, Marciniszyn, Spöhel, and Steger [33] analyze the game of avoiding monochromatic cliques K_ℓ of any fixed size ℓ, and they exhibit a threshold for the number of moves at which the player loses a.s. It turns out that the optimal strategy is to play greedily—using the first color whenever possible, and the second one only to prevent losing immediately. The colorings obtained by following this strategy are typically unbalanced. A natural question arising is: If the player is forced to keep his coloring balanced, how long can he survive without losing? We try to give an answer to this question by looking at the analogous game in which the coloring of the graph is balanced. As it turns out, several thresholds we obtain in the balanced game are not the same as for the unbalanced game, showing that the balancedness condition makes a difference. For instance, applying a general criterion from [33] to cycles C_ℓ of length ℓ yields the threshold $n^{(\ell+1)/\ell}$ in the unbalanced case, whereas we derive the threshold $n^{2\ell/(2\ell-1)}$ from our results for balanced online colorings. Hence, the balanced online cycle avoidance game will end substantially earlier than the unbalanced game.

Another motivation comes from Beck's Chooser-Picker games on graphs [3, 4]. During the game, the balanced coloring of the subset of edges of K_n is maintained. In the "misère" version of the game, Chooser wins if at the end of the game (when all edges are colored) there is no red copy of a fixed graph F. Otherwise, Picker wins.

When Picker is playing randomly, this game is quite similar to the balanced avoidance games that we introduce here. A balanced two coloring of the edges of the random graph process is maintained by Chooser, and he colors them two at a time. The only difference that in our game a monochromatic copy of F (in any of the colors, not just red) is avoided.

In both [22] and [33], the generalizations of games in which the player uses more than two colors are mentioned, and analyzed in some cases. We can use the same approach for generalizing the games we introduce here.

I.e., we can assume that the edges in a random graph process are introduced to Painter s at a time, where $s \geq 2$ is a fixed integer. He then immediately colors them with s colors such that each edge is colored with a different color. Painter's goal remains the same, to avoid creating a monochromatic copy of F for as long as possible.

5.2 Games

5.2.1 Cycle game

In this section we give two generic theorems which together give matching upper and lower bounds for the threshold, for a class of graphs that satisfy certain conditions. As a consequence of this result, we will prove the exact threshold for the cycle game.

Theorem 48 *For any fixed integer $\ell \geq 3$, the threshold for balanced online avoidance game for cycles exists, and is*

$$m_{C_\ell} = n^{\frac{2\ell}{2\ell-1}}.$$

Let the random variable $X(G_i, H)$ count the number of subgraphs isomorphic to H in G_i, where G_i is the graph consisting of the first i edges in the random graph process. Similarly, let $X(G(n,p), H)$ count the number of subgraphs isomorphic to H in the random graph $G(n,p)$. Let $\mathcal{Q}(H,x)$ denote the property that a given graph contains at least x subgraphs isomorphic to H. Clearly, $\mathcal{Q}(H,x)$ is a monotone increasing property. In order to bound the probability of $G_i \in \mathcal{Q}(H,x)$ for H being F with one edge removed, we need the following lemma.

Lemma 49 *For $p = 4m'/n^2$ and $0 \leq i \leq m'$, we have*

$$\Pr[G_{2i} \in \mathcal{Q}(H,x)] \leq \Pr[G(n,p) \in \mathcal{Q}(H,x)] + e^{-\Theta(m')}.$$

Proof. Observe that each graph G_{2i}, $0 \leq i \leq m'$, appearing in the random process is distributed like $G(n, 2i)$, the uniform random graph with exactly $2i$ edges. Hence, we get

$$\Pr[G_{2i} \in \mathcal{Q}(H,x)] \quad \leq \quad \Pr[G(n, 2m') \in \mathcal{Q}(H,x)]\,.$$

Now we bound $\Pr[G(n,2m') \in \mathcal{Q}(H,x)]$ again using the monotonicity of $\mathcal{Q}(H,x)$,

$$\begin{aligned}\Pr[G(n,p) \in \mathcal{Q}(H,x)] &= \sum_{m=0}^{\binom{n}{2}} \Pr[G_{n,m} \in \mathcal{Q}(H,x)] \cdot \Pr[|E(G(n,p))| = m] \\ &\geq \Pr[G(n,2m') \in \mathcal{Q}(H,x)] \cdot \Pr[|E(G(n,p))| \geq 2m'] .\end{aligned}$$

If we set $p = 4m'/n^2$, Chernoff bounds imply that $\Pr[|E(G(n,p))| \geq 2m'] \geq 1 - e^{\Theta(m')}$. Hence,

$$\begin{aligned}\Pr[G_{2i} \in \mathcal{Q}(H,x)] &\leq \frac{\Pr[G(n,p) \in \mathcal{Q}(H,x)]}{1 - e^{-\Theta(m')}} \\ &\leq \Pr[G(n,p) \in \mathcal{Q}(H,x)] + e^{-\Theta(m')},\end{aligned}$$

and the lemma is proved. □

Now we can give a lower bound on the number of moves Painter can play without losing, provided that F satisfies certain conditions. For a graph F we define

$$\underline{m}(F) = \max\left\{\frac{2e_H - 1}{2v_H - 2} : H \subseteq F \text{ and } v_H \geq 2\right\}.$$

We say that F is strictly balanced with respect to $\underline{m}$ if

$$\underline{m}(F) = \frac{2e_F - 1}{2v_F - 2} \text{ and } \forall H \subsetneq F : \frac{2e_H - 1}{2v_H - 2} < \frac{2e_F - 1}{2v_F - 2}.$$

Theorem 50 *Let F be a graph with $v_F \geq 2$, and let H be a subgraph of F such that $\underline{m}(F) = \underline{m}(H)$ and H is strictly balanced with respect to $\underline{m}$. If every proper subgraph of H with $e_H - 1$ edges is balanced, then Painter can a.s. play $m'(n) = o(n^{2-1/\underline{m}(F)})$ moves in the balanced online game without creating a monochromatic copy of F.*

Proof. We have to argue that there exists a strategy for Painter that enables him to a.s. succeed in avoiding monochromatic copies of F in every step of the random process up to $G_{2m'}$. W.l.o.g. F is strictly balanced with respect to $\underline{m}(F)$. We may assume this since otherwise we restrict our attention to H. Clearly, if one can avoid H, then one can avoid F as well.

Painter's strategy is the following. If one of the two possibilities to play a move will lead to a monochromatic copy of F, then he chooses the other one. Otherwise, he plays arbitrarily.

Let $\mathcal{F}^-$ denote the family of pairwise non-isomorphic subgraphs of F with $e_F - 1$ edges. For $F^- \in \mathcal{F}^-$, we have that $v(F^-) = v(F)$ and $e(F^-) = e(F) - 1$. Since all edges in the random graph process appear independently uniformly at random, the probability of losing the game in one particular step is determined by the number of edges u, v that close a monochromatic copy of F^- to F. As we already mentioned, there are two different configurations that force Painter to create a monochromatic copy of F. In the first case a new edge may appear as a vertex pair uv that is covered by both a red and a blue copy of F^-. But this implies the existence of a graph $F^{(2)}$ in $G(n, 2m')$ consisting of two subgraphs isomorphic to F, which share one edge. It is well known [28] that this fixed subgraph $F^{(2)}$ will a.s. not appear if $m' = o(n^{2-v(F^{(2)})/e(F^{(2)})})$. Since $v(F^{(2)}) = 2v(F) - 2$ and $e(F^{(2)}) = 2e(F) - 1$ and since F is balanced with respect to $\underline{m}(F)$, this event will occur with probability $o(1)$.

The other case to consider is when two edges v_1v_2 and v_3v_4 to be colored are both covered by a monochromatic copy of F^- of the *same* color. We refer to $\{v_1v_2, v_3v_4\}$ as a threat. An upper bound on the number of threats in the graph can be computed by counting the number of subgraphs isomorphic to F^- and taking its square. Note that not every such threat is actually dangerous to Painter since we disregard the coloring of the surrounding structure. In this way we overestimate the risk of losing the game.

For all $F^- \in \mathcal{F}^-$, we have

$$\lambda(F^-) = \mathbf{E}\left[X(G(n,p), F^-)\right] = \Theta(n^{v_F} p^{e_F - 1}).$$

Let $m' = n^{2-(2v_F-2)/(2e_F-1)}/\nu(n)$, where $\nu(n)$ tends to infinity. W.l.o.g. we can assume that $\nu(n) = o(\log(n))$. Hence, we have

$$\lambda(F^-) \geq n^{v_F} \left(n^{-\frac{2v_F-2}{2e_F-1}} / \log(n)\right)^{e_F-1} = n^{\frac{v_F+2e_F-2}{2e_F-1}} \log(n)^{1-e_F} = \Omega(n^{\varepsilon})$$

for a suitable $\varepsilon = \varepsilon(F) > 0$. By Theorem 9 we have for each $F^- \in \mathcal{F}^-$,

$$\Pr\left[G(n,p) \in \mathcal{Q}(F^-, 2\lambda(F^-))\right] \leq \exp\left(-\Omega\left(n^{\frac{\varepsilon}{v_H - 1}}\right)\right).$$

Let Z_i be a random variable indicating that the two new edges close a monochromatic threat in step i and let Z denote the sum over all steps. From the previous calculations and Lemma 49, we conclude that

$$\begin{aligned}
\Pr[Z > 0] &\leq \sum_{i=1}^{m'} \Pr[Z_i > 0] \\
&\leq \sum_{i=1}^{m'} \left\{ \Pr\left[Z_i > 0 \,\middle|\, \bigwedge_{F^- \in \mathcal{F}^-} G_{2i-2} \notin \mathcal{Q}(F^-, 2\lambda(F^-)) \right] \right. \\
&\quad \left. + \left(\sum_{F^- \in \mathcal{F}^-} \Pr\left[G(n,p) \in \mathcal{Q}(F^-, 2\lambda(F^-))\right] \right) + e^{-\Theta(m')} \right\} \\
&\leq m' \left[\frac{\left(\sum_{F^- \in \mathcal{F}^-} 2\lambda(F^-)\right)^2}{\frac{1}{4}\left(\binom{n}{2} - 2m'\right)^2} + e^{-\Omega\left(n^{\varepsilon/(v_H - 1)}\right)} + e^{-\Theta(m')} \right] \\
&\leq m' \left[O\left(n^{-2(2e_F - v_F)/(2e_F - 1)}\right) + e^{-\Omega\left(n^{\varepsilon/(v_H-1)}\right)} + e^{-\Theta(m')} \right] \\
&= o(1),
\end{aligned}$$

since

$$m' = \omega(n^{2-(2v_F-2)/(2e_F-1)}/\log(n)),$$

and $m' = o(n^{2-(2v_F-2)/(2e_F-1)})$. □

The following theorem is a counting version of the main result of Rödl and Ruciński.

Theorem 51 *[39, Theorem 3] Let $r \geq 1$, F be a non-empty graph with $v_F \geq 3$ and set*

$$m_2(F) = \max\left\{ \frac{e_H - 1}{v_H - 2} : H \subseteq F \text{ and } v_H \geq 3 \right\}.$$

Then there exist constants $C = C(F,r)$ and $a = a(F,r)$ such that for

$$p(n) \geq Cn^{-1/m_2(F)}$$

a.s. in every r-edge-coloring of the random graph $G(n,p)$, one color contains at least $an^{v_F}p^{e_F}$ copies of F.

Using this result, we can give an upper bound for the number of moves Painter can play.

Theorem 52 *Let F be a graph with $v_F \geq 3$ such that there exists a subgraph F^- of F with $e_F - 1$ edges satisfying $3/2 > \varepsilon(F) \geq m_2(F^-)$, where*

$$\varepsilon(F) = \frac{2e_F - 1}{2v_F - 2}.$$

Moreover, suppose that every proper subgraph $H \subset F$ with $v_H \geq 2$ satisfies

$$v_F - \frac{e_F}{\varepsilon(F)} < v_H - \frac{e_H}{\varepsilon(F)}. \tag{5.1}$$

Then, after playing $m'(n) = \omega(n^{2-1/\varepsilon(F)})$ moves of balanced online game Painter will a.s. create a monochromatic copy of F, regardless of his strategy.

Proof. Let F^- be fixed. We switch between the models $G(n,p)$ and $G(n,m)$, exploiting their asymptotic equivalence via $p = \Theta(m/n^2)$.

Claim 53 *There exists $C > 0$ such that a.s. in every 2-edge-coloring of the random graph $G(n, m_1')$, $m_1' = Cn^{2-(2v_F-2)/(2e_F-1)}$, there are $\Omega((n^4/m_1')^{1/2})$ pairs $uv \in \binom{[n]}{2} \setminus E(G(n, m_1'))$ that complete a monochromatic copy of F^- in the same color, say red, to F.*

Proof. Set $C = C(F^-, 2)$ according to Theorem 51, and let Y denote the number of copies of F^- in $G(n, m_1')$. The expected number of such copies is

$$\begin{aligned}
\mathbf{E}[Y] &= \Theta\left(n^{v(F^-)}\left(\frac{m_1'}{n^2}\right)^{e(F^-)}\right) \\
&= \Theta\left(n^{v(F^-)-\frac{e(F^-)(2v(F)-2)}{2e(F)-1}}\right) \\
&= \Theta\left(n^{\frac{2e(F)+v(F)+2}{2e(F)-1}}\right) = \Theta\left(n^{\frac{1}{2\varepsilon(F)}+1}\right) \\
&= \Theta\left(\left(\frac{m_1'}{n^4}\right)^{-\frac{1}{2}}\right).
\end{aligned}$$

Here we used that $v(F^-) = v(F)$ and $e(F^-) = e_F - 1$. We call an edge critical, if it completes an entirely red copy of F^- to F. Theorem 51 yields asymptotically the same number of copies of (w.l.o.g.) red monochromatic F^-, since

$$m_1' = Cn^{2-(2v_F-2)/(2e_F-1)} \geq Cn^{2-1/m_2(F^-)}$$

due to the assumption of Theorem 52. Every such copy induces one critical edge in $G(n, m_1')$, but we may over-count if there are many pairs of monochromatic copies of F^- that cover the same vertex pair.

If one critical edge $e = uv$ is induced by multiple copies of F^-, then $G(n, m_1')$ contains a subgraph $(F^-)_H$ of the following structure: $(F^-)_H$ is the union of two graphs isomorphic to F^- such that their intersection complemented with e is a copy of a proper subgraph $H \subset F$. For any graph $(F^-)_H$, we have

$$e\left((F^-)_H\right) = e(F^-) + e(F^-) - (e_H - 1) = e(F^-) + e_F - e_H,$$

and

$$v\left((F^-)_H\right) = v(F^-) + v_F - v_H.$$

We denote the number of subgraphs isomorphic to $(F^-)_H$ in $G(n, m_1')$ by Y_H. It follows that

$$\begin{aligned}
\mathbf{E}[Y_H] &= \Theta\left(n^{v(F^-)+v_F-v_H}\left(\frac{m_1'}{n^2}\right)^{e(F^-)+e_F-e_H}\right) \\
&= \Theta\left(\mathbf{E}[Y]\,\frac{n^{v_F-e_F/\varepsilon(F)}}{n^{v_H-e_H/\varepsilon(F)}}\right) \\
&\stackrel{(5.1)}{=} o(\mathbf{E}[Y]).
\end{aligned}$$

The Markov inequality now yields $\Pr[Y_H \geq c\mathbf{E}[Y]] = o(1)$ for any $c > 0$. Since the number of critical edges induced by a fixed occurrence of a graph $(F^-)_H$ is bounded by a constant only depending on F, the multiply counted copies of F are a.s. of lower order of magnitude than $\mathbf{E}[Y]$ and thus negligible. This concludes the proof of Claim 53. $\square$

We fix m_1' as in Claim 53. Continuing the proof of Theorem 52, we apply the claim to show that the game a.s. stops for any duration $m' = \omega(m_1')$. Let thus $R\dot{\cup}B$ be a coloring assigned by Painter to the first m_1' edges. Since $m' = \omega(m_1')$, we have $m_2'' = m' - m_1' = \omega(m_1')$ in the second round. By Claim 53, there are $M_1 = \Omega\left((n^4/m_1')^{1/2}\right)$ critical edges in $\binom{[n]}{2} \setminus E(G(n, m_1'))$. If two of these pairs are simultaneously presented to Painter, he loses the game. In every step i, the probability of this event is determined by the number of critical edges M_i. Observe that $m_1' = o(M_1)$ by the condition $\varepsilon(F) < 3/2$ and we may choose the length of the second phase such that $m_2'' = o(M_1)$ as well. Hence, even if one of the critical edges of $G(n, m_1')$ is closed in every step of m_2'' properly, we are left with at least $M_1/2$ critical edges in the very last step. Let X_i be the random variable indicating that the game was lost in step i of the second phase. Then we have

$$\Pr[X_i = 0 | X_1 = 0 \wedge \ldots \wedge X_{i-1} = 0] \leq 1 - \binom{M_i}{2} \Big/ \binom{\binom{n}{2}}{2} \leq 1 - \frac{M_1^2}{n^4}.$$

Let $X = \sum_{i=1}^{m_2''} X_i$. The game ends within the second round if $X \geq 1$. Hence,

$$\begin{aligned}
\Pr[X \geq 1] &= 1 - \Pr[X = 0] \\
&= 1 - \Pr[X_1 = 0] \prod_{i=2}^{m_2''} \Pr[X_i = 0 | X_1 = 0 \wedge \ldots \wedge X_{i-1} = 0] \\
&\geq 1 - \left(1 - \frac{M_1^2}{n^4}\right)^{m_2''} \geq 1 - \exp\left(-\frac{m_2''}{m_1'}\right) = 1 - o(1).
\end{aligned}$$

This concludes the proof of Theorem 52. □

Some graphs F satisfy conditions of both Theorem 50 and Theorem 52. In that case the lower bound obtained in Theorem 50 matches the upper bound from Theorem 52, giving the exact threshold probability for such graphs. In particular, we obtain the threshold for the cycle game.

Proof. (Theorem 48) Observe that for a cycle C_ℓ, the only member in the family $\mathcal{F}^-$ is a path P_ℓ with $\ell - 1$ edges. Clearly, this graph is balanced with respect to $m(P_\ell)$ and

$$m_2(P_\ell) = \frac{\ell - 1 - 1}{\ell - 2} = 1.$$

Moreover, every proper subgraph of $H \subset C_\ell$ with $k \leq \ell$ vertices has at most $k - 1$ edges. Hence, we have

$$\frac{n^{\ell - \frac{\ell(2\ell-2)}{2\ell-1}}}{n^{v_H - \frac{e_H(2\ell-2)}{2\ell-1}}} \leq \frac{n^{\ell - \frac{\ell(2\ell-2)}{2\ell-1}}}{n^{k - \frac{(k-1)(2\ell-2)}{2\ell-1}}} = n^{-\frac{l+k-2}{2\ell-1}} = o(1).$$

Since

$$\frac{3}{2} > \frac{2\ell - 1}{2\ell - 2} > 1$$

for all $\ell \geq 3$, and we can apply both Theorem 50 and Theorem 52. □

Remark. Note that Theorem 48 gives the exact threshold for the triangle avoidance game, but unfortunately we cannot apply Theorem 50 and Theorem 52 to cliques with more than three vertices.

5.2.2 Star game

In this section we consider the star game. The goal of Painter is to avoid a monochromatic k-star S_k, for some fixed integer k, for as long as possible. Equivalently, we can say that Painter wants to keep the maximum degree in both red and blue graph below k. The next theorem gives more than a threshold for this game, since at the same time it gives (the order of) the number of k' stars for all $k' < k$.

Theorem 54 *Let $\alpha > 0$ be a constant, and let $m = n^{1-\alpha}$. For $k_0 = \lfloor \frac{1}{2}\left(\frac{1}{\alpha}+1\right)\rfloor$ we have*

(i) After $m' = \omega(m)$ moves of the balanced online game, for every integer $k \leq k_0$ Painter has created $\omega(m^{2k-1}n^{-2k+2})$ monochromatic stars of size k a.s.

(ii) For $m'' = o(m)$, there is a strategy for Painter that enables him to create $o(m^{2k-1}n^{-2k+2})$ monochromatic stars of size k, for all $k \leq k_0$, in the first m'' moves of the balanced online game a.s.

When we assume that $m' = \omega(m)$ (part (i) of the statement) or that $m'' = o(m)$ (part (ii)), we actually assume that m' and m'' are concrete functions satisfying these conditions, fixed before the game starts.

Written down in a strict mathematical notation, the statement of the first part of the theorem reads as follows. Let $\alpha > 0$, let $\nu : \mathbb{N} \to \mathbb{N}$ be a function with $\frac{\nu(n)}{n} \to \infty$ as n tends to infinity, and let $k \leq k_0$. Then there exists a function $\mu : \mathbb{N} \to \mathbb{N}$ with $\frac{\mu(n)}{n} \to \infty$ as n tends to infinity, such that for every strategy of Painter we have

$$\lim_{n\to\infty} \Pr\Big[(\#\text{ monochr. } k\text{-stars after } \nu(n^{1-\alpha}) \text{ moves}) \geq \mu\left(\left(n^{1-\alpha}\right)^{2k-1} n^{-2k+2}\right)\Big] = 1.$$

All the conclusions that we draw throughout the proof about the number of certain structures appearing during the game are also concrete and depend on the actual value of m' and m'', not just their asymptotics. But in order to make the proof more readable we just describe the most of the values using o and ω notation.

Proof. *(i)* It is enough to prove the statement if $m' = o(n^{1-\beta})$, for some $0 < \beta < \alpha$. Then the order of connected components of the random graph $G(n, m')$ is a.s. bounded from above by a constant $c' = c'(\beta) > 0$.

After m' moves of the balanced online game there are $\omega(m^{2k-1}n^{-2k+2})$ copies of $(2k-1)$-stars in $G_{m'}$ a.s. No matter how Painter colored the present edges, each of those stars has to contain a monochromatic k-star. Since the order of connected components is bounded by c' a.s., each monochromatic k-star can be contained in at most $\binom{c'-k-1}{k-1}$ (so, at most constantly many) $(2k-1)$-stars a.s. Therefore, there is $\omega(m^{2k-1}n^{-2k+2})$ monochromatic k-stars after m moves a.s.

(ii) Painter's strategy is the following. Whenever he should color edges v_1v_2 and v_3v_4, he spots the largest monochromatic star that is centered at one of vertices $v_1, \ldots, v_4$ at that moment. There may be more than one star with that property in which case he chooses one of them arbitrarily. He colors the edge adjacent to the center of the largest monochromatic star using the color complementary to the color of the star, in order to prevent the monochromatic star from increasing in size. The other edge is colored accordingly.

Let $s_k(m,n)$ being a function defined by $s_0(m,n) = m''$ and $s_k(m,n) = \frac{m''}{n^2}\left(s_{k-1}(m,n)\right)^2$. Note that $s_k(m,n) = o(m^{2k-1}n^{-2k+2})$. Using induction on k we will prove that the probability that after m'' moves there are "too many" different monochromatic k-star centers, for all $k < k_0$.

More precisely, the inductive statement for k reads as follows. There exist constants $\gamma_k > 0$ and $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$ we have that

$$\Pr[(\#\text{ of } k\text{-star centers after } m'' \text{ moves}) > s_k(m,n)] \leq e^{-n^{\gamma_k}}.$$

Note that at this point we do not care about the number of monochromatic k-stars centered at any of the vertices we count.

The statement holds for $k = 1$, since we have $m'' = o(m)$ edges and every colored edge is a monochromatic 1-star, so there is not more than $2m''$ monochromatic 1-star centers. Hence, the probability we are interested in is identically zero, and we can choose an arbitrary value for γ_1.

Assume that the statement is true for $k-1$, $k < k_0$. Suppose that in a move the number of monochromatic k-star centers is increased. We distinguish two cases.

Case 1. If one of the edges that was to be colored was not adjacent to any monochromatic $(k-1)$-star, then the other edge was adjacent to both blue and red $(k-1)$-star, meaning that in this move at least one new subgraph of size $2(k-1)+1 = 2k-1$ is created. Therefore, after m'' moves, we can not have more monochromatic k-star centers created in this fashion then the total number of subgraphs of size $2k-1$. Let B be a random variable counting the number of subgraphs of size $2k-1$ of $G_{2m''}$. The value of B is an upper bound for the number of moves $i \leq m''$ in which Painter created a new monochromatic k-star center in this fashion.

Next, we show that $B > \frac{1}{8}s_k(m,n)$ holds only with exponentially small probability. For $p = 4m''/n^2$, from Lemma 49 we get that

$$\Pr[B > s_k(m,n)] \leq \Pr[G_{n,p} \in \mathcal{Q}(S_{2k-1}, s_k(m,n))] + e^{-\Theta(m'')}.$$

The expected number of copies of S_{2k-1} appearing in $G_{n,p}$ is $p^{2k-1}n^{2k}$. If $k < k_0$, then Theorem 9 implies that

$$\Pr[G_{n,p} \in \mathcal{Q}(S_{2k-1}, s_k(m,n))] \leq e^{-n^{\gamma'_k}},$$

for some $\gamma'_k > 0$. Note that for $k = k_0$ Theorem 9 gives just

$$\Pr[G_{n,p} \in \mathcal{Q}(S_{2k-1}, s_k(m,n))] = o(1),$$

but since that is the last step of the induction we do not need the exponential bound.

Case 2. The other possibility is that each of the edges to be colored is adjacent to a monochromatic $(k-1)$-star of the same color. By C_i we denote the indicator random variable which has value 1 if the number of monochromatic k-star centers is increased in ith move, $i \leq m''$, in this way.

Next, for every move $i \leq m''$ we define the following indicator random variables

$$D_i = [(\#\text{ monochromatic } (k-1)\text{-star centers in } G_{2i}) > s_{k-1}(m,n)].$$

Finally, we define an auxiliary sequence of indicator random variables C_i', $i \le m''$. Our goal is to define them in such a way that, on one hand, the value of C_i' is less than the value of C_i only for a "reasonably small" number of graph processes (actually, only when $D_{i-1} = 1$), and on the other hand, they are mutually independent and we can apply Chernoff bounds on their sum.

For every graph process we look at the set containing all possible pairs of edges of G_{2i-2} that, if they appear in ith step, increase the number of monochromatic k-star centers. Denote this set by $T(G_{2i-2})$. Note that $C_i = 1$ if and only if the pair of edges that is to be colored in ith move is in $T(G_{2i-2})$.

If $|T(G_{2i-2})| \le (n \cdot s_{k-1}(m,n))^2$, then we construct the set $T'(G_{2i-2})$ by starting from $T(G_{2i-2})$, and adding another $(n \cdot s_{k-1}(m,n))^2 - T(G_{2i-2})$ pairs of edges from $E(K_n) \backslash E(G_{2i-2})$, by some arbitrary (but deterministic) rule.

On the other hand, if $|T(G_{2i-2})| > (n \cdot s_{k-1}(m,n))^2$, we construct the set $T'(G_{2i-2})$ by starting from $T(G_{2i-2})$, and removing $T(G_{2i-2}) - (n \cdot s_{k-1}(m,n))^2$ pairs of edges from $E(K_n) \setminus E(G_{2i-2})$, by some arbitrary (but deterministic) rule.

Hence, we always have $|T'(G_{2i-2})| = (n \cdot s_{k-1}(m,n))^2$. We define C_i' to be 1 if and only if the pair of edges that is to be colored in ith move is in $T'(G_{2i-2})$. Crucially, $C_i' < C_i$ only when $D_{i-1} = 1$ and therefore we have

$$\sum_{i=1}^{m''} C_i \le \sum_{i=1}^{m''} (C_i' + D_{i-1}) .$$

Since we know the exact size of $T'(G_{2i-2})$, for every i we get

$$\Pr[C_i' = 1] = \frac{(n \cdot s_{k-1}(m,n))^2}{\binom{\binom{n}{2}-2i+2}{2}},$$

and this probability does not change if we fix the value of a variable C_j' for any other j. Therefore, the variables $\{C_i'\}_i$ are independent and we can apply Chernoff bounds to get

$$\sum_{i=1}^{m''} C_i' \le 8m'' \frac{(n \cdot s_{k-1}(m,n))^2}{n^4} \le \frac{1}{12} s_k(m,n),$$

with probability $1 - e^{-n^{\gamma_k''}}$, for some $\gamma_k'' > 0$.

From the induction hypothesis, there is a constant $\gamma_{k-1} > 0$ such that the probability that $D_{i-1} = 1$ is at most $e^{-n^{\gamma_{k-1}}}$. Then, $\sum_{i=1}^{m''} D_{i-1} \neq 0$ with probability at most $m'' e^{-n^{\gamma_{k-1}}}$.

Since in one move we create at most 4 new star centers, if we denote the total number of monochromatic k-stars after m'' moves by A, we have

$$\begin{aligned} A &\leq 4B + 4\sum_{i=1}^{m''} C_i \\ &\leq 4B + 4\sum_{i=1}^{m''} C_i' + 4\sum_{i=1}^{m''} D_{i-1} \\ &\leq \frac{1}{2}s_k(m,n) + \frac{1}{2}s_k(m,n) + 0, \end{aligned}$$

with probability at least $1 - \left(e^{-n^{\gamma_k'}} + e^{-n^{\gamma_k''}} + m'' e^{-n^{\gamma_{k-1}}}\right)$, and thus also at least $1 - e^{-n^{\gamma_k}}$ for some $\gamma_k > 0$. This completes the induction step, if $k < k_0$. For $k = k_0$, the same holds with probability $1 - o(1)$.

We proved that after m'' moves the number of monochromatic k-star centers is $o(m^{2k-1} n^{-2k+2})$ for all $k \leq k_0$, a.s. On the other hand, since $m = \Theta(n^{1-\alpha})$, the order of every connected component of $G_{2m''}$ is bounded by a constant a.s. If that holds, then every vertex is center for at most constantly many monochromatic k-stars. Therefore, after playing m'' moves, Painter has created at most $o(m^{2k-1} n^{-2k+2})$ monochromatic k-stars a.s. □

As an immediate corollary we get the threshold for the star game.

Corollary 55 *For any fixed integer $k \geq 2$, the threshold for balanced online avoidance game for stars exists, and is $m_{S_k} = n^{\frac{2k-2}{2k-1}}$.*

Proof. We set $\alpha = \frac{1}{2k-1}$ and apply Theorem 54. For $k_0 = k$ we have that $n^{1-\alpha} = n^{\frac{2k-2}{2k-1}}$ is the exact threshold in the k-star game. □

Note that Theorem 50 cannot be applied to stars since a graph obtained from a star by removing an edge is not balanced.

5.2.3 Path game

In the path game, the goal of Painter is to avoid creating a monochromatic l-path, for a fixed integer l, for as long as possible. Here, we do not manage to give the exact threshold for arbitrary l, but just an upper bound. We show that this upper bound is sharp for $l = 2$ and $l = 3$. On the other hand, already in the case $l = 4$ we can exhibit a better upper bound.

We define $s_j := \sum_{i=2}^{j} 2^{\lfloor \log i \rfloor}$, for all $j \in \mathbb{N}$.

Lemma 56 *For all $m \in \mathbb{N}$, we have*

(i) $4s_m + 4 = s_{2m+1}$, $m \geq 1$,

(ii) $2s_m + 2s_{m-1} + 4 = s_{2m}$, $m \geq 2$,

(iii) $s_j = \Theta(j^2)$.

Proof. *(i)* We prove the statement by induction on m. For $m = 1$, $s_1 = 0$ and $s_3 = 4$ and the equality is satisfied. Now suppose the statement is true for all $i \leq m$ and we want to prove it for $m+1$. To simplify the notation, let x denote $\lfloor \log(m+1) \rfloor$. Then $\lfloor \log 2(m+1) \rfloor = 1 + \lfloor \log(m+1) \rfloor = 1 + x$. Since $2m+3$ is odd, it is not a power of two and hence $\lfloor \log(2m+3) \rfloor = \lfloor \log(2m+2) \rfloor = 1 + x$. Using the definition of s_j and the induction hypothesis, $s_{2(m+1)+1} = s_{2m+1} + 2^{1+x} + 2^{1+x} = 4s_m + 4 + 2^{1+x} + 2^{1+x}$ should be equal to $4s_{m+1} + 4$. Canceling $4s_m + 4$ on both sides gives $2^2 2^x = 2^{1+x} + 2^{1+x}$, and the equality is satisfied.

(ii) We make use of *(i)*: We know that $4s_m + 4 = s_{2m+1}$ and $4s_{m-1} + 4 = s_{2m-1}$. Combining these two equalities we get $2s_{m-1} + 2s_m + 4 = \frac{1}{2}(s_{2m+1} + s_{2m-1})$. Hence we only have to show that $s_{2m} = \frac{1}{2}(s_{2m-1} + s_{2m+1})$. Writing out the definitions and canceling common terms we obtain $2^{\lfloor \log(2m) \rfloor} = 2^{\lfloor \log(2m+1) \rfloor}$, which is clearly true since $2m + 1$ is not a power of two.

(iii) Since

$$\sum_{i=2}^{j} 2^{\log i - 1} \leq s_j \leq \sum_{i=2}^{j} 2^{\log i},$$

we have

$$\frac{1}{2}\left(\sum_{i=2}^{j} i\right) \leq s_j \leq \sum_{i=2}^{j} i = \Theta(j^2).$$

□

Theorem 57 *Using the notation from above, for all $k, l \in \mathbb{N}$, the number of red k-paths Painter has created after playing $m' = \omega\left(n^{\frac{s_l}{s_l+1}}\right)$ edges is $\omega\left(n^{(s_k+1)\frac{s_l}{s_l+1}-s_k}\right)$ a.s.*

Written down in a strict mathematical notation, the statement of the theorem reads as follows. Let $k, l \in \mathbb{N}$, and let $\nu : \mathbb{N} \to \mathbb{N}$ be a function with $\frac{\nu(n)}{n} \to \infty$ as n tends to infinity. Then there exists a function $\mu : \mathbb{N} \to \mathbb{N}$ with $\frac{\mu(n)}{n} \to \infty$ as n tends to infinity, such that for every strategy of Painter we have

$$\lim_{n\to\infty} \Pr\left[(\#\text{ red } k\text{-paths after } \nu\left(n^{\frac{s_l}{s_l+1}}\right) \text{ moves}) \geq \mu\left(n^{(s_k+1)\frac{s_l}{s_l+1}-s_k}\right)\right] = 1.$$

Proof. It is enough to prove the statement for $m' = o(n^{1-\alpha})$ for some $\alpha > 0$. Consider an arbitrary, but fixed $l \in \mathbb{N}$. The proof proceeds by induction on k.

There exists a constant $C > 0$ such that the random graph with m' edges contains a connected component of order larger than C with probability n^{-2}. Now we proceed similarly to the proof of Theorem 54 (ii). Namely, if we can prove that the statement of the theorem holds with probability $1 - \nu(n)$, $\nu(n) = o(1)$, using the assumption that there is no component of order more than C before every move, then the statement of the theorem holds with probability at least $1 - \nu(n) - m'n^{-2} = 1 - o(1)$. Note that the bounded component order before every move implies that addition of a new edge creates at most constantly many new connected subgraphs. We frequently rely on that fact throughout the proof.

We make use of the method of l-round exposure. This means that we split the game into l rounds, where in each of the rounds $\omega\left(n^{\frac{s_l}{s_l+1}}\right)$ edges are played. This is useful for the analysis, because we can lower-bound the number of copies of a (colored) subgraph during the $(i+1)$st round by the number of copies of this (colored) subgraph after the ith round. More precisely, we are going to lower-bound the number of i-paths after i rounds by the number of i-paths created during the ith round.

In the base case, for $k = 1$, we have $s_k = 0$. After playing $\omega\left(n^{\frac{s_l}{s_l+1}}\right)$ edges the number of red edges is $\omega\left(n^{\frac{s_l}{s_l+1}}\right)$ as well. Since every red edge is a red 1-path, the statement is true. In the base case for $k = 2$ we have $s_k = 2$, and two rounds with $\omega\left(n^{\frac{s_l}{s_l+1}}\right)$ edges each are played. An edge can be extended to a 2-path in $2(n-1)$ ways. Therefore, throughout the second round there is $\omega\left(n^{\frac{s_l}{s_l+1}}\right)\cdot\Theta(n)$ edges that are not played yet and are adjacent to a red edge. Here we use the assumption on bounded component order to infer that every edge can extend only constantly many red edges to a 2-path. If both edges played in a move are adjacent to a red edge then Painter surely creates a red 2-path. The probability for such an event to happen in one move of the second round is

$$\omega\left(\left(\frac{n^{\frac{s_l}{s_l+1}}n}{n^2}\right)^2\right).$$

The random choice of edge to come in each move in the random process is uniformly distributed on all free edges. Hence, using a similar coupling argument as in Theorem 54 (ii), we can apply Chernoff bounds and get that after $\omega\left(n^{\frac{s_l}{s_l+1}}\right)$ moves the number of red 2-paths will be

$$\omega\left(n^{\frac{s_l}{s_l+1}}n^{2\left(\frac{s_l}{s_l+1}-1\right)}\right) = \omega\left(n^{3\frac{s_l}{s_l+1}-2}\right)$$

a.s.

Now, let us assume that the induction hypothesis holds for all $1 \leq i \leq k-1$. To prove the statement for k, we make a case analysis depending on the parity of k.

Case k odd, $k = 2m+1$, $m \in \mathbb{N}$, $m \geq 1$. If each of the edges being played in one move of the k-th round connects endpoints of two red m-paths, then Painter is forced to create a red k-path. Since the number of red m-paths after the m-th round by the induction hypothesis is

$$\omega\left(n^{(s_m+1)\frac{s_l}{s_l+1}-s_m}\right)$$

a.s., it clearly cannot be less after the $(k-1)$-st round because the number of red m-paths does not decrease during the course of the game. Note that at most constantly many of the unplayed edges that connect two m-paths can overlap, since we assume that the component order before every move is bounded.

Hence the probability that each of the edges in a move of the k-th round connects endpoints of two red m-paths is

$$\omega\left(\left(\left(\frac{n^{(s_m+1)\frac{s_l}{s_l+1}-s_m}n}{n^2}\right)^2\right)^2\right)=\omega\left(n^{4((s_m+1)\frac{s_l}{s_l+1}-s_m-1)}\right),$$

since the choice of each edge in the random process is made uniformly at random. Crucially, this lower bound on the probability holds independently of the previous moves in the same stage, and thus, applying the same coupling argument as before we can use Chernoff bounds to get that the number of red $k=2m+1$-paths after playing (k times) $\omega\left(n^{\frac{s_l}{s_l+1}}\right)$ edges is

$$\omega\left(n^{\frac{s_l}{s_l+1}(4s_m+5)-4s_m-4}\right)=\omega\left(n^{(s_{2m+1}+1)\frac{s_l}{s_l+1}-s_{2m+1}}\right)$$

a.s., where the last equality follows from *(i)* of Lemma 56. This proves the induction step in the case k odd.

Case k even, $k=2m$, $m\in\mathbb{N}$, $m\geq 2$. If each of the edges being played in a move of the k-th round connects endpoints of a red m-path and a red $(m-1)$-path, then Painter is forced to create a red k-path. By the same argument as above, probability for such an event to happen in a move of the k-th round is

$$\begin{aligned}\omega&\left(\left(\frac{n^{(s_m+1)\frac{s_l}{s_l+1}-s_m}n}{n^2}\frac{n^{(s_{m-1}+1)\frac{s_l}{s_l+1}-s_{m-1}}n}{n^2}\right)^2\right)\\&=\omega\left(n^{2\left(\left(\frac{s_l}{s_l+1}(s_m+s_{m-1}+2)\right)-s_m-s_{m-1}-2\right)}\right)\end{aligned}$$

a.s. Hence the number of red $k=(2m)$-paths after playing (k rounds of) $\omega\left(n^{\frac{s_l}{s_l+1}}\right)$ edges is

$$\omega\left(n^{(2s_m+2s_{m-1}+5)\frac{s_l}{s_l+1}-2s_m-2s_{m-1}-4}\right)=\omega\left(n^{(s_{2m}+1)\frac{s_l}{s_l+1}-s_{2m}}\right)$$

a.s., where the last equality follows from *(ii)* of Lemma 56. This proves the induction step in the case k even. □

As a corollary we obtain an upper bound for the number of edges to be played until Painter loses the l-path game, $l \in \mathbb{N}$, a.s.

Corollary 58 *After* $\omega\left(n^{\frac{s_l}{s_l+1}}\right)$ *moves of the balanced online game, Painter has created a monochromatic l-path a.s.*

Proof. Using the last theorem with $l = k$, we directly obtain the statement of the corollary. □

Since a 2-path is also a 2-star, Theorem 54 implies that the last theorem gives the exact value in the case $l = 2$. As we will see from the next theorem, the bound is also tight for $k = 3$. But already for $k = 4$, the threshold is not of the same order, as Theorem 61 will show.

By $P_{a,b}$ we denote a *colored* path of length $a + b$ whose first a edges are colored red, and the remaining b edges are colored blue. The vertex of $P_{a,b}$ which is adjacent to both red and blue edge is called the *middle vertex* of $P_{a,b}$.

Theorem 59 *For $m'' = o(n^{4/5})$, there is a strategy for Painter that enables him to avoid monochromatic 3-paths in the first m'' moves of the balanced online game a.s.*

Proof. Suppose that in ith move, $i \leq m''$, Painter should color edges v_1v_2 and v_3v_4. If the number of the edges of G_{2i-2} that are in the same connected component as one of vertices $\{v_1, \dots, v_4\}$ is more than 3, then there is a subgraph H of G_{2i-2} with 8 vertices and 4 edges with $\{v_1, \dots, v_4\} \subset V(H)$. On the other hand, if this number of edges is 3 or less, then Painter can avoid loosing the game in ith move. So, his strategy in each move is just to avoid creating a monochromatic 3-path. If both possibilities to color the edges in a move are not dangerous, he plays arbitrarily (according to some deterministic strategy).

We define the following indicator random variables

$$
\begin{aligned}
A_i &= [\text{Painter creates a monochromatic 3-path in move } i], \\
D_i &= [\#\text{ subgraphs of } G_{2i} \text{ with 4 edges and 8 vertices is } \leq n^{16/5}],
\end{aligned}
$$

and we have
$$\sum_{i=1}^{m''} A_i \leq \sum_{i=1}^{m''} A_i D_{i-1} + \sum_{i=1}^{m''} (1 - D_{i-1})$$

Since
$$\mathbf{E}[\#\text{ subgraphs of } G_{2i} \text{ with 4 edges and 8 vertices}] = o(n^{16/5})$$

for all $i \leq m''$, Theorem 9 implies that the probability for $D_i = 1$ is exponentially small, and therefore we have $\sum_{i=1}^{m''}(1 - D_{i-1}) > 0$ with probability $o(1)$.

On the other hand, we get
$$\mathbf{E}\left[\sum_{i=1}^{m''} A_i D_{i-1}\right] = \Theta\left(\frac{n^{16/5}}{n^4}\right) \cdot m'' = o(1),$$

and by the first moment method
$$\sum_{i=1}^{m''} A_i D_{i-1} = 0$$

holds a.s.

Putting everything together, we get that $\sum_{i=1}^{m''} A_i = 0$ a.s., which means that Painter will not lose the balanced 3-path avoidance game in the first m'' moves a.s. □

From the last Theorem and Corollary 58 for $l = 3$, we get

Corollary 60 $m_{P_3} = n^{4/5}$.

Next, we prove an upper bound for the 4-path game. Note that unlike in games with 2-paths and 3-paths, this bound shows that Theorem 57 does not give tight upper bounds in general.

Theorem 61 *After $m' = \omega(n^{7/8})$ moves of the balanced online game, Painter has created a monochromatic 4-path a.s.*

Proof. There exists a constant $C > 0$ such that the random graph with m' edges contains a connected component of order larger than C with probability n^{-2}. Similarly to the proof of Theorem 57, if we can prove that the statement of the theorem holds with probability $1 - \nu(n)$, $\nu(n) = o(1)$, using the assumption that there is no component of order more than C before every move, then the statement of the theorem holds with probability at least $1 - \nu(n) - m'n^{-2} = 1 - o(1)$.

We split the game into five stages. In each of the first four stages $n^{7/8}$ moves are played, and in the fifth stage Painter plays all the remaining $m' - 4n^{7/8}$ moves. By $\overline{G}_i$, $i = 1, \ldots, 4$ we denote the (colored) graph after the ith stage of the game.

$\overline{G}_1$ contains $n^{7/8}$ red and $n^{7/8}$ blue edges. If both edges to be played in one move have a red adjacent edge, then in that move Painter has to create one new $P_{2,0}$. We claim that the density of unplayed edges of this kind during the second stage is $\Omega\left(\frac{n^{7/8} \cdot n}{n^2}\right)$. This holds since the size of all connected components before every move is bounded, and each new edge can appear in at most constantly many 2-paths. Also, the total number of edges played is asymptotically less then the number of edges that extend a red edge to a 2-path. Therefore, almost every such edge must be unplayed. We can use a coupling argument as in the proof of Theorem 54, to apply Chernoff bounds. We get that in the second stage Painter creates

$$\Omega\left(n^{7/8}\left(\frac{n^{7/8} \cdot n}{n^2}\right)^2\right) = \Omega(n^{5/8})$$

of $P_{2,0}$ a.s. We can prove analogously that Painter creates the same number of $P_{0,2}$ a.s.

We estimated the number of edges that close a red 2-path by counting the number of possible extensions of each red edge to a 2-path, showing that this way each edge is counted at most constantly many times, and almost all of such edges are unplayed. This argument can be applied analogously in each of the following stages. We omit it from now on and just give the estimate of expectation.

When Painter colors an edge whose one endpoint is adjacent to a red edge and the other to a blue edge, he creates either a $P_{1,2}$ or a $P_{2,1}$. During the second stage, the density of unplayed edges of this kind is $\Omega\left(\frac{n^{7/8}\cdot n^{7/8}}{n^2}\right)$, implying that w.l.o.g. we can assume that $\overline{G}_2$ contains

$$\Omega\left(n^{7/8}\frac{n^{7/8}\cdot n^{7/8}}{n^2}\right) = \Omega(n^{5/8})$$

of $P_{1,2}$.

When Painter colors an edge adjacent to an endpoint of $P_{2,0}$, he creates either a $P_{2,1}$ or a $P_{3,0}$. During the third stage, the density of unplayed edges of this kind is $\Omega\left(\frac{n^{5/8}\cdot n}{n^2}\right)$. $\overline{G}_3$ must contain

$$\Omega\left(n^{7/8}\frac{n^{5/8}\cdot n}{n^2}\right) = \Omega(n^{4/8})$$

of either $P_{2,1}$ or $P_{3,0}$ a.s. We distinguish two cases.
1. $\overline{G}_3$ contains $\Omega(n^{4/8})$ of $P_{3,0}$. To avoid losing right away, Painter should color blue every edge connecting an endpoint of $P_{3,0}$ with a blue edge. Unplayed edges of this kind have density $\Omega\left(\frac{n^{4/8}\cdot n^{7/8}}{n^2}\right)$ during the fourth stage, which means that $\overline{G}_4$ contains

$$\Omega\left(n^{7/8}\frac{n^{4/8}\cdot n^{7/8}}{n^2}\right) = \Omega(n^{2/8})$$

of $P_{3,2}$ a.s.

In the last stage density of unplayed edges connecting the middle vertex of a $P_{3,2}$ with a blue edge is

$$\Omega\left(\frac{n^{2/8}\cdot n^{7/8}}{n^2}\right) = \Omega(n^{-7/8}).$$

As soon as Painter gets to color one of edges of this kind he loses, and that happens in the last stage a.s. since the number of moves played is $\omega(n^{7/8})$.
2. $\overline{G}_3$ contains $\Omega(n^{4/8})$ of $P_{2,1}$. Then, in the last stage density of unplayed edges connecting the middle vertex of a $P_{2,1}$ and the middle vertex of a $P_{1,2}$ is

$$\Omega\left(\frac{n^{4/8}\cdot n^{5/8}}{n^2}\right) = \Omega(n^{-7/8}).$$

As soon as Painter gets to color one of edges of this kind he loses, and that happens in the last stage a.s. since the number of moves played is $\omega(n^{7/8})$.
$\square$

Bibliography

[1] J. Beck: Random graphs and positional games on the complete graph, *Ann. Discrete Math.* 28(1985), 7–13.

[2] J. Beck: Remarks on positional games, *Acta Math. Acad. Sci. Hungar.* 40(1982), 65–71.

[3] J. Beck: Positional games and the second moment method, *Combinatorica* 22(2002), 169–216.

[4] J. Beck: On positional games, *Journal of Combinatorial Theory, ser. A* 30(1981), 117–133.

[5] J. Beck: Van der Waerden and Ramsey type games, *Combinatorica* 3(1981), 103–116.

[6] J. Beck: An algorithmic approach to the Lovász Local Lemma. I., *Random Structures and Algorithms* 2(1991), 343–365.

[7] J. Beck: Deterministic graph games and a probabilistic intuition, *Combinat. Probab. Comput.* 3(1993), 13–26.

[8] J. Beck: Foundations of positional games, *Random Structures and Algorithms* 9(1996), 15–47.

[9] J. Beck: Ramsey games, *Discrete Math.* 249(2002), 3–30.

[10] J. Beck: The Erdős-Selfridge theorem in positional game theory, *Bolyai Society Math. Studies, 11: Paul Erdős and His Mathematics*, Budapest, 2002, 33–77.

[11] J. Beck: Tic-Tac-Toe, *Contemporary combinatorics, Bolyai Soc. Math. Stud.* 10(2002), 93–137.

[12] J. Beck: Lectures on Positional Games, Algorithms and Complexity, manuscript, ETH Zurich, 2003.

[13] M. Bednarska, T. Łuczak: Biased positional games for which random strategies are nearly optimal, *Combinatorica* 20(2000), 477–488.

[14] M. Bednarska, T. Łuczak: Biased positional games and the phase transition, *Random Structures & Algorithms* 18(2001), 141–152.

[15] M. Bednarska, O. Pikhurko: Biased positional games on matroids, *European J. Combin.* 26(2005), 271–285.

[16] B. Bollobás, A. Thomason: Threshold functions, *Combinatorica* 7(1987), 35–38.

[17] B. Bollobás: *Random graphs*, Cambridge University Press, London, 1985.

[18] V. Chvátal, P. Erdős: Biased positional games, *Annals of Discrete Math.* 2(1978), 221–228.

[19] V. Chvátal, P. Erdős: A note on Hamiltonian circuits, *Discrete Math.* 2(1972), 111–113.

[20] E.R. Berlekamp, J.H. Conway, R.K. Guy: Winning ways, Academic Press, London, 1982.

[21] P. Erdős, J. Selfridge: On a combinatorial game, *J. Combinatorial Theory* 14(1973), 298–301.

[22] E. Friedgut, Y. Kohayakawa, V. Rödl, A. Ruciński, P. Tetali: Ramsey games against a one-armed bandit, *Combin. Probab. Comput.* 12(2003), 515–545.

[23] A. Frieze, M. Krivelevich: Hamilton cycles in random subgraphs of pseudo-random graphs, *Discrete Math.* 256(2002), 137–150.

[24] A. Frieze, M. Krivelevich, O. Pikhurko, T. Szabó: The game of JumbleG, *Combinat. Probab. Comput.* (2005), to appear.

[25] A.W. Hales, R.I. Jewett: On regularity and positional games, *Trans. Amer. Math. Soc.* 106(1963), 222–229.

[26] D. Hefetz, M. Krivelevich, T. Szabó: Avoider-Enforcer games, manuscript.

[27] D. Hefetz, M. Krivelevich, M. Stojaković, T. Szabó: Planarity game and k-coloring game, manuscript.

[28] S. Janson, T. Łuczak, A. Rucinski: *Random Graphs*, Wiley, New York, 2000.

[29] J.H. Kim: On Brooks' Theorem for sparse graphs, *Combin. Probab. Comput.* 4(1995), 97-132.

[30] M. Krivelevich, M. Stojaković, T. Szabó: On unbiased games on random graphs, manuscript.

[31] A. Lehman, A solution of the Shannon switching game, *J. Soc. Indust. Appl. Math.* 12(1964), 687–725.

[32] T. Łuczak: On the equivalence of two basic models of random graphs, In *Random Graphs '87*, Proceedings, Poznań, 1987, John Wiley & Sons, eds. Karoński, Jaworski, Ruciński, 151–158.

[33] M. Marciniszyn, R. Spöhel, A. Steger: The Online Clique Avoidance Game on Random Graphs, manuscript, (available at: http://www.ti.inf.ethz.ch/as/people/marciniszyn/).

[34] M. Marciniszyn, D. Mitsche, M. Stojaković: Balanced avoidance games on random graphs, manuscript.

[35] R. Motwani, P. Raghavan, *Randomized Algorithms*, Cambridge University Press, Cambridge, 1995.

[36] E.M. Palmer, J.J. Spencer: Hitting time for k edge-disjoint spanning trees in a random graph. *Period. Math. Hungar.* 31(1995), 235–240.

[37] A. Pekec: A winning strategy for the Ramsey graph game, *Combinat. Probab. Comput.* 5(1996), 267-276.

[38] L. Pósa: Hamilton circuits in random graphs, *Discrete Math.* 14(1976), 359-364.

[39] V. Rödl, A. Ruciński: Threshold functions for Ramsey properties, *J. Amer. Math. Soc.* 8(1995), 917–942.

[40] S. Shelah: Primitive recursive bounds for Van der Waerden numbers, *Journal of the American Math. Soc.* 1(1988), 683–697.

[41] M. Stojaković, T. Szabó: Positional games on random graphs, *Random Structures Algorithms* 26(2005), 204–223.

[42] V.H. Vu: A large deviation result on the number of small subgraphs of a random graph, *Combin. Probab. Comput.* 10(2001), 79–94.

Curriculum Vitae

Miloš Stojaković

born on August 26, 1976 in Novi Sad, Serbia & Montenegro

1991-1995	Highschool, "Gymnasium J.J. Zmaj", Novi Sad, Serbia & Montenegro,
1995-1999	Studies at University of Novi Sad, Serbia & Montenegro, Degree: Bachelor of Mathematics,
1996-1999	Studies at University of Novi Sad, Serbia & Montenegro, Degree: Bachelor of Computer Science,
1999-2001	Masters studies at University of Novi Sad, Serbia & Montenegro, Degree: Master of Computer Science,
2001-2002	Pre-doc program at ETH Zurich, Switzerland,
since 2002	PhD student at ETH Zurich, Switzerland.