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On k-Sets and Applications

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Abstract

In this thesis, we study the notion of k-sets from discrete geometry and its applications to other mathematical problems.

We prove that the number of k-sets of the set $\mathbf{N}_0^d$ of nonnegative lattice points is between $k^{d-1} \log k$ and $k^{d-1}(\log k)^{d-1}$ in order of magnitude.

Next, we consider the continuous counterparts of h-vectors of simplicial polytopes that are known as *h-functions*. We prove that an important fact about h-vectors of polytopes, the so-called *Generalized Lower Bound Theorem*, carries over to h-functions.

We also describe an application of h-vectors and h-functions: We give an alternative proof of the *First Selection Lemma*, which asserts that for every finite set S in d-dimensional space, there exists a point which is contained in a positive fraction of all full-dimensional simplices spanned by S. Specifically, we show that every *centerpoint* of S has this property. Our proof immediately extends to the corresponding statement for continuous probability distributions.

Finally, we consider, for a continuous probability measure μ in the plane, the probability $\square(\mu)$ that four random points independently and identically distributed according to μ form a convex quadrilaeral. This question, which is known as *Sylvester's Four-Point Problem* was completely solved by Blaschke for the case of uniform distributions on convex bodies. For general distributions, however, it is still unknown which distributions minimize $\square(\mu)$ or what the value of $\inf_\mu \square(\mu)$ is.

We improve the lower bound to $\inf_\mu \square(\mu) > 3/8 + 10^{-5} \approx 0.37501$. This comes quite close to the best upper bound known to date, which is $\inf_\mu \square(\mu) < 0.38074$. The Four-Point Problem can be equivalently reformulated in terms of finite point sets. In this discrete context, it is also known as the problem of determining the *rectilinear crossing number* of complete graphs. We observe that this discrete reformulation of the Four-Point Problem is closely related to the distribution of k-sets, and as a main tool, we show that for every finite point set in the plane, the number of $(\leq k)$-sets is at least $3\binom{k+1}{2}$.

Zusammenfassung

Der Schwerpunkt dieser Arbeit liegt auf dem Begriff der k-*Menge* aus der diskreten Geometrie sowie dessen Implikationen für andere mathematische Fragestellungen.

Zunächst beschäftigen wir uns mit der asymptotischen Größenordung der Anzahl $a_k(\mathbf{N}_0^d)$ von k-Mengen von $\mathbf{N}_0^d$. Wir beweisen, daß diese zwischen $k^{d-1} \log k$ und $k^{d-1}(\log k)^{d-1}$ liegt.

Sodann wenden wir uns den sogenannten h-Funktionen zu, die stetige Gegenstücke zu h-Vektoren simplizialer Polytope darstellen. Wir zeigen, daß ein wichtiger Satz über h-Vektoren von Polytopen, das sogenannte *Generalized Lower Bound Theorem*, sich auf h-Funktionen überträgt.

Ferner beschreiben wir eine Anwendung von h-Vektoren, bzw. h-Funktionen: Wir präsentieren einen neuen Beweis der als *First Selection Lemma* bekannten Tatsache, daß es zu einer gegebenen endlichen Punktemenge S im d-dimensionalen Euklidischen Raum immer einen Punkt gibt, der in einem positiven Prozentsatz aller von S aufgespannten volldimensionalen Simplices liegt. Genauer gesagt zeigen wir, daß jeder *Centerpunkt* von S diese Eigenschaft hat. Unsere Beweismethode erlaubt uns auch unmittelbar, den analogen Satz über stetige Wahrscheinlichkeitsverteilungen herzuleiten.

Schließlich betrachten wir für stetige Wahrscheinlichkeitsmaße μ in der Ebene die Wahrscheinlichkeit $\Box(\mu)$, daß vier unabhängige μ-verteilte Zufallspunkte ein konvexes Viereck bilden. Dieses als *Sylvesters Vierpunktproblem* bekannte Problem wurde zwar von Blaschke für den Fall einer Gleichverteilung auf einer beschränkten konvexen Menge vollständig gelöst, jedoch ist für allgemeinere Verteilungen die Frage noch unbeantwortet, welche Verteilungen $\Box(\mu)$ minimieren bzw. was $\inf_\mu \Box(\mu)$ ist.

Wir kommen der Lösung dieses Problems einen Schritt näher, indem wir die untere Schranke auf $\inf_\mu \Box(\mu) > 3/8 + 10^{-5} \approx 0.37501$ verbessern, was der besten bisher bekannten oberen Schranke von $\inf_\mu \Box(\mu) < 0.38074$ recht nahe kommt. Das Vierpunktproblem lässt sich äquivalent als Fragestellung über endliche Punktemengen in der Ebene reformulieren und ist in diesem diskreten Kontext auch als das Problem bekannt, die *rektilineare Kreuzungszahl* vollständiger Graphen zu bestimmen. Wir zeigen, daß diese diskrete Reformulierung des Vierpunktproblems in engem Zusammenhang zur Verteilung der k-Mengen steht, und beweisen als wichtigstes Hilfsmittel, daß für jede endliche Punktmenge in der Ebene die Anzahl der $(\leq k)$-Mengen mindestens $3\binom{k+1}{2}$ beträgt.

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Contents

Chapter 1

Introduction

In this thesis, we study the notions of k-*sets* and k-*facets* from discrete geometry and their applications to other mathematical problems.

Consider a set S of points in d-dimensional Euclidean space. A subset $T \subseteq S$ is called a k-*set* of S, for integer k, if $|T| = k$ and T can be strictly separated from its complement by a hyperplane.

The question known as the k-*set problem* concerns the number of k-sets of a finite ground set. The maximum number $a_k^d(n)$ of k-sets of any n-element set $S \subseteq \mathbf{R}^d$ has numerous applications in the analysis of geometric algorithms and the complexity of other geometric objects, such as k^{th}-order Voronoi diagrams. The question is to determine the asymptotic behaviour of the function $a_k^d(n)$ for fixed dimension d and $k, n \to \infty$. This appears to be extremely difficult and is regarded as one of the most challenging problems in discrete geometry. Despite considerable efforts by numerous researchers over the last thirty-odd years, and despite recent significant progress in dimensions two and three, the gap between the known upper and lower bounds remains quite large, even in the plane.

A closely related notion is that of a k-*facet*. Suppose that $S \subseteq \mathbf{R}^d$ is a set of points in general position (i.e., every subset of cardinality at most $d+1$ is affinely independent), and consider an oriented $(d-1)$-dimensional simplex σ spanned by points from S. Such a simplex is called a k-*facet* of S, for integer k, if there are precisely k points of S in the positive open halfspace determined by σ. It is known that the number of k-sets is maximized for point sets in general position, and that up to constant factors, $a_k^d(n)$ is also the maximum number of k-facets of any set of n points in general position in dimension d.

We will discuss these notions and the known bounds in somewhat more detail (yet still quite tersely) in Chapter 2.

Interestingly, k-sets and k-facets also appear in other contexts which, at first sight, seem rather unrelated. For instance, Onn and Sturmfels [60] considered the k-sets of the infinite set $\mathbf{N}_0^d$ and showed that these are in one-to-one correspondence with the Gröbner bases of a certain kind of ideal in the polynomial ring $K[x_1, \ldots, x_d]$, K any infinite field.

Another example is McMullen's [52] *Upper Bound Theorem (UBT)* for convex polytopes. This theorem gives exact upper bounds for the face numbers of a convex polytope, and as shown by Welzl [86], there is an equivalent reformulation of the UBT in terms of k-facets. This reformulation formed the basis for an analogue of the UBT for continuous probability distributions, which was developed in [84].

In this thesis, we will further investigate such connections and also study some new ones. In particular, we will be concerned with the following questions:

1. What is the asymptotic order of magnitude of the number $a_k(\mathbf{N}_0^d)$ of k-sets of $\mathbf{N}_0^d$ (for fixed d and $k \to \infty$). Onn and Sturmfels gave a first upper bound of $O(k^{2\frac{d-1}{d+1}})$. We will prove in Chapter 3 that the correct order of magnitude is between $k^{d-1} \log k$ and $(k \log k)^{d-1}$.

2. Certain linear combinations of the face numbers of a simplicial polytope $\mathcal{P}$ form the entries of the *h-vector* of $\mathcal{P}$, which is a fundamental invariant of the polytope. In [84], continuous counterparts of h-vectors, so-called *h-functions,* were introduced. The h-function associated with a continuous probability measure μ and a base point $\boldsymbol{o}$ in $\mathbf{R}^d$ is a certain continuous functions $h_{\mu,\boldsymbol{o}} : [0,1] \to \mathbf{R}_{\geq 0}$ which is determined by the following property: For each integer $k \geq 0$, the probability $f_k(\mu, \boldsymbol{o})$ that $\boldsymbol{o}$ is contained in the convex hull of $d+1+k$ independent μ-distributed random points can be expressed (up to constant factors depending on k and d only) as the k^{th} moment $\int_0^1 y^k h_{\mu,\boldsymbol{o}}(y)\mathrm{d}y$. It was shown that h-functions enjoy various properties that are in direct analogy to important theorems about h-vectors. Most notably, continuous analogues of the *Dehn-Sommerville Equations* and of the UBT were proved. The continuous version of the former asserts that h-functions are symmetric about $1/2$, i.e., $h(y) = h(1-y)$ for all y. The latter gives exact pointwise upper bounds for the values of the h-function. These, in turn imply exact upper bounds on the probabilities $f_k(\mu, \boldsymbol{o})$. In particular,

these probabilities are essentially maximized by distributions that are symmetric about $\boldsymbol{o}$.

After reviewing the definitions and basic facts, we will see in Chapter 4 that another prominent theorem about h-vectors also carries over: we will prove a continuous version of the *Generalized Lower Bound Theorem*, to the extent that h-functions are monotonically increasing on the interval $[0, 1/2]$ (and hence decreasing on $[1/2, 1]$, by the Dehn-Sommerville Equation).

Furthermore, we will use the technique of h-vectors and h-functions to give an alternative proof of the *First Selection Lemma* and to establish a continuous analogue of it. The latter guarantees that for any probability distribution in $\mathbf{R}^d$, we can find a point $\boldsymbol{o} \in \mathbf{R}^d$ (namely, a centerpoint of μ) such that the probability $f_k(\mu, \boldsymbol{o})$ is at least some constant $s(d, k) > 0$ which depends only on k and d. Thus, it can in some sense be considered a converse of the Continuous Upper Bound Theorem.

3. The simplest interesting instance for the probabilities $f_k(\mu, \boldsymbol{o})$ is the case $d = 2$ and $k = 0$: Given a continuous probability distribution μ and a point $\boldsymbol{o}$ in the plane, what is the probability that $\boldsymbol{o}$ is contained in the triangle spanned by three random points P_1, P_2, P_3 i.i.d. $\sim \mu$?

   Now suppose that instead of a point fixed in advance, we consider a fourth independent random point, i.e., the probability

$$\Pr[P_4 \in \operatorname{conv}\{P_1, P_2, P_3\}] = \frac{1}{4} \Pr[\operatorname{conv}\{P_1, P_2, P_3, P_4\} \text{ is a triangle}],$$

   or equivalently, the complementary probability

$$\Box(\mu) := \Pr[\operatorname{conv}\{P_1, P_2, P_3, P_4\} \text{ is a convex quadrilateral}].$$

   This is the well-known *Four-Point Problem* of J.J. Sylvester [77]. While this problem was completely solved by Blaschke [18] for uniform distributions on convex bodies, for the general case it is still unknown which distributions minimize $\Box(\mu)$, or what the true value $\Box_* := \inf_\mu \Box(\mu)$ is. It is known [66] that the problem can be equivalently stated as a question about discrete point sets: if we denote by $\Box(n)$ the minimum number of convex 4-element subsets of any set of n points in general position in the plane, then

$$\Box_* = \lim_{n \to \infty} \frac{\Box(n)}{\binom{n}{4}}.$$

In this context, the problem is also known as that of determining the *rectilinear crossing number* of complete graphs.

In Chapter 5, we will work towards closing the gap of our knowledge about $\Box_*$. We will first prove the lower bound $\Box_* > 0.3288$ by a method inspired by the h-function approach. After that, we improve this to $\Box_* \geq 3/8 + \varepsilon$, with $\varepsilon \approx 10^{-5}$ by a more direct connection to k-sets: We express the number of convex quadrilaterals in a point set $S \subseteq \mathbf{R}^2$ as a positive linear combination of the numbers $a_k(S)$ of k-sets of S. The immediate strategy of substituting lower bounds for the a_k's fails, since for each k, there are point sets with very few k-sets. However, these examples are very attuned to the specific k at hand, and we can save our approach by doing "integration by parts", i.e., by passing to the numbers $A_k := \sum_{i=1}^{k} a_k$. For an n-point set, the number $\Box$ of convex quadrilaterals can also be expressed as a positive linear combination of the numbers A_k, $1 \leq k < n/2$. We then combine the lower bound $A_k \geq 3\binom{k+1}{2}$, which we prove in Chapter 6 and which is tight for $k < n/3$, with a result of Welzl that implies better estimates for k close to $n/2$, and obtain the bound for $\Box_*$ as advertised.

Chapter 2

Basics

The purpose of this chapter is to review the central notions of our investigations, k-sets and j-facets. Along the way, we will introduce the terminology and notation used throughout this thesis. Furthermore, we compile a number of well-known facts which we will need in what follows. For a more thorough and extensive survey of the landscape of k-sets, including many of the proofs, see Chapter 11 of Matoušek's textbook [51].

2.1 k-Sets

Let $S \subseteq \mathbf{R}^d$. A subset $T \subseteq S$ is called a *k-set* of S, for integer k, if $|T| = k$ and there is a hyperplane H that strictly separates T and $S \setminus T$, i.e., T lies in one of the open halfspaces bounded by H and $S \setminus T$ in the other.

The number of k-sets of S will be denoted by $a_k(S)$, or just by a_k if S is understood from the context. We will only be concerned with sets for which these numbers are finite, and always implicitly assume so.

If S is a finite set and $n := |S|$ then $T \subseteq S$ is a k-set iff $S \setminus T$ is an $(n-k)$-set. Thus, the numbers $a_k(S)$ are symmetric about $n/2$, i.e.,

$$a_k(S) = a_{n-k}(S).$$

Usually, the notion of k-sets is defined only for finite ground sets, but it makes sense and is of interest also in other contexts. For instance, Chapter 3 will be concerned with the k-sets of the infinite set $\mathbf{N}_0^d$, which also go under the suggestive name of *corner cuts* and turn out to have applications in computational commutative algebra [60].

Example 2.1. If S is a set of n points in convex position in the plane, then $a_k(S) = n$ for $1 \leq k \leq n-1$ (any consecutive k points along the boundary of the convex hull form a k-set, and vice versa, see Figure 2.1).

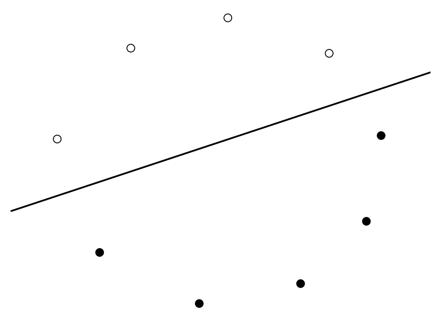


Figure 2.1: *A set of* 9 *points in convex position and a typical* 4*-set.*

Example 2.2. Figure 2.2 shows a "tripod shaped" set of nine points in the plane and its 4-sets: Each 4-set consists either of the two outermost points from each of two "spokes" (3 possibilities), or of all three points from one spoke and one point from another ($3 \cdot 5 = 15$ possibilities). Thus, $a_4(S) = 18$. This will be a useful example to keep in mind. Suitable generalizations of it will appear later.

It is not difficult to derive tight bounds for the total number of partitions of a finite point set S, i.e., for the sum $\sum_k a_k(S)$, using the following form of duality.

Point-Hyperplane Duality and Arrangements. For a point $\boldsymbol{a} = (a_1, \ldots, a_d) \in \mathbf{R}^d$, the *dual hyperplane* $\boldsymbol{a}^*$ is defined by

$$\boldsymbol{a}^* := \{\boldsymbol{x} \in \mathbf{R}^d : x_d = a_1 x_1 + \ldots + a_{d-1} x_{d-1} - a_d\}.$$

Conversely, if a hyperplane $\mathrm{H} \subseteq \mathbf{R}^d$ is not *vertical*, i.e., is not parallel to the x_d-axis, then it can be uniquely written as $\mathrm{H} = \{\boldsymbol{x} \in \mathbf{R}^d : x_d = a_1 x_1 + \ldots + a_{d-1} x_{d-1} - a_d\}$, and we set

$$\mathrm{H}^* := (a_1, \ldots, a_d).$$

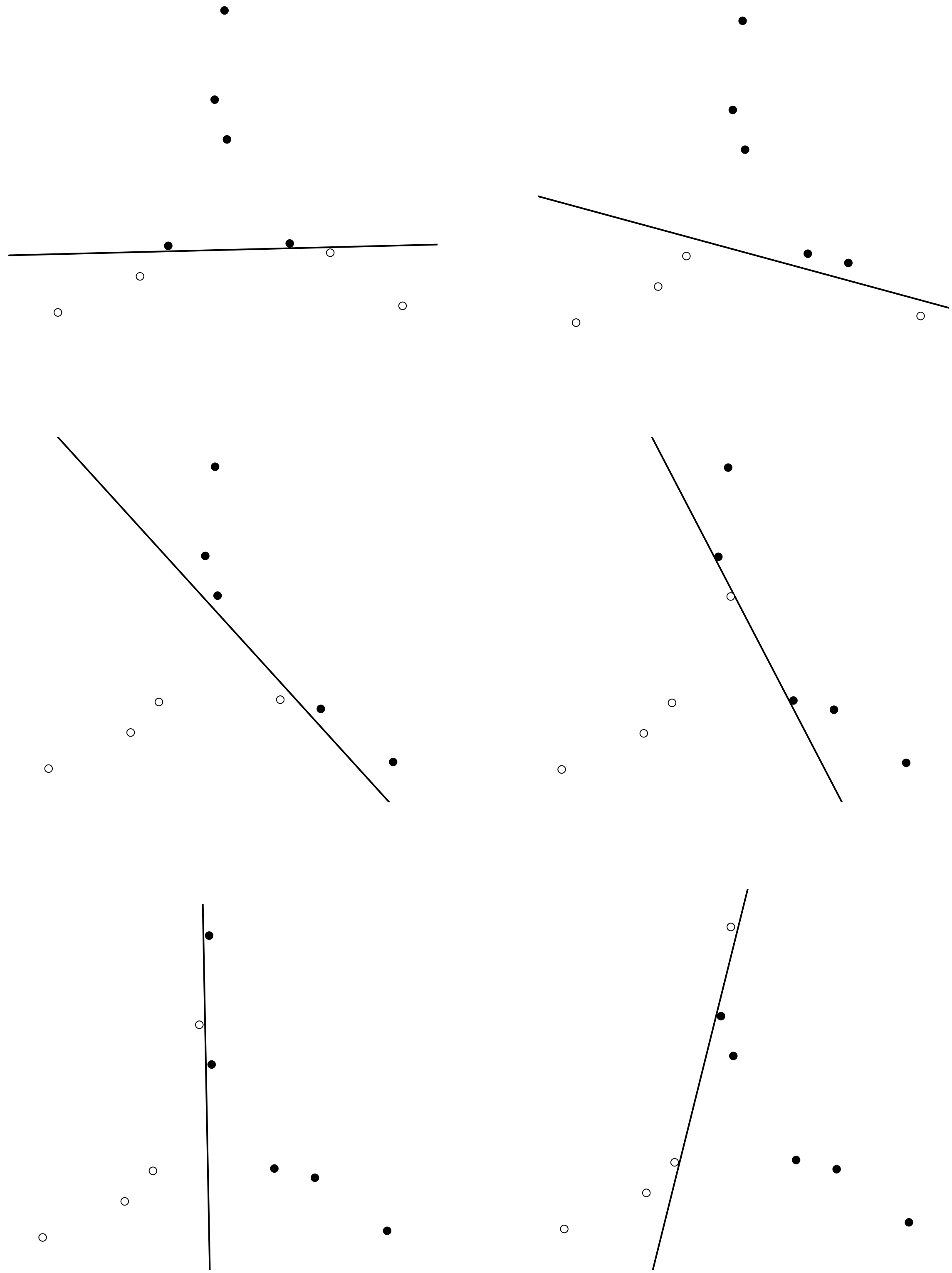


Figure 2.2: *Another set of* 9 *points and its* 4*-sets (up to symmetry).*

It is easy to see that for all $\boldsymbol{a}$ and all nonvertical $\textsc{h}$, we have

1. $(\boldsymbol{a}^*)^* = \boldsymbol{a}$ and $(\textsc{h}^*)^* = \textsc{h}$.
2. $\boldsymbol{a} \in \textsc{h}$ iff $\textsc{h}^* \in \boldsymbol{a}^*$.
3. $\boldsymbol{a}$ lies above $\textsc{h}$ iff the point $\textsc{h}^*$ lies above the hyperplane $\boldsymbol{a}^*$.

Now, consider a finite set $S \subseteq \mathbf{R}^d$. Dualizing S, we get a set of hyperplanes $S^* = \{\boldsymbol{p}^* : \boldsymbol{p} \in S\}$.

If $\textsc{h}$ is a hyperplane disjoint from S then we may assume that it is not vertical; otherwise, we can perturb $\textsc{h}$ without changing the partition it induces on S. Then $\textsc{h}^*$ is a point which is disjoint from all hyperplanes in S^*, i.e., $\textsc{h}^*$ lies in the complement $\mathbf{R}^d \setminus \bigcup_{\boldsymbol{p} \in S} \boldsymbol{p}^*$.

The connected components of this complement are called the *full-dimensional faces* of the *arrangement* S^* of hyperplanes. More generally, a finite set H of hyperplanes in $\mathbf{R}^d$ defines a partition of $\mathbf{R}^d$ into relatively open convex subsets, called *faces*, of various dimensions $i = 0, 1, \ldots, d$ (see Chapter 6 of [51]). This partition is called the *arrangement* induced by H and denoted by $\mathcal{A}(H)$ or sometimes simply by H.

The full-dimensional faces of the arrangement S^* are relevant in our context for the following reason: two nonvertical hyperplanes $\textsc{h}_1$ and $\textsc{h}_2$ disjoint from S induce the same partition on S if and only if their duals $\textsc{h}_1^*$ and $\textsc{h}^*$ either lie in the same full-dimensional face of S^* or lie in antipodal unbounded full-dimensional faces. Here, two full-dimensional faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of an arrangement of hyperplanes are called *antipodal* if for every hyperplane $\textsc{h}$ defining the arrangement, $\mathcal{F}_1$ lies above $\textsc{h}$ iff $\mathcal{F}_2$ lies below $\textsc{h}$ and vice versa.

It is not difficult to see by induction on the dimension d that the number of full-dimensional faces in an arrangement of n hyperplanes in $\mathbf{R}^d$ is at most $\sum_{i=0}^{d} \binom{n}{i} = O(n^d)$, and that the number of pairs of antipodal unbounded full-dimensional faces is at most $\sum_{i=0}^{d-1} \binom{n-1}{i}$. Moreover, both maxima are attained iff the arrangement is *simple* in the following sense.

Definition 2.3 (Various Non-Degeneracy Notions).

1. An arrangement of hyperplanes in $\mathbf{R}^d$ is called *simple* if for $1 \leq i \leq d+1$, any i of the hyperplanes intersect in an affine flat of dimension $d - i$ (in particular, there is no point common to any $d+1$ of them).
2. Further, a point set $S \subseteq \mathbf{R}^d$ is said to be in *general position* if every subset of S of cardinality at most $d+1$ is affinely independent (the “at

most" is only a precaution to avoid a vacuous condition in the case that $|S| \leq d$).

3. For future reference, we also define the analogous notion for probability distributions in $\mathbf{R}^d$ (which, for us, will always mean probability measures on the σ-algebra of Lebesgue measurable subsets of $\mathbf{R}^d$): We say that a probability distribution μ on $\mathbf{R}^d$ is *continuous* if every hyperplane has μ-measure zero.

Note that S^* is simple iff S is in general position. Moreover, μ is continuous if and only if any $d+1$ mutually independent μ-random points are almost surely (i.e., with probability 1) in general position, and this then holds for any countable set of mutually independent μ-distributed random points.

But back to k-sets. How can we interpret these in terms of arrangement of hyperplanes? For an arrangement of nonvertical hyperplanes and a point $\boldsymbol{p} \in \mathbf{R}^d$, let us define the *level* of $\boldsymbol{p}$ to be the number of hyperplanes strictly below $\boldsymbol{p}$. Suppose now that T is a k-set of S and that H is a separating hyperplane for T, which we can always take to be nonvertical. If T lies above H, then H^* is a point of level k, and if T lies below H, then H^* is a point of level $n-k$. Thus, the k-sets of S correspond to the full-dimensional faces of level k or $n-k$ in S^* (with antipodal faces defining the same k-sets).

Apart from offering a different viewpoint on k-sets, this also leads to the study of levels in arrangements of other geometric objects than hyperplanes, for instance of algebraic surfaces. We refer to the survey [2], to the book [69], or to Chapters 6 and 7 of [51] as starting points for the study of more general arrangements, and restrict our attention to k-sets in what follows.

Bounds for the Number of k-Sets. As we have seen, it is easy to give exact bounds for the number $\sum_k a_k(S)$ of all partitions of a finite set by hyperplanes. If we consider the numbers a_k individually, however, then finite point sets in convex position in the plane are among the very few classes of point sets for which these numbers are easy to analyze. In general, this appears to be very difficult, and understanding the asymptotic behavior of the maximum number

$$a_k^d(n) := \max_{\substack{S \subset \mathbf{R}^d \\ |S|=n}} a_k(S) \tag{2.1}$$

of k-sets of any n-point set in d-space is considered one of the most challenging problems in discrete geometry. To be more precise, the question known as the *k-set problem* is to find good upper and lower bounds (if possible tight up to constant factors) for $a_k^d(n)$ if d is fixed and $k, n \to \infty$.

One motivation for studying k-sets is that they have found various applications in the analysis of geometric algorithms, see [25, 27, 36], We will encounter further applications of a different, mostly non-algorithmic nature, in the following chapters, most notably in Chapter 5.

However, the main interest may simply lie in the intellectual challenge itself: to understand this particular aspect of the combinatorial structure of finite point sets. Moreover, despite considerable efforts by numerous researchers over the last thirty-odd years, the gap between the known upper and lower bounds is still quite large, even in the plane.

The k-set problem was first posed (in the slightly different guise of *halving edges*, which we will define in the following section) by Simmons (unpublished). Straus (also unpublished) found a lower bound of

$$a_{n/2}^2(n) = \Omega(n \log n), \quad n \text{ even}, \tag{2.2}$$

and Lovász [50] proved an upper bound of

$$a_{n/2}^2(n) = O(n^{3/2}), \quad n \text{ even}.$$

An extension of this to general k,

$$a_k^2(n) = O(n\sqrt{k}) \tag{2.3}$$

appeared together with Straus' lower bound in Erdös, Lovász, Simmons, and Straus [37].

We refer to the notes at the end of Section 11.1 in [51] for a summary and bibliography of the subsequent progress on the problem, and just state the currently best bounds, which are as follows:

In the plane,

$$a_k^2(n) = O(nk^{1/3}), \tag{2.4}$$

as was shown by Dey [32]. In three dimensions,

$$a_k^3(n) = O(nk^{3/2}), \tag{2.5}$$

which was proved by Sharir, Smorodinsky, and Tardos [70]. In general dimension d, Alon, Bárány, Füredi, and Kleitman [4] (following and extending a method developed by Bárány, Füredi, and Lovász [12] for $d = 3$) obtained

$$a_k^d(n) = O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - c_d}), \tag{2.6}$$

where $c_d > 0$ is a small number which depends only on d and tends to zero very fast as d grows. The crucial ingredient of the proof is the so-called *colored Tverberg Theorem*, which was proved by Živaljević and Vrećica [82].

On the other hand, Tóth [78] proved the lower bound

$$a_k^d(n) = nk^{d-2}e^{\Omega(\sqrt{\log k})}. \qquad (2.7)$$

Note that $e^{\sqrt{\log k}}$ is asymptotically larger than any fixed power of $\log k$, but smaller than k^ε for any constant $\varepsilon > 0$.

All of these bounds were shown for the case of even n and $k = n/2$, to which the general case reduces, as we will see below. Moreover, all proofs proceed in terms of objects that are slightly different from k-sets, but closely related and technically more convenient to handle.

2.2 j-Facets

One of the first observations when studying the k-set problem is that not only the sum $\sum_k a_k$ but also each single a_k is maximized by point sets that are in general position.

Observation 2.4. *For every finite set S in $\mathbf{R}^d$, there is another set $S' \subset \mathbf{R}^d$ of the same cardinality and in general position, such that $a_k(S) \leq a_k(S')$ for all k. In fact, any set S' arising from suitable small perturbations of the points in S will do.*

To see why this is, consider all possible partitions of S by hyperplanes disjoint from S. For each of these partitions, choose a hyperplane witnessing it. In this way, we obtain a finite collection H of hyperplanes, and each point of S is contained in some full-dimensional face of the resulting arrangement. These faces are open sets, and by moving the points within them, we can ensure general position without affecting any of the partitions. Thus, the number of k-sets can only grow (and if S was not in general position, then it will, for some k).

Now suppose that S is a finite set in general position in the plane. Let T is a k-set of S, $1 \leq k \leq n-1$, and let ℓ be a separating line for T.

It is not hard to see that there is a unique pair of points $\boldsymbol{p} \in T$ and $\boldsymbol{q} \in S \setminus T$ with the following property (see Figure 2.3): $T \setminus \boldsymbol{p}$ lies to completely in the open halfplane $\mathrm{H}^+(\boldsymbol{p}, \boldsymbol{q})$ to the left of the oriented line from $\boldsymbol{p}$ through $\boldsymbol{q}$, and $(S \setminus T) \setminus \boldsymbol{q}$ is contained in the open halfplane $\mathrm{H}^-(\boldsymbol{p}, \boldsymbol{q})$ to the right of that line. Thus, the oriented edge $[\boldsymbol{p}, \boldsymbol{q}]$ contains exactly $k-1$ points of S on its left side.

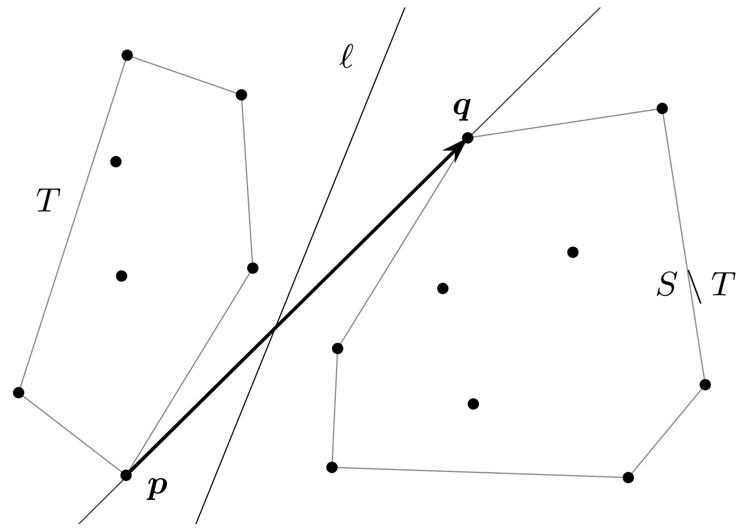


Figure 2.3: *k-Sets and $(k-1)$-Edges.*

Conversely, let $[\boldsymbol{p}, \boldsymbol{q}]$ be a $(k-1)$-*edge* of S, i.e., an oriented edge spanned by points $\boldsymbol{p}, \boldsymbol{q} \in S$ that contains exactly $(k-1)$ points from S on its left side. If L denotes the set of these $k-1$ points, then $L \cup \{\boldsymbol{p}\}$ is a k-set of S: a small counterclockwise rotation of the line through $\boldsymbol{p}$ and $\boldsymbol{q}$ about the midpoint of the edge $[\boldsymbol{p}, \boldsymbol{q}]$ produces a line ℓ that strictly separates $L \cup \{\boldsymbol{p}\}$ from $S \setminus (L \cup \{\boldsymbol{p}\})$.

The two operations just described yield a bijection between the k-sets of S and the $(k-1)$-edges of a point set S in general position in the plane.

More generally, let us make the following

Definition 2.5 (j-Facets). Let S be a set of n points in general position in $\mathbf{R}^d$, and let σ be an oriented $(d-1)$-dimensional simplex spanned by points of S, where the orientation just means that one of the open halfspaces bounded by the affine hull of σ is appointed the *positive side* of σ, denoted by $\mathrm{H}^+(\sigma)$.

If σ contains precisely j points from S on its positive side then σ is called a *j-facet* of S.

We denote the number of j-facets of S by $e_j(S)$ or simply by e_j. As in the case of the numbers a_k, the e_j's are symmetric, this time about $\frac{n-d}{2}$ (which we see by reversing the orientation, i.e., exchanging the roles of the positive and the negative side of a simplex):

$$e_j(S) = e_{n-d-j}(S).$$

We will also use the somewhat sloppy notation $e_{1/2}(S)$ for the number of *halving facets* of S. These are the $\frac{n-d}{2}$-facets of S (and correspondingly only exist if $n-d$ is even).

We note that in the dual setting of hyperplane arrangements, the j-facets of S correspond to the *vertices* of level (as defined above) j or $n-d-j$ in the arrangement S^*.

The above correspondence between k-sets and $(k-1)$-edges immediately tells us that in the plane, $a_k = e_{k-1}$. In higher dimensions, the relationship between the a_k's and the e_j's is more subtle: In three dimension, these numbers still determine each other via the linear relations $a_k = \frac{1}{2}(e_{k-2}+e_{k-1})+2$, for $1 \leq k \leq n-1$ and $n \geq 4$ (see [6, 8]) but starting from dimension four, this is in general no longer true (see [8]).

It remains true, however, that these quantities are equivalent as far as their order of magnitude is concerned: It is not hard to see (confer [51], for instance) that for a set of n points in in general position in dimension d,

$$a_k \leq \sum_{j=k-d-1}^{k} e_j + O(n^{d-1})$$

and

$$e_j \leq \sum_{k=j}^{j+d-1} a_k.$$

Thus, for asymptotic bounds like the ones in (2.2)–(2.7), it does not matter whether we speak about k-sets or j-facets.

2.3 Polytopes

In this section, we review very quickly a bit of standard terminology of convex polytopes and polyhedra. All of this (and the proofs of the various assertions we just make) can be found in much more detail in Ziegler's textbook [91].

A subset of $\mathbf{R}^d$ is called a *convex polyhedron* if it can be written as the intersection of finitely many closed halfspaces. A subset $\mathcal{F}$ of a convex polyhedron $\mathcal{P}$ is called a *face* of $\mathcal{P}$ if either $\mathcal{F} = \mathcal{P}$, or $\mathcal{F} = \emptyset$, or there is a hyperplane H such that $\mathcal{F} = \mathcal{P} \cap \mathrm{H}$ and $\mathcal{P}$ is completely contained in one of the closed halfspaces bounded by H. The faces $\mathcal{F} \neq \emptyset, \mathcal{P}$ are called *proper faces* of $\mathcal{P}$. Clearly, all faces are convex polyhedra themselves.

The *dimension* $\dim \mathcal{F}$ of a face is defined as the dimension of the affine hull of $\mathcal{F}$. By convention, $\dim \emptyset = -1$. The faces of dimensions $0, 1, \dim \mathcal{P} - 2,$

and $\dim \mathcal{P} - 1$ have special names: They are called *vertices*, *edges*, *ridges*, and *facets* of $\mathcal{P}$, respectively. We denote the number of i-dimensional faces of a polyhedron $\mathcal{P}$ by $f_i(\mathcal{P})$, $i = -1, 0, 1, \ldots, \dim \mathcal{P}$.

We will mostly be concerned with convex polyhedra that are bounded. These are called *convex polytopes*. It is a fundamental fact of life that convex polytopes can be equivalently characterized as convex hulls of finite point sets: a bounded set $\mathcal{P} \subset \mathbf{R}^d$ can be written as the intersection of finitely many closed halfspaces if and only if $\mathcal{P}$ is the convex hull of some finite set $V \subset \mathbf{R}^d$. In fact, there is a unique inclusion-minimal such set, namely the set of vertices of $\mathcal{P}$.

If we order the faces of a polytope by inclusion, then the resulting poset is called the *face lattice* of $\mathcal{P}$, often denoted by $L(\mathcal{P})$. Two polytopes $\mathcal{P}, \mathcal{Q}$ are called *combinatorially equivalent* if their face lattices are isomorphic, i.e., if if there is an inclusion-preserving bijection $L(\mathcal{P}) \to L(\mathcal{Q})$.

If, on the other hand, there is an inclusion-*reversing* bijection $L(\mathcal{P}) \to L(\mathcal{Q})$, then $\mathcal{P}$ and $\mathcal{Q}$ are called *polars* (or *duals*) of each other, and we write $\mathcal{Q} = \mathcal{P}^*$ (and $\mathcal{P} = \mathcal{Q}^*$). Every polytope has a polar, which can be constructed geometrically using the point-hyperplane duality mentioned earlier: Suppose that $\mathcal{P} \subseteq \mathbf{R}^d$ is d-dimensional and contains the origin $\mathbf{0}$ in its interior (we can assume this by passing to the affine hull of $\mathcal{P}$, if necessary, and by an appropriate translation). Let V be the set of vertices of $\mathcal{P}$, and consider the dual hyperplanes $\mathtt{H}_{\boldsymbol{v}} := \boldsymbol{v}^*$, $\boldsymbol{v} \in V$. If we orient all these hyperplanes consistently so that the origin is on their negative side, then the intersection $\mathcal{Q}$ of the closed negative halfspaces $\overline{\mathtt{H}_{\boldsymbol{v}}^-}$, $\boldsymbol{v} \in V$ can be shown to be a convex polytope that is polar to $\mathcal{P}$.

Two special classes of polytopes deserve mentioning: A polytope $\mathcal{P}$ is called *simplicial* if all its proper faces are simplices of the appropriate dimension. A polytope is called *simple* if its polar is simplicial.

We conclude this section with a few words about a particular kind of polytope that is closely related to the leitmotif of this thesis.

The k-Set Polytope. For $S \subset \mathbf{R}^d$, define

$$\mathcal{P}_k(S) := \operatorname{conv}\left\{\sum X : X \subseteq S, |X| = k\right\},$$

where $\sum X$ is a shorthand for $\sum_{\boldsymbol{x} \in X} \boldsymbol{x}$. If S is finite, then $\mathcal{P}_k(S)$ is a convex polytope, which is known as the *k-set polytope* because of Fact 2.6 below, and it is for this finite case that $\mathcal{P}_k(S)$ was first defined by Edelsbrunner, Valtr, and Welzl [35]. But $\mathcal{P}_k(S)$ is also of interest in the infinite case. For instance, in Chapter 3, we will encounter $\mathcal{P}_k(\mathbf{N}_0^d)$, which turns out to be a convex polyhedron, called the *corner cut polyhedron*. It was studied by Onn and Sturmfels [60] in relation with computational commutative algebra.

It is not difficult to prove the following characterization of the vertices and facets of $\mathcal{P}_k(S)$:

Fact 2.6. *1. A point $v \in \mathbf{R}^d$ is a vertex of $\mathcal{P}_k(S)$ iff $v = \sum T$ for some k-set T of S.*

2. A facet $\mathcal{F}$ of $\mathcal{P}_k(S)$ corresponds to a hyperplane H*, spanned by points from S, such that $|\mathrm{H}^- \cap S| = j < k$ and $|\overline{\mathrm{H}^-} \cap S| > k$. More precisely,*

$$\mathcal{F} = \sum(\mathrm{H}^- \cap S) + \mathcal{P}_{k-j}(\mathrm{H} \cap S),$$

every facet of $\mathcal{P}_k(S)$ is of this form, and conversely, each H *as above gives rise to a facet.*

Thus, if S is a finite set in general position, then each facet of $\mathcal{P}_k(S)$ corresponds to a j-facet of S with $k - d < j < k$. The faces of intermediate dimension can be characterized in terms of so-called (i, j)*-partitions*; see [8] for the definition of this notion and a detailed analysis.

In dimension $d \leq 3$, $\mathcal{P}_k(S)$ is simplicial, and together with Euler's formula, this implies that the numbers of k-sets respectively j-facets can be expressed as linear combinations of each other. As mentioned at the end of the previous section, this breaks down in higher dimensions.

The k-set polytope was first used [35] to derive the improved (compared to (2.6)) upper bound

$$e_{1/2}(S) = O(n^{d-2/d}) \tag{2.8}$$

if S is a so-called *dense* set of n points in $\mathbf{R}^d$, $d \geq 3$. Here, a point set is called *dense* if the ratio of the largest over the smallest distance between any two points from is $O(n^{1/d})$ (the constant in (2.8) depends on the implicit constant in the definition of density). Let us digress for a moment to outline the proof of (2.8), which proceeds along the following lines:

1. Assume that S is a dense set of n points in $\mathbf{R}^d$, $n - d$ even, and set $j := (n - d)/2$ and $k := j + 1$. By (2), every j-facet σ of S gives rise to a facet $\mathcal{F}(\sigma)$ of $\mathcal{P}_k(S)$, and since $k - j = 1$, $\mathcal{F}(\sigma)$ is just a translated copy of $|\sigma|$. Therefore, the total $(d - 1)$-dimensional area of all j-facets is bounded from above by the $(d - 1)$-dimensional surface area of $\mathcal{P}_k(S)$.

2. The homothetic copy $\frac{1}{k}\mathcal{P}_k(S)$ is contained in the convex hull of S. Therefore, the projection of $\frac{1}{k}\mathcal{P}_k(S)$ onto any coordinate hyperplane is contained in the convex hull of the corresponding projection of S, and hence, by density, has $(d-1)$-dimensional area at most $O(n^{\frac{d-1}{d}})$. The total $(d-1)$-dimensional surface area of a convex body is at most two times the sum of the $(d-1)$-dimensional areas of its projections onto the coordinate hyperplanes. Therefore, the $(d-1)$ dimensional surface area of $\mathcal{P}_k(S)$ is "not too large", namely $O(k^{d-1}n^{1-1/d})$. By the first step, the same holds for the total area of all j-facets.

3. On the other hand, any collection of "many" $(d-1)$-dimensional simplices spanned by points from a dense set necessarily has "large" total $(d-1)$-dimensional area. (The precise statement and the proof of this lemma are somewhat technical, see [35] for the details.) Therefore, if there were too many j-facets of S (more than $Cn^{d-2/d}$ for some suitable constant C), then their total area would have to be too large, i.e. would exceed the bound derived in the second step.

Remark 2.7. Edelsbrunner et al. [35] also showed, by a more direct approach, that for a dense set S of n points in the plane,

$$e_{1/2}(S) = O(\sqrt{n}e_{1/2}(\sqrt{n})).$$

In particular, any general bound $e_{1/2}(n) = O(n^{1+c})$ implies a bound of $O(n^{1+c/2})$ for dense point sets. If the number of halving edges was maximized by dense point sets, by bootstrapping, this would lead to $e_{1/2}(n) = O(n \operatorname{polylog} n)$, contradicting Tóth's lower bound (2.7).

Chapter 3

Corner Cuts

In this chapter, we study the k-sets of the infinite set $\mathbf{N}_0^d$. These objects, which also go under the suggestive names of *corner cuts*, were investigated by Onn and Sturmfels [60] in connection with computational commutative algebra: They showed that the corner cuts of a given size k, or k*-cuts*, for short, in dimension d are in one-to-one correspondence with the Gröbner bases of a certain kind of ideal in the polynomial ring $K[x_1, \ldots, x_d]$, K any infinite field.

Apart from this algebraic connection, which we briefly review in Section 3.1, corner cuts seem to be a very natural special instance of the k-set problem. Onn and Sturmfels prove an upper bound of $O(k^{2d\frac{d-1}{d+1}})$ for the number $a_k(\mathbf{N}_0^d)$ of corner cuts of cardinality k in dimension d (as usual, the dimension is considered fixed). We will see in Section 3.2 that this can be quite easily improved upon by restricting our attention to a suitable finite subset of $\mathbf{N}_0^d$ and applying some general k-set bounds. However, applying methods that were devised for point sets in general position does not do justice to corner cuts, because of the massive affine dependencies within the set $\mathbf{N}_0^d$. We cannot afford to pass to general position (by invoking some perturbation arguments, say), lest we risk to increase the number of k-sets dramatically: We will prove in Section 3.4 that

$$a_k(\mathbf{N}_0^d) = O((k \log k)^{d-1}) \tag{3.1}$$

for any fixed dimension d. Yet, as we will see below, the number $n = n(k,d)$ of nonnegative integer points belonging to some k-cut roughly equals $k(\log k)^{d-1}$, and from Chapter 2 we know that there are examples of n-point sets in $\mathbf{R}^d$ that have $nk^{d-2}e^{\Omega(\sqrt{k})}$ many k-sets. Thus, since $e^{\sqrt{\log k}}$ grows faster than any given power of $\log k$, general k-set estimates are of no avail if we want to establish an upper bound of the form $k^{d-1}\operatorname{polylog}(k)$.

For the planar case, (3.1) specializes to the upper bound part of

$$a_k(\mathbf{N}_0^2) = \Theta(k \log k),$$

which was proved by Corteel et al. [28]. In Section 3.3, we use this planar result to derive a general lower bound

$$a_k(\mathbf{N}_0^d) = \Omega(k^{d-1} \log k), \tag{3.2}$$

which shows that the upper bound (3.1) is quite tight. (It has been communicated to me that the bounds (3.1) and (3.2) have been found independently by Gaël Rémond.)

Finally, in Section 3.5, we discuss some algorithmic issues concerning corner cuts, and mention a related open problem.

3.1 A Glimpse of the Algebraic Background

Let K be a field and $K[x_1, \ldots, x_d]$ the ring of polynomials in d indeterminates over K. Recall that a set I of polynomials is called an *ideal* if I contains the zero polynomial 0 and if for all $f, g \in I$ and for any $h \in K[x_1, \ldots, x_d]$, we have $f + g \in I$ and $h \cdot f \in I$.

For any set $F \subseteq K[x_1, \ldots, x_d]$ of polynomials, there is a unique inclusion-minimal ideal containing F, which is called the *ideal generated by* F and denoted by $\langle F \rangle$. It is a fundamental fact about polynomial rings (in a finite number of indeterminates) over fields (or, more generally, over so-called commutative Noetherian rings with unity) that every ideal I in $K[x_1 \ldots, x_d]$ is *finitely generated*, i.e., there exist a finite number of polynomials $f_1, \ldots, f_s$ such that $I = \langle f_1, \ldots, f_s \rangle$. This assertion is known as the *Hilbert Basis Theorem* (for a proof of this, and as references for the material discussed in this Section, see the books by Cox, Little, and O'Shea [29, 30] or by Sturmfels [75, 76]).

Ideals are closely related to basic geometric objects, namely all those defined by polynomial equations, such as affine subspaces (linear equations), conics (ellipses, hyperbolas, parabolas) in the plane or quadrics in higher dimensions (quadratic equations), and so forth. In general, a set $V \subseteq K^d$ is called an *algebraic variety* if it is the set of zeros of a collection of polynomials, i.e., if if for some $F \subseteq K[x_1, \ldots, x_d]$, we have $V = V(F)$, where

$$V(F) := \{\boldsymbol{a} \in K^d : f(\boldsymbol{a}) = 0 \text{ for all } f \in F\}.$$

Observe that for any F, we have $V(F) = V(\langle F \rangle)$, so by the Hilbert Basis Theorem, every variety is, in fact, defined by a finite number of polynomials.

Classic examples of ideals are *vanishing ideals*: For an arbitrary subset $A \subseteq K^d$, the vanishing ideal of A is the set of polynomials that evaluate to zero for all points $\boldsymbol{a} = (a_1, \ldots, a_d) \in A$,

$$I(A) := \{f \in K[x_1, \ldots, K_d] : f(\boldsymbol{a}) = 0 \text{ for all } \boldsymbol{a} \in A\}.$$

It is straightforward to check that this is indeed an ideal.

There are various basic algorithmic questions concerning ideals. For instance, given an ideal $I = \langle f_1, \ldots, f_s \rangle$ and a polynomial f, we can ask whether $f \in I$. This is called the *Ideal Membership Problem*.

In one indeterminate, this is easy. For univariate polynomials $f, g \in K[x_1]$ with $g \neq 0$, the standard algorithm for division with remainder produces unique polynomials $q, r \in K[x_1]$ such that

$$f = qg + r \quad \text{and} \quad \deg(r) < \deg(g).$$

Having division with remainder at our disposal, it is not difficult to see that every ideal $I = \langle f_1, \ldots, f_s \rangle$ in $K[x_1]$ is in fact generated by one single polynomial, namely by the greatest common divisor (GCD) of the f_i's. Recall that the GCD is defined as the unique (up to multiplication by a nonzero constant) polynomial in $K[x_1]$ that divides every f_i and is itself divisible by every other polynomial dividing all f_i. The essential observation for finding the GCD is that for two polynomials f, g, either $g = 0$, in which case $\mathrm{GCD}(f, g) = f$, or $\mathrm{GCD}(f, g) = \mathrm{GCD}(f, r)$, where r is the remainder upon dividing f by g. Since $\deg(r) < \deg(g)$, this gives an efficient algorithm to compute the GCD of two polynomials, the *Euclidean Algorithm*, which extends immediately to any finite number of polynomials since $\mathrm{GCD}(f_1, \ldots, f_s) = \mathrm{GCD}(\mathrm{GCD}(f_1, \ldots, f_{s-1}), f_s)$. Thus, for one indeterminate, the Ideal Membership Problem "$f \in \langle f_1, \ldots, f_s \rangle$?" reduces to the question whether f lies in the ideal generated by $g := \mathrm{GCD}(f_1, \ldots, f_s)$, which is the case iff g divides f, i.e., iff the remainder upon dividing f by g is zero.

The extension to more indeterminates requires a measure of progress that takes the role of the degree for one indeterminate.

Definition 3.1 (Monomial orderings). A *monomial ordering* on $K[x_1, \ldots, x_d]$ is a linear ordering $\prec$ on the set of monomials $\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ that

1. is compatible with multiplication, i.e., $\boldsymbol{x}^{\boldsymbol{\alpha}} \prec \boldsymbol{x}^{\boldsymbol{\beta}} \Rightarrow \boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\gamma}} \prec \boldsymbol{x}^{\boldsymbol{\beta}}\boldsymbol{x}^{\boldsymbol{\gamma}}$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbf{N}_0^d$, and

2. is a *well-ordering*, i.e., there is no infinite descending chain $\boldsymbol{x}^{\boldsymbol{\alpha_1}} \succ \boldsymbol{x}^{\boldsymbol{\alpha_2}} \succ \boldsymbol{x}^{\boldsymbol{\alpha_3}} \succ \ldots$

Equivalently, we can view a monomial ordering as a well-ordering on $\mathbf{N}_0^d$ that is compatible with addition.

For one indeterminate, the ordering $1 = x^0 \prec x^1 \prec x^2 \prec \ldots$ by degree is the only monomial order. For several indeterminates, there are infinitely many (see below). One standard example is the lexicographic order $\prec_{\text{lex}}$ which for two monomials first compares their degrees in the first variable x_1, in case of a tie compares the degrees in the second variable, and so forth. (Formally, $\boldsymbol{x}^{\boldsymbol{a}} \prec_{\text{lex}} \boldsymbol{x}^{\boldsymbol{b}}$ iff for $j := \min\{i : a_i \neq b_i\}$, we have $a_j < b_j$.)

For a polynomial $f = \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}}$, the *(multi)degree* of f w.r.t. to a monomial ordering $\prec$ is

$$\deg_{\prec}(f) := \max\{\boldsymbol{a} \in \mathbf{N}_0^d : c_{\boldsymbol{a}} \neq 0\},$$

where the maximum is taken w.r.t. $\prec$. Further,

$$in_{\prec}(f) := c_{\deg_{\prec}(f)} \boldsymbol{x}^{\deg_{\prec}(f)}$$

is called the *initial* or *leading term* of f.

With this terminology, the extension of polynomial division to several variables can be phrased as follows. Fix a monomial order $\prec$ on $K[x_1, \ldots, x_d]$. Given polynomials $f_1, \ldots, f_s$ and f in $K[x_1, \ldots, x_d]$, there are polynomials $q_1, \ldots, q_s$ (the “quotients”) and r (the “remainder”) such that

$$f = q_1 f_1 + \ldots + q_s f_s + r \tag{3.3}$$

and the remainder r is either zero, or is a K-linear combination of monomials none of which is divisible by any of the initial terms $in_{\prec}(f_i)$. Division by several polynomials at once is necessary because for more than one indeterminate, ideals in the polynomial ring are in general not generated by a single polynomial.

The basic idea is the same as in the case of one indeterminate: cancel the leading term of f by multiplying one of the f_i’s by an appropriate monomial and subtracting. However, there are various subtleties that arise for several indeterminates. For a discussion of these, as well as a precise algorithmic description of how to find the expression (3.3), see Chapter 2 of [29].

One problem is that the expression (3.3) need not be unique. (In general, it depends on the monomial order, on the particular implementation of the division algorithm, and on the order in which the f_i’s are considered.) For instance, if $f_1 = xy + 1,\ f_2 = y^2 - 1 \in K[x, y]$, then $f = xy^2 - x$ can be written in two ways,

$$f = y(xy + 1) + 0(y^2 - 1) + (-x - y) = 0(xy + 1) + x(y^2 - 1) + 0.$$

This is particularly unpleasant, since the remainder in the first expression is nonzero, while the second expression shows that actually $f \in \langle f_1, f_2 \rangle$. Thus, membership in an ideal $I = \langle f_1, \ldots, f_s \rangle$ is no longer characterized by the vanishing of the remainder upon division by the f_i's (it is still a sufficient, but not a necessary condition).

Luckily, this difficulty can be resolved by passing to generating sets with special properties.

Definition 3.2 (Gröbner Bases). Let I be an ideal in $K[x_1, \ldots, x_d]$, and let $\prec$ be a monomial ordering. A finite subset $G = \{g_1, \ldots, g_s\} \subseteq I$ is called a *Gröbner basis* of I w.r.t. $\prec$ if for every $f \in I$, the leading term $in_\prec(f)$ is divisible by one of the $in_\prec(g_i)$'s.

Gröbner bases were introduced by Buchberger in his dissertation [24] in 1965 (and named after his adviser, Wolfgang Gröbner). A Gröbner basis is always a generating set for the ideal, and it turns out that every ideal has a Gröbner basis w.r.t. any given monomial order. Moreover, if $\{g_1, \ldots, g_s\}$ is a Gröbner basis of an ideal I, then the remainder r upon dividing any polynomial $f \in K[x_1, \ldots, x_d]$ by the g_i's is *unique*. In particular, $f \in I$ iff $r = 0$. (We note that even for a Gröbner basis, the "quotients" in the polynomial division are not uniquely determined.)

The Gröbner basis of an ideal w.r.t. a monomial ordering is not unique. For instance, if we add some elements of I to a Gröbner basis, we get a Gröbner basis again. However, it turns out that for every $\prec$ and every I, there is a unique Gröbner basis G that is *reduced*, in the following sense: For every $g \in G$, we require that $c_{in_\prec(g)} = 1$, and that no monomial of g be divisible by the leading term $in_\prec(g')$ for any $g' \in G \setminus \{g\}$. The usefulness of Gröbner bases for many applications rests upon the fact that they not only exist, but can be computed. Buchberger devised an algorithm which takes a term order and a finite set $\{f_1, \ldots, f_s\} \subseteq K[x_1, \ldots, x_d]$ as input and outputs the reduced Gröbner basis w.r.t. $\prec$ for the ideal $\langle f_1, \ldots, f_s \rangle$.

Again, we can use this to solve the Ideal Membership Problem. Given a generating set $f_1, \ldots, f_s$, choose a monomial ordering $\prec$ and compute the (reduced) Gröbner basis $g_1, \ldots, g_r$ for the ideal $I = \langle f_1, \ldots, f_s \rangle$ w.r.t. $\prec$. As mentioned above, a polynomial f lies in I iff the remainder upon dividing f by $g_1, \ldots, g_r$ is zero.

This is only a very basic one among a host of applications of Gröbner bases, see [29, 30] and [75]. Often, it plays an impotant role which monomial ordering is chosen. For instance, Gröbner bases with respect to the lexicographic monomial order are very well suited for solving polynomial equations through elimination of variables (see [29], Chapter 3). Other monomial orders have other advantages. For example, the so-called graded reverse lexicographic order (first order the monomials by their total degree, then by their degree in x_d, then by their degree in x_{d-1}, and so forth) usually leads to small Gröbner bases and that Buchberger's algorithm often performs faster (a short discussion of the known results about the complexity of Buchberger's algorithm, and further references, can be found in [29], Chapter 2, § 9).

As mentioned above, there are infinitely many monomial orders if the number of indeterminates is larger than one. To see this, let us identify the set of all monomials in d indeterminates with $\mathbf{N}_0^d$. Any vector $\boldsymbol{w} \in \mathbf{R}^d$ induces a partial order on $\mathbf{N}_0^d$ by comparing values of scalar products, $\boldsymbol{a} \preccurlyeq_{\boldsymbol{w}} \boldsymbol{b} :\Leftrightarrow \langle \boldsymbol{w}, \boldsymbol{a} \rangle \leq \langle \boldsymbol{w}, \boldsymbol{b} \rangle$. This partial order is compatible with addition. Moreover, if $\boldsymbol{w} \geq 0$ componentwise, then there is no infinite descending chain, and if $\boldsymbol{w}$ is sufficiently generic (namely, if the entries of $\boldsymbol{w}$ are linearly independent over the field $\mathbf{Q}$ of rational numbers), then there are no ties, i.e., $\preccurlyeq_{\boldsymbol{w}}$ is a linear ordering. Thus, any such $\boldsymbol{w} \in \mathbf{R}_{\geq 0}^d$ yields a monomial ordering $\prec_{\boldsymbol{w}}$ (and these are all distinct). Note that not all monomial orderings on $K[x_1, \ldots, x_d]$ are of this form. The lexicographic ordering, for instance, is not.

However, if we fix an ideal I, then all monomial orders can be grouped into finitely many equivalence classes, as described below, and in each equivalence class, there will be a representative of the form $\prec_{\boldsymbol{w}}$. For an ideal I and a monomial order $\prec$, let $in_{\prec}(I)$ be the ideal generated by the leading terms $in_{\prec}(f)$ of polynomials $f \in I$. The condition for G being a Gröbner basis can be rephrased as $in_{\prec}(I) = \langle in_{\prec}(g) : g \in G \rangle$. Further, if $in_{\prec}(I) = in_{\prec'}(I)$, for two monomial orderings $\prec, \prec'$ then the reduced Gröbner bases of I with respect to these orderings coincide. Thus, we can declare $\prec$ and $\prec'$ to be equivalent (more precisely, equivalent w.r.t. I) in this case. It can be shown that for every ideal I, there are only finitely many equivalence classes, and for each equivalence class, there is a representative of the form $\prec_{\boldsymbol{w}}$ (see [75]).

One corollary is that I has only finitely many distinct reduced Gröbner bases. Hence, I even has a *universal Gröbner basis*, i.e., a finite set $U \subseteq I$ that is simultaneously a Gröbner basis with respect to all monomial orderings: simply take the union of all reduced Gröbner bases.

Another consequence is that we can look at the different Gröbner bases of an ideal from a geometric viewpoint. Let us generalize the notion of leading term to non-generic $\boldsymbol{w}$ by defining, for a polynomial $f = \sum_{\boldsymbol{a}} c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}}$, the “initial form” $in_{\boldsymbol{w}}(f)$ as the sum over all terms $c_{\boldsymbol{a}} \boldsymbol{x}^{\boldsymbol{a}}$ such that $\langle \boldsymbol{w}, \boldsymbol{a} \rangle$ is maximal (among the exponents $\boldsymbol{a}$ with $c_{\boldsymbol{a}} \neq 0$). We then define $in_{\boldsymbol{w}}(I) := \langle in_{\boldsymbol{w}}(f) : f \in I \rangle$, and say that $\boldsymbol{w}, \boldsymbol{w}' \in \mathbf{R}^d_{\geq 0}$ are equivalent if $in_{\boldsymbol{w}}(I) = in_{\boldsymbol{w}'}(I)$. This defines a partition of $\mathbf{R}^d_{\geq 0}$ into equivalence classes, and this partition turns out to be a polyhedral fan, called the *Gröbner fan* of I (see Mora and Robbiano [58]). That is, each equivalence class is a relatively open polyhedral cone, and the closures of these cones form a fan in the sense that along with every cone, all its faces are present, and that two closed cones intersect in a common face. The full-dimensional cones of this fan correspond to the non-equivalent monomial orders, i.e., to the reduced Gröbner bases of I.

The link to polyhedral combinatorics is formed by the notion of a state polyhedron. If $\mathcal{F}$ is a face of a convex polyhedron $\mathcal{P} \subseteq \mathbf{R}^d$, let us define the *(minimizing) normal cone* $\mathcal{N}_{\mathcal{P}}(\mathcal{F})$ as the set of all $\boldsymbol{w} \in \mathbf{R}^d$ that are minimized on $\mathcal{F}$, i.e., $\langle \boldsymbol{w}, \boldsymbol{x} \rangle = \min_{\boldsymbol{y} \in \mathcal{P}} \langle \boldsymbol{w}, \boldsymbol{y} \rangle$ for all $\boldsymbol{x} \in \mathcal{F}$. The *normal fan* $\mathcal{N}(\mathcal{P})$ is the collection of all normal cones $\mathcal{N}_{\mathcal{P}}(\mathcal{F})$, $\mathcal{F}$ any face of $\mathcal{P}$. Note that the full-dimensional cones in $\mathcal{N}(\mathcal{P})$ are precisely the normal cones of the vertices of $\mathcal{P}$. A polyhedron $\mathcal{P}$ is called a *state polyhedron* of an ideal I (see [14]) if the normal fan of $\mathcal{P}$ equals the Gröbner fan of I.

Onn and Sturmfels [60] studied the k-set polyhedron (see Section 2.3)

$$\mathcal{P}_k(\mathbf{N}_0^d) = \operatorname{conv}\{\sum_{\boldsymbol{x} \in X} \boldsymbol{x} \mid X \subseteq \mathbf{N}_0^d \text{ and } |X| = k\},$$

which they name the *corner cut polyhedron*, and showed that it is the state polyhedron of a particular class of ideals. Namely, if $I = I(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k)$ is the vanishing ideal of a set of k points $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k \in K^d$, and if the $\boldsymbol{q}_i$'s are in a certain sense “generic” then $\mathcal{P}_k(\mathbf{N}_0^d)$ is the state polyhedron of I. Thus, the distinct reduced Gröbner bases are in one-to-one correspondence with the vertices of $\mathcal{P}_k(\mathbf{N}_0^d)$, hence, by Fact 2.6, with the corner cuts of cardinality k in dimension d. This can be used, for instance, to compute a universal Gröbner basis of I in polynomial time (for fixed d).

These results have been extended to a considerably broader class of ideals (namely, all those for which the quotient $K[x_1, \ldots, x_d]/I$ has finite dimension as a vector space over K) by Babson, Onn, and Thomas [9].

We leave the realm of computational commutative algebra at this point and return to the combinatorial question of estimating the number of corner cuts.

3.2 How Many Lattice Points Are Involved?

Unless explicitly stated otherwise, let us assume for the remainder of this chapter that whenever we encounter a hyperplane H not containing the point $-\mathbf{1} = (-1, \ldots, -1)$, it is oriented in such a way that $-\mathbf{1}$ lies in the negative halfspace H^-.

Suppose that a corner cut T of size k contains a lattice point $\boldsymbol{u} = (u_1, \ldots, u_d)$. Then the whole "lattice box" $Q_{\boldsymbol{u}} = \{0 \ldots u_1\} \times \ldots \times \{0 \ldots u_d\}$ is contained in T, and therefore $k \geq \prod_{i=1}^{d}(1 + u_i)$. In other words, all corner cuts of cardinality k are subsets of the finite set $S_k^d := \{\boldsymbol{u} \in \mathbf{N}_0^d \mid \prod_{i=1}^{d}(1+u_i) \leq k\}$.

Observation 3.3. *For any real number $y \geq 1$,*

$$|S_y^d| \leq y(1 + \log y)^{d-1}.$$

Proof. We proceed by induction on d: For $d = 1$, $|S_y^1| = \lfloor y \rfloor \leq y$. Moreover, for $d > 1$ and integer $1 \leq j \leq y$, $|\{\boldsymbol{u} \in S_y^d \mid 1 + u_d = j\}| = |S_{y/j}^{d-1}| \leq (y/j)(1 + \log(y/j))^{d-2}$. Therefore,

$$|S_y^d| \leq \sum_{j=1}^{\lfloor y \rfloor}(y/j)(1 + \log(y/j))^{d-2} \leq y(1 + \log y)^{d-2} \underbrace{\sum_{j=1}^{\lfloor y \rfloor} \frac{1}{j}}_{\leq 1+\log y} .$$

□

Observation 3.4. *Every k-set T of S_k^d that contains the origin is a k-cut.*

Proof. Suppose for a contradiction that $T \ni \mathbf{0}$ is a k-set of S_k^d such that, for every hyperplane H with $T = \mathrm{H}^- \cap S_k^d$, there exists some nonnegative lattice point $\boldsymbol{u} \notin S_k^d$ with $u \in \mathrm{H}^-$; call such a point $\boldsymbol{u}$ a *violator* and H a *witness* for $\boldsymbol{u}$. Clearly, $d > 1$, and T is not contained in any coordinate hyperplane (else we are done by induction).

Now, consider a violator $\boldsymbol{u}$ that minimizes $\prod_{j=1}^{d}(1 + u_j)$ and a witness hyperplane $\mathrm{H} = \{\boldsymbol{x} \in \mathbf{R}^d \mid \langle \boldsymbol{\nu}, \boldsymbol{x} \rangle = t\}$ for $\boldsymbol{u}$. If $Q = \{\boldsymbol{m} \in \mathbf{N}_0^d \mid m_j \leq u_j \text{ for } 1 \leq j \leq d\}$, then $Q \setminus \{\boldsymbol{u}\}$ contains at least k points since $\boldsymbol{u} \notin S_k^d$ Moreover, $Q \subset \mathrm{H}^- = \{\boldsymbol{x} : \langle \boldsymbol{\nu}, \boldsymbol{x} \rangle < t\}$: By assumption, $\mathbf{0} \in \mathrm{H}^-$, and so $t > 0$; moreover, all entries of $\boldsymbol{\nu}$ must be positive, for if $\nu_j \leq 0$, then all the points $i\boldsymbol{e}_j = (0, \ldots, i, \ldots, 0)$, $i \in \{0 \ldots k-1\}$, would belong to $\mathrm{H}^- \cap S_k^d = T$, and hence they would constitute it, contradicting the fact that

T is not contained in any coordinate hyperplane. Since $\boldsymbol{\nu}$ and t are positive, $\boldsymbol{u} \in \mathrm{H}^-$ implies $Q \subset \mathrm{H}^-$.

Therefore, $Q \setminus \{\boldsymbol{u}\} = T$. But then all $u_j \geq 1$ (otherwise T would be contained in some coordinate hyperplane) and so, for some j, H must intersect the x_j-axis at a distance greater than $u_j + 1$ from the origin. This, however, contradicts the assumption that H separates T from $S_k^d \setminus T$ because the point $(u_j + 1)\boldsymbol{e}_j$ belongs to $S_k^d \setminus T$ and lies below H. $\square$

Together with the general k-set bound (2.6), the first observation immediately yields $a_k(\mathbf{N}_0^d) \leq a_k(S_k^d) = O(k^{d-c'_d})$ for some small constant $c'_d > 0$ (which, for simplicity, is adjusted in such a way that the various $\log k$ factors are absorbed) as a first improvement over the the upper bound of $O(k^{2d\frac{d-1}{d+1}})$.

Furthermore, both observations together imply that the corner cuts of size k are precisely the k-sets of S_k^d that contain $\mathbf{0}$. This makes it easy to enumerate all k-cuts by applying a known k-set enumeration method by Andrzejak and Fukuda [7] to the finite set S_k^d, see Section 3.5.

3.3 The Lower Bound

Lemma 3.5. *For every d and k, the number of corner cuts of cardinality k in dimension d satisfies*

$$a_k(\mathbf{N}_0^d) \geq \sum_{i=1}^{k} a_k(\mathbf{N}_0^{d-1}).$$

Proof. We will show that for every $(d-1)$-dimensional corner cut T of size j, $1 \leq j \leq k$, there is some corner cut $\hat{T}$ of cardinality k in dimension d with $T = \mathbf{N}_0^{d-1} \cap \hat{T}$.

Take some T of size j as above with separating $(d-2)$-dimensional flat $\mathrm{F} \subseteq \mathbf{R}^{d-1}$. Let us identify $\mathbf{R}^{d-1}$ with $\mathbf{R}^{d-1} \times \{0\} \subseteq \mathbf{R}^d$, and for a real parameter $t > 0$, consider the hyperplane H_t spanned by the flat F and the the point $(0, \ldots, 0, t)$, see Figure 3.1.

Observe that by choosing F in a sufficiently generic manner, we may assume that no H_t contains more than one lattice point.

Each of the hyperplanes H_t defines a certain d-dimensional corner cut T_t. For $t < 1$, this is exactly our original T. But as t grows, more and more lattice points will be included, one at a time by our assumption on f, until for some

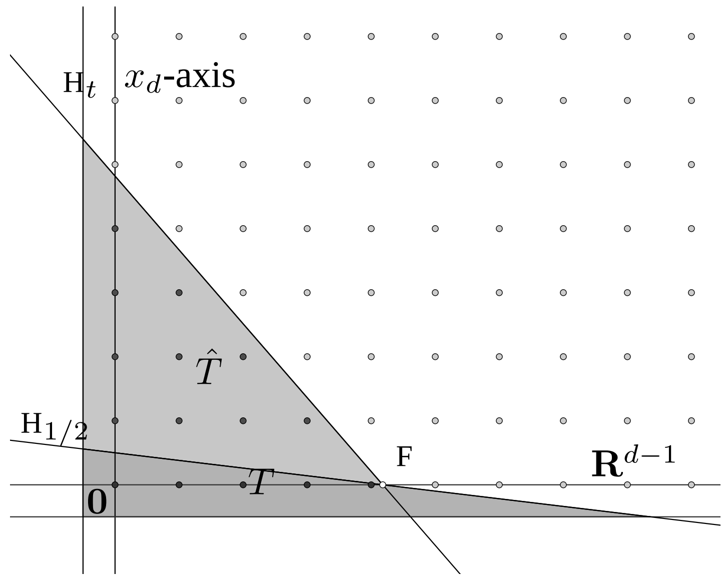


Figure 3.1: *Lifting a corner cut.*

appropriate $\hat{t}$, we obtain a d-dimensional corner cut $\hat{T} := T_{\hat{t}}$ of size k as advertised. (Note that $\hat{T}$ depends on the choice of a separating flat F.) $\square$

Since we know that $a_k(\mathbf{N}_0^2) = \Omega(k \log k)$, and since $\sum_{j=1}^{k} j^{d-2} \log j \geq \int_1^k x^{d-2} \log x \, dx = \frac{1}{d-1} k^{d-1} \log k - \int_1^k x^{d-2} \sim \frac{1}{d-1} k^{d-1} \log k$, we conclude inductively:

Theorem 3.6. *The number of k-cuts in d dimensions satisfies*

$$a_k(\mathbf{N}_0^d) = \Omega_d(k^{d-1} \log k).$$

3.4 The Upper Bound

For a point set in general position, it is often more convenient to consider k-facets instead k-sets. Let us introduce a notion that will serve a similar purpose for degenerate point sets.

Definition 3.7. An oriented hyperplane H is called a *k-hyperplane* of a set $S \subseteq \mathbf{R}^d$ if H is spanned by points from S and moreover, $|\mathrm{H}^- \cap S| < k$ and $|\overline{\mathrm{H}^-} \cap S| \geq k$.

Note that, for a given vector $\boldsymbol{\nu} \neq \mathbf{0}$, there is at most one k-hyperplane H of S with outer normal $\boldsymbol{\nu}$. Observe also that H may be a k-hyperplane for more than one value of k.

For instance, according to our definition, the coordinate hyperplanes are k-hyperplanes of $\mathbf{N}_0^d$ for every $k > 0$. On the other hand, consider a *proper* k-hyperplane H of $\mathbf{N}_0^d$, i.e., one that is not one of the coordinate hyperplanes. We claim that the outer normal vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_d)$ of H must be strictly positive, i.e., all $\nu_i > 0$. Clearly, we have $\nu_i \geq 0$ (else H^- would contain all sufficiently large integer multiples $m\boldsymbol{e}_i$ of the i^{th} coordinate vector and thus not be finite). Further, by assumption, there are d points in $\mathbf{N}_0^d$ that span H, and for each $1 \leq i \leq d$, one of these points, call it $\boldsymbol{u}$, must have i^{th} coordinate $u_i > 0$, else H would be just the i^{th} coordinate hyperplane. Since $\boldsymbol{\nu}$ is componentwise nonnegative, it follows that $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d \in \overline{\mathrm{H}^-}$ and that $\mathbf{0} \in \mathrm{H}^-$. Thus, $\boldsymbol{\nu}$ must in fact be strictly positive, for if $\nu_i = 0$, then along with $\mathbf{0}$, the whole "ray" $\mathbf{N}_0\boldsymbol{e}_i$ would lie in H^-.

Definition 3.8. We say that a k-hyperplane H of a set $S \subseteq \mathbf{R}^d$ is *incident* to a k-set T of S if $S \cap \mathrm{H}^- \subseteq T \subseteq \overline{\mathrm{H}^-}$.

The basic idea is that if H is k-hyperplane incident to a k-set T of S, then we can "encode" T by means of H plus some additional information, which will be specified below. In order to use this to bound the number of corner cuts, we need the following:

Lemma 3.9. *If $S \subseteq \mathbf{R}^d$ is finite and at least $(d-1)$-dimensional, or if $S = \mathbf{N}_0^d$, then for $k > 0$, every k-set T of S is incident to some k-hyperplane of S.*

Admittedly, this lemma is rather obvious, but it nonetheless deserves a proof, since there are malicious point sets for which the conclusion of the Lemma does not hold. For instance, if $S = \mathbf{Z} \times \{0\} \cup \{(0,1)\} \in \mathbf{R}^2$, then $(0,1)$ is a 1-set of S that is not incident to any 1-hyperplane of S (the problem is that we insist that these be spanned by points from S).

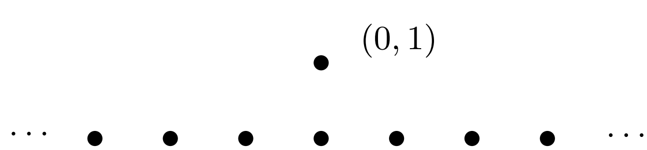


Figure 3.2: *A malicious point set.*

We also note that if S is finite or $S = \mathbf{N}_0^d$ and if H is a k-hyperplane of S, then the intersection $S \cap \mathrm{H}$ is again either finite or $\mathbf{N}_0^{d-1}$. This will be important since we will need to apply the lemma recursively.

Proof of Lemma 3.9. First observe that if $S = \mathbf{N}_0^d$, then we may assume that T is a *proper* corner cut, i.e., is not contained in one of the coordinate hyperplanes. Otherwise, that coordinate hyperplane will serve as an incident k-hyperplane for T. Let then H_0 be a separating hyperplane for T. Move this hyperplane in parallel towards T until we hit the first point, i.e., until we obtain the unique parallel translate H_1 of H_0 such that $T \subseteq \overline{\mathrm{H}^-}_1$, $T \cap \mathrm{H}_1 \neq \emptyset$, and $S \setminus T \subseteq \mathrm{H}_1^+$. Observe that the only reason why H_1 might not be a k-hyperplane is that it might not yet be spanned by points from S. If this is problem occurs, pick a $(d-2)$-dimensional flat F such that $S_1 \subseteq \mathrm{F} \subseteq \mathrm{H}_1$. We want to argue that a suitable rotation of H_1 about F will produce a hyperplane H_2 such that (i) $S \cap \mathrm{H}_2^- \subseteq S \cap \mathrm{H}_1^-$, (ii) $S \cap \mathrm{H}_2^+ \subseteq S \cap \mathrm{H}_1^+$, and (iii) $S_1 = S \cap \mathrm{H}_1 \subsetneq S \cap \mathrm{H}_2$. That is, we can rotate until we catch a new point without traversing any points. If we can convince ourselves that this claim is true, then we are done, since we can successively increase the dimension of $S \cap \mathrm{H}_i$ through a sequence of such rotations, until H_i is spanned by points from S and hence a k-hyperplane. The claim is clear for finite S, but for $S = \mathbf{N}_0^d$, we need a little argument. Observe that since T is a proper corner cut, every hyperplane H such that $T \subset \overline{\mathrm{H}^-}$ and $\mathrm{H}^- \cap \mathbf{N}_0^d$ is finite has strictly positive normal vector.

Observation 3.10. *For any hyperplane* H *with strictly positive normal vector, there are only finitely many componentwise minimal elements of* $N_0^d \cap \mathrm{H}^+$ *(i.e., elements* $\boldsymbol{u} \in \mathbf{N}_0^d \cap \mathrm{H}^+$ *such that for any other* $\boldsymbol{v} \in \mathbf{N}_0^d \cap \mathrm{H}^+$*, there is some coordinate* i *with* $u_i < v_i$*).*

Let M be the set of componentwise minimal elements of $\mathbf{N}_0^d \cap \mathrm{H}_1^+$, and let $N := S \cap \mathrm{H}_1^-$. We rotate H_1 about F (in an arbitrary direction) until we hit the first point $\boldsymbol{p}$ in $N \cup M$ (we can do this since this is a finite set). We claim that the resulting hyperplane H_2 spanned by F and $\boldsymbol{p}$ does the job. By construction, condition (iii) is fulfilled and we have $\overline{\mathrm{H}_1^-} \cap \mathbf{N}_0^d \subseteq \overline{\mathrm{H}_2^-} \cap \mathbf{N}_0^d$. We only need to verify that during the rotation, we did not inadvertedly traverse any of the non-minimal points from $\mathbf{N}_0^d \cap \mathrm{H}_1^+$. Let us write $\mathrm{H}_2 = \{\boldsymbol{x} \in \mathbf{R}^d : \langle \boldsymbol{\nu}, \boldsymbol{x} \rangle = t\}$. It suffices to show that the outer normal vector ν is componentwise nonnegative: because then, $\boldsymbol{v} \in \mathrm{H}_2^-$ for some $\boldsymbol{v} \in \mathbf{N}_0^d \mathrm{H}_1^+$ would imply that also $\boldsymbol{u} \in \mathrm{H}_2^-$ for some componentwise minimal $\boldsymbol{u} \in \mathbf{N}_0^d \cap \mathrm{H}_1^+$, a contradiction. So, why is $\boldsymbol{\nu} \geq 0$? Observe that $\{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\} \subseteq T \subseteq \overline{\mathrm{H}_2^-}$. Thus, in the defining equation of H_2, we have $t \geq 0$, and if some $\nu_i < 0$, then $m\boldsymbol{e}_i \in \mathrm{H}_2^-$ for all integers $m > 0$, contradicting the fact that for some m, $m\boldsymbol{e}_i \in M$, and $M \subseteq \overline{\mathrm{H}^+}_2$ by construction. □

Moreover, if H is a k-hyperplane of S, and S is finite , then $S \cap \mathrm{H}$ is finite, or $S \cap \mathrm{H} = \mathbf{N}_0^{d-1}$, so $S \cap \mathrm{H}$ is again a non-malicious $(d-1)$-dimensional set.

For the remainder of this section, we only consider point sets that are "non-malicious", i.e., that satisfy the conclusion of Lemma 3.9.

Observe that if $\mathtt{H}$ is a k-hyperplane incident to a k-set T and if $\mathtt{H}_0$ is a separating hyperplane for T then the open wedge $(\mathtt{H}^- \cap \mathtt{H}_0^+) \cup (\mathtt{H}^+ \cap \mathtt{H}_0^-)$ contains no points from S; that is, we can get from $\mathtt{H}_0$ to $\mathtt{H}$ by a rotation (about the $(d-2)$-dimensional axis $\mathtt{H} \cap \mathtt{H}_0$) without traversing any points from S.

Now, consider $J := \mathtt{H}_0^- \cap \mathtt{H} \cap S$. This is a j-set (as witnessed by the separating flat $\mathtt{H}_0 \cap \mathtt{H}$) of the $(d-1)$-dimensional point set $S \cap \mathtt{H}$, where $j = k - |\mathtt{H}^- \cap S|$. What is more, we can recover T from $\mathtt{H}$ and J: Take any $(d-2)$-dimensional separating flat $\mathtt{F}$ for J in $\mathtt{H}$; then a small rotation of $\mathtt{H}$ about $\mathtt{F}$ gives a separating hyperplane $\mathtt{H}_0$ for T.

Lemma 3.11. *Let T be a k-set of a (non-malicious) d-dimensional point set S. Set $\mathtt{F}_d := \mathbf{R}^d$ and $k_d := k$. Then T can be uniquely represented by a sequence $(\mathtt{F}_{d-1}, \ldots, \mathtt{F}_1)$, where, for $1 \leq i \leq d-1$, $\mathtt{F}_i$ is an i-dimensional oriented flat, spanned by points from $S \cap \mathtt{F}_{i+1}$, that forms a k_{i+1}-hyperplane within $\mathtt{F}_{i+1}$, and $k_i = k_{i+1} - |S \cap \mathtt{F}_{i+1} \cap \mathtt{F}_i^-|$.*

Proof. Applying the above observation recursively, we see that each k-set T of S can be represented by a sequence $(h = \mathtt{F}_{d-1}, \mathtt{F}_{d-2}, \ldots, \mathtt{F}_2)$ as specified (up to the last entry) together with a $j := k_2$-set J of the two-dimensional point configuration $S \cap \mathtt{F}_2$.

But in two dimensions, every j-set J can be uniquely represented by a j-line (that is, a j-hyperplane w.r.t. the surrounding 2-dimensional space): If ℓ_0 is a separating line for J take a j-line $\mathtt{F}_1$ incident to J such that the angle α of $\mathtt{F}_1$ w.r.t. ℓ_0 is negative (that is, ℓ arises from ℓ_0 by a clockwise rotation).

Conversely, given $\mathtt{F}_1$ and j, let $i = j - |S \cap \mathtt{F}_1^-|$ and let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the i^{th} and $(i+1)^{\text{th}}$ point of S on $\mathtt{F}_1$, respectively (in the direction of $\mathtt{F}_1$). Then a small counterclockwise rotation of $\mathtt{F}_1$ about the midpoint of $\boldsymbol{a}$ and $\boldsymbol{b}$ gives a separating line ℓ_0 for J (see Figure 3.3). Observe that it is crucial to know both $\mathtt{F}_1$ *and* j (we can reproduce the latter from our knowledge of k and $(\mathtt{F}_{d-1}, \ldots, \mathtt{F}_2)$). ☐

Corollary 3.12. *Every k-set T of a discrete d-dimensional set S can be encoded by a sign vector $\boldsymbol{\varepsilon} \in \{+1, -1\}^{d-1}$ together with a $(d-1)$-tuple $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d-1})$ of vectors of the following kind: There exist points $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_{d-1} \in S$ that span a k-hyperplane $\mathtt{H}$ incident to T and such that $\boldsymbol{v}_i = (\boldsymbol{p}_i - \boldsymbol{p}_0)$ for $i \in \{1 \ldots d-1\}$.*

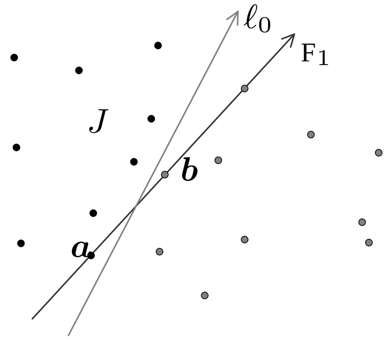


Figure 3.3: *An example with $j = 9$ and $i = 1$.*

Proof. Given T, represent it by $(\mathrm{F}_{d-1}, \ldots, \mathrm{F}_1)$ as above. Now, pick a pair $(\boldsymbol{p}_0, \boldsymbol{p}_1)$ of points that span the k_2-line F_1. Inductively, we construct a sequence of points $\boldsymbol{p}_i \in S$, $i \in \{0 \ldots d-1\}$ such that $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_i \in S$ span the flat F_i. Hence, if we set $\boldsymbol{v}_i := (\boldsymbol{p}_i - \boldsymbol{p}_0)$, then the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i$ span an i-dimensional linear flat parallel to F_i, and by choosing appropriate signs $\varepsilon_i \in \{+, -\}$ we can also record the orientation of F_i within F_{i+1}.

Moreover, this encoding is one-to-one: Given $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d-1}$ and $\varepsilon_1, \ldots, \varepsilon_{d-1}$, we get the outer normal of the k-hyperplane F_{d-1} of S, and hence F_{d-1} itself, since we know k. Then $k_{d-1} = k - S \cap \mathrm{F}^-_{d-1}$, and by induction, we can reconstruct the sequence $(\mathrm{F}_{d-2}, \ldots, \mathrm{F}_1)$ from $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d-2}, \varepsilon_1, \ldots, \varepsilon_{d-2}$ and k_{d-1}. But once we know all F_i's, the set T is uniquely determined. $\square$

Applying this to $S = \mathbf{N}_0^d$, we see that every k-cut T can be uniquely encoded by a sign vector $\boldsymbol{\varepsilon} \in \{+, -\}^{d-1}$ and the $d \times (d-1)$-matrix $V = [v_{ij}]$ whose columns are the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d-1}$ constructed above. Now, we derive some properties of V that will allow us to estimate the number of such matrices.

If H is one of the coordinate hyperplanes, then $T \subseteq \mathrm{H}$ is essentially a corner cut in dimension $d-1$, and we can handle the number of these inductively. Thus, we may assume that H is a proper k-hyperplane, i.e. that its outer normal vector is strictly positive.

Observation 3.13. *A "corner simplex" $\Delta = \mathbf{R}^d_{\geq 0} \cap \mathrm{H}^-$ that contains r lattice points has volume $\mathrm{vol}_d(\Delta) \leq r$.*

Proof. For each lattice point $\boldsymbol{u} \in \Delta$, consider the unit cube $\{\boldsymbol{x} \mid u_j \leq x_j \leq u_j + 1 \text{ for all } j\}$. Clearly, Δ is contained in the union of these boxes, whose volume is r. $\square$

Thus, the open simplex bounded by our proper k-hyperplane H and the coordinate hyperplanes has volume at most k; it follows that the same is true for its closure.

Let B be the *bounding box* of the points $\boldsymbol{p}_i$, $i \in \{0 \ldots d-1\}$, i.e. the smallest axis-parallel box $[a_1, b_1] \times \ldots \times [a_d, b_d]$ containing them. Since the hyperplane H spanned by the $\boldsymbol{p}_i$ is not one of the coordinate hyperplanes, B is a full-dimensional box, i.e. $b_i - a_i > 0$ for all $i \in \{1 \ldots d\}$.

Observation 3.14. *Let $\overline{\text{H}^-}$ be any halfspace containing all $\boldsymbol{p}_i$'s. Then*

$$\text{vol}_d(B \cap \overline{\text{H}^-}) \geq \frac{1}{d!} \text{vol}_d(B).$$

Proof. Let $\overline{\text{H}^-} = \{\boldsymbol{x} \in \mathbf{R}^d \mid \boldsymbol{\nu} \cdot \boldsymbol{x} \leq t\}$. Suppose without loss of generality $\nu_d \geq 0$, and consider the projection of the points onto the the hyperplane $\{\boldsymbol{x} \in \mathbf{R}^d \mid x_d = a_d\}$. Then these projected points are contained in $\overline{\text{H}^-} \cap \{x_d = a_d\}$. By induction, $\mathcal{P} = \overline{\text{H}^-} \cap B \cap \{x_d = a_d\}$ has $(d-1)$-dimensional volume at least $\frac{1}{(d-1)!} \prod_{i=1}^{d-1} (b_i - a_i)$. But then, the pyramid whose base is $\mathcal{P}$ and whose apex is any $\boldsymbol{p}_j$ maximizing the x_d-coordinate is contained in $B \cap \overline{\text{H}^-}$ and has volume as guaranteed. $\square$

For each row index $i \in \{1 \ldots d\}$ of the matrix $V = [v_{ij}]$, choose a column index $j(i) \in \{1 \ldots d-1\}$ such that $|v_{ij(i)}| = \max_j |v_{ij}|$. Then the i^{th} side of the bounding box B has length $b_i - a_i \geq |v_{ij(i)}| \geq 1$, whence $\prod_{i=1}^{d} |v_{ij(i)}| \leq \text{vol}_d(B) \leq d! \, \text{vol}(\overline{h^-} \cap B) \leq d!k$, by Observation 3.14.

Now, fix a sequence $(j(1), \ldots, j(d))$ and positive integers $m_1, \ldots, m_d$ such that $\prod_{i=1}^{d} m_i \leq d!k$, that is, $(m_1 - 1, \ldots, m_d - 1) \in S_{d!k}^d$. What is the number of integer matrices $V = [v_{ij}]$ such that $m_i = |v_{ij(i)}| = \max_j |v_{ij}|$ for all $i \in \{1 \ldots d\}$? Well, for each entry $v_{ij(i)}$ we get to choose a sign from $\{+, -\}$, while for the entries v_{ij} with $j \neq j(i)$ we may select any integer from $\{-m_i \ldots m_i\}$. Thus, we have $2^d \prod_{i=1}^{d} (2m_i + 1)^{d-2} = O(k^{d-2})$ possibilities to choose the entries when $j(1), \ldots, j(d)$ and $m_1, \ldots, m_d$ are fixed. Since there are $(d-1)^d = O(1)$ choices for the $j(i)$'s and, by Observation 3.3, $O(k(\log k)^{d-1})$ choices for the m_i's, we get a total of at most $O((k \log k)^{d-1})$ candidate matrices V. These, together with sign tuples $\boldsymbol{\varepsilon} \in \{+, -\}^{d-1}$ suffice to encode all k-cuts for which we have picked an incident k-hyperplane h that is not one of the coordinate hyperplanes, hence there are at most $O((k \log k)^{d-1})$ such k-cuts. But the remaining k-cuts correspond to lower-dimensional corner cuts, and by induction, there are at most $O((k \log k)^{d-2})$ of those. We have proved:

Theorem 3.15. *The number of k-cuts in d dimensions satisfies*

$$a_k(\mathbf{N}_0^d) = O((k \log k)^{d-1})$$

3.5 Remarks

Extensions of the upper bound. The above proof of the upper bound of $O((k \log k)^{d-1})$ immediately extends to the total number of faces of the corner cut polyhedron $\mathcal{P}_k(\mathbf{N}_0^d)$, and indeed even to the number of *flags* of $\mathcal{P}_k(\mathbf{N}_0^d)$, where a flag of a polytope $\mathcal{P}$ is a chain $\mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_s$ of faces of the polytope. In fact, the proof really is about flags. To see this, let $\mathcal{F} = \mathrm{conv}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ be an r-dimensional face of $\mathcal{P}_k(\mathbf{N}_0^d)$, where each $\boldsymbol{v}_i = \sum T_i$ for some k-cut T_i (these are the bounded faces of $\mathcal{P}_k(\mathbf{N}_0^d)$; on the other hand each unbounded r-face is contained in some r-dimensional intersection of coordinate hyperplanes, so there are only 2^d unbounded faces altogether). Then $\mathcal{F}$ corresponds, in a one-to-one fashion, to an r-dimensional affine flat F that is parallel to $\mathcal{F}$ and spanned by points from $\mathbf{N}_0^d$, such that there exists a hyperplane H with the following properties:

1. $\mathrm{F} \subseteq \mathrm{H}$, and $\mathrm{H} \cap \mathbf{N}_0^d = \mathrm{F} \cap \mathbf{N}_0^d$,
2. $\mathrm{H}^- \cap \mathbf{N}_0^d = T_1 \cap \ldots \cap T_m$,
3. each $T_i \subseteq \mathrm{H}^- \cup \mathrm{F}$, and the sets $T_i \cap \mathrm{F}$ are precisely the j-sets of $\mathbf{N}_0^d \cap \mathrm{F}$, where $j = k - |\mathrm{H}^- \cap \mathbf{N}_0^d|$.

(H is an appropriate translate of the supporting hyperplane of $\mathcal{P}_k(\mathbf{N}_0^d)$ that defines $\mathcal{F}$, and vice versa.)

Now, the total number of flags is at most 2^d times the number of maximal flags. Thus, consider such a maximal chain $\boldsymbol{v} = \mathcal{F}_0 \subsetneq \ldots \subsetneq \mathcal{F}_{d-1}$ of bounded faces of $\mathcal{P}_k(\mathbf{N}_0^d)$. This corresponds to a chain $\mathrm{F}_0 \subsetneq \ldots \subsetneq \mathrm{F}_{d-1}$ of flats as above, where $\dim \mathrm{F}_i = \dim \mathcal{F}_i = i$ for $0 \leq i < d$. Now, F_{d-1} is a k-hyperplane of $\mathbf{N}_0^d$ (with the additional property that $|\overline{\mathrm{F}_{d-1}^-} \cap \mathbf{N}_0^d| > k$), and $\mathrm{F}_{d-1}, \ldots, \mathrm{F}_1$ is a descending sequence of nested flats like the one constructed in Lemma 3.11. But as we have seen, there are at most $O((k \log k)^{d-1})$ such sequences, hence at most that many flags.

The proof can also be adapted to show that there are at most $O((k \log k)^{d-1})$ many pairs ("incidences") $(\mathrm{H}, \boldsymbol{p})$ where H is a proper k-hyperplane and $\boldsymbol{p} \in \mathrm{H} \cap \mathbf{N}_0^d$.

Weakness of the lower bound. For the lower bound, we considered a k-cut T in dimension $\mathbf{N}_0^d$ and showed that that for each $0 \leq i \leq k$, the "fiber" $\{J \subseteq \mathbf{N}_0^{d-1} : J = T \cap \mathbf{N}_0^{d-1}\}$ is non-empty. Since $a_j(\mathbf{N}_0^1) = 1$ for all j in dimension 1, this argument would only give a linear number of corner cuts of size k in dimension 2. But this number is $k \log k$, so the "average" fiber must have size $\log k$. This might be considered as evidence that the lower bound in Theorem 3.6 is not optimal. (To get a lower bound that matches the upper one we have, we would have to show that the average fiber is of size $\log k$ in every dimension.)

The algorithmic side. As noted in Section 3.2, enumerating the k-cuts in dimension d reduces to enumerating the k-sets of $S = S_k^d$ that contain the origin $\mathbf{0}$. For this purpose, we can use the k-set enumeration algorithm of Andrzejak and Fukuda [7], which is based on the paradigm of reverse search. Conceptually, the algorithm implicitly builds a subtree of the edge graph of the corner-cut polyhedron $\mathcal{P}_k(\mathbf{N}_0^d)$ and traverses this graph in a depth first manner. The crucial issue is how to find the neighbors of a given vertex $\boldsymbol{v} = \sum T$. As shown in [7], this reduces to solving certain linear programs. Some care is needed to ensure that the algorithm handles degenerate point configurations correctly, but on the other hand, the corner cut set-up is particularly nice in that the coefficients of the LP's will be integers in the range $\{0 \ldots k\}$. In total, the running time is (cf. [7])

$$a_k(\mathbf{N}_0^d) k n^2 \operatorname{lp}_k(d, |S_k^d|)\,, \tag{3.4}$$

where $\operatorname{lp}_k(d, n)$ denotes the time required to solve a linear program in d variables, with n constraints and coefficients from $\{0, \ldots, k\}$. If we substitute the upper bound of Theorem 3.15 and $|S_k^d| = O(k(\log k)^{d-1})$ in (3.4), we obtain, for fixed dimension d, a time complexity of

$$k^{d+2}(\log k)^{3d-3} \operatorname{lp}_k(d, k(\log k)^{d-1})\,.$$

Recognizing vertices of $\mathcal{P}_k(\mathbf{N}_0^d)$. To conclude this section and this chapter, let us mention a related open problem, posed by Onn and Sturmfels [60]. Suppose we are given a point $\boldsymbol{v} \in \mathbf{N}_0^d$ and an integer k and want to know whether $\boldsymbol{v}$ is a vertex of the corner cut polyhedron $\mathcal{P}_k(\mathbf{N}_0^d)$. If the dimension d is fixed, we can decide this in polynomial time by enumerating all corner cuts T and checking for each of them whether $\boldsymbol{v} = \sum T$. In the case of a positive answer, this also provides a witness that $\boldsymbol{v}$ is a vertex. The question is, whether the problem can be solved in a number of steps that is bounded by a polynomial in k and d.

Chapter 4

Origin-Embracing Distributions

Suppose we choose $n \geq 3$ random points, $P_1, \ldots, P_n$ in the unit disk centered at the origin, independently and identically distributed according to the uniform distribution. What is the probability that the convex hull of these points contains the origin? There is an elegant way of determining this probability, due to Wendel [87], see also [5]: First choose n random points $Q_1, \ldots, Q_n$ independently and uniformly distributed in the disk. For each i independently, set P_i to Q_i or to $-Q_i$ with equal probability $1/2$. The points $P_1, \ldots, P_n$ are again independently and uniformly distributed random points in the disk. Observe that almost surely (a.s.), no two of the Q_i lie on a common line through the origin,. Moreover, for any choice of the Q_i's that satisfies this condition, there are exactly $2n$ possibilities to choose the signs for the P_i's such that the origin can be separated from these points by a line: every partition of the Q_i's by a line through the origin gives two such possibilities. Therefore, the probability for the convex hull of the P_i's to contain the origin is precisely $1 - 2n/2^n$.

A second glance at the proof shows that all we used are the facts that the distribution is symmetric about the origin and that every line through the origin has mass zero. More generally, the same argument shows that if μ is a continuous probability distribution in $\mathbf{R}^d$ which is centrally symmetric about the origin, i.e. $\mu(B) = \mu(-B)$ for all measurable sets, then the probability that the origin is contained in the convex hull of $n \geq 1$ independent μ-distributed random points is

$$1 - \frac{\sum_{i=0}^{d-1} \binom{n-1}{i}}{2^{n-1}}. \tag{4.1}$$

To see this, observe that the number of ways to partition a finite set $\{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$ of points in $\mathbf{R}^d$ by a hyperplane through the origin equals the number of full-dimensional cells in the arrangement of linear hyperplanes $\boldsymbol{q}_i^* := \{\boldsymbol{x} \in \mathbf{R}^d : \langle \boldsymbol{q}_i, \boldsymbol{x} \rangle = 0\}$ dual to the points. It is not hard to show by induction on d that for $n \geq 1$, the latter is exactly $2\sum_{i=0}^{d-1} \binom{n-1}{i}$ if the points are in linearly general position.

What happens if we choose the points from a distribution that is not centrally symmetric? Given a continuous probability distribution μ and a point $\boldsymbol{o}$ in $\mathbf{R}^d$, let us denote the probability that $\boldsymbol{o}$ is contained in the convex hull of n independent μ-random points by $f_{n-d-1}(\mu, \boldsymbol{o})$ (the reason for the shift in the index will become apparent later).

It was shown by Wagner and Welzl [84] that for any μ and $\boldsymbol{o}$,

1. the probability $f_{n-d-1}(\mu, \boldsymbol{o})$ is at most the one given in (4.1).

2. Furthermore, this upper bound is attained iff μ is *balanced about* $\boldsymbol{o}$, in the sense that every hyperplane through $\boldsymbol{o}$ equipartitions μ. (We note that balancedness, in turn, can be equivalently characterized by saying that the radial projection of μ onto the unit sphere centered at $\boldsymbol{o}$ is centrally symmetric about $\boldsymbol{o}$, see Schneider [67].)

Somewhat surprisingly at first sight, this result can be considered a continuous analogue of McMullen's [52] *Upper Bound Theorem (UBT)* for convex polytopes, which gives exact upper bounds for the face numbers of a d-dimensional convex polytope with a prescribed number of vertices.

The proof of the Upper Bound Theorem rests on the notion of h-*vectors*. These are fundamental invariants of simplicial convex polytopes (simple perturbation arguments show that the numbers of faces of all dimensions are maximized by such polytopes). The face numbers of a simplicial polytope can be expressed as positive linear combinations of the entries of its h-vector, and what the UBT really does is to give exact upper bounds for these entries of the h-vector. This implies the bounds for the face numbers.

For the the continuous analogue above, so-called h-*functions* were introduced. The h-function associated with a distribution μ in $\mathbf{R}^d$ is a certain continuous function $h = h_{\mu,\boldsymbol{o}} : [0,1] \to \mathbf{R}_{\geq 0}$, and as it turns out, up to a factor depending only on k and d, the probability $f_k(\mu, \boldsymbol{o})$ is given by the k^{th} moment $\int_0^1 x^k h_{\mu,\boldsymbol{o}}(x)\mathrm{d}x$. Therefore, a pointwise upper bound for h-functions, the *Continuous Upper Bound Theorem (CUBT)*, implies the bound (4.1) for $f_{n-d-1}(\mu, \boldsymbol{o})$ stated above.

Rather than just being a formal coincidence of proof strategies, however, the definition of h-functions was motivated by a geometric re-interpretation by Welzl [86] (and in similar form already noted by Lee [49], Clarkson [26], and Mulmuley [59]) of h-vectors and their properties under Gale duality, so that h-functions can truly be considered continuous counterparts of h-vectors.

In Section 4.1, we review this duality, and the interpretation of h-vectors and their properties in both, the polytope set-up and the dual set-up. We also summarize the results from [84] about h-functions.

We then proceed to prove that another important result about h-vectors also carry over to the continuous set-up. To this end, we need a few facts about particular sequences of probability distributions on the unit interval that in a certain sense converge to point masses. We collect these facts in Section 4.2. In Section 4.3, we use these findings to prove a *Continuous Generalized Lower Bound Theorem* to the extent that h-functions are monotonically increasing on the interval $[0, 1/2]$ and decreasing on $[1/2, 1]$.

In Section 4.4, we describe an application of h-vectors and h-functions. We use h-functions (respectively, h-vectors) to prove a continuous analogue of the so-called *First Selection Lemma* (respectively, to give an alternative proof of the discrete version).

4.1 h-Vectors and h-Functions

Gale duality, which we summarize in Section 4.1.2, involves a shift in dimension. Therefore, we will denote the dimension by D when speaking about the polytope setting, and by d when speaking about the dual setting.

4.1.1 h-Vectors of Simplicial Polytopes

Let $\mathcal{P}$ be a simplicial D-dimensional polytope. The *h-vector*

$$h(\mathcal{P}) = (h_0(\mathcal{P}), \ldots, h_D(\mathcal{P}))$$

of $\mathcal{P}$ is defined by

$$h_j(\mathcal{P}) := \sum_k (-1)^{j-k} \binom{D-k}{D-j} f_{k-1}(\mathcal{P}), \tag{4.2}$$

where $f_k(\mathcal{P})$ denotes the number of k-dimensional faces of $\mathcal{P}$.

The motivation for this at first sight rather mysterious definition comes from the following geometric interpretation (see Kalai [47]): Let $\mathcal{P}^*$ be the polar polytope of $\mathcal{P}$ (i.e., the face lattice of $\mathcal{P}^*$ is obtained by turning that of $\mathcal{P}$ upside down). Any linear functional $\boldsymbol{c}$ on $\mathbf{R}^D$ such that the values $\langle c, v\rangle$ of the vertices of $\mathcal{P}^*$ are distinct induces an orientation of the edge graph of $\mathcal{P}^*$: orient every edge from the endpoint with smaller towards the endpoint with larger c-value.

Since $\mathcal{P}$ is simplicial, its polar $\mathcal{P}^*$ is simple, i.e., every vertex v of $\mathcal{P}^*$ of is incident to exactly D edges, and any k of these edges span a k-dimensional face of $\mathcal{P}^*$ incident to the vertex v and these edges. Moreover, every nonempty face $\mathcal{F}$ of $\mathcal{P}^*$ has a unique sink with respect to the orientation induced by c, i.e., a unique vertex $v \in \mathcal{F}$ such all edges of $\mathcal{F}$ incident to v are directed towards v.

Let us double-count the pairs $(\mathcal{F}, v)$, where $\mathcal{F}$ is a k-dimensional face of $\mathcal{P}^*$. On the one hand, we count exactly the number $f_k(\mathcal{P}^*)$ of k-dimensional faces of $\mathcal{P}^*$. On the other hand, if a vertex v has in-degree j_v, then v is the sink of precisely $\binom{j_v}{k}$ faces of dimension k (any k of the incoming edges span such a face, and any face is of that form). Thus, if we denote the number of vertices with given in-degree j by h_j (at first sight, this number depends on c), we obtain

$$f_k(\mathcal{P}^*) = \sum_j \binom{j}{k} h_j. \tag{4.3}$$

Now we observe that if two sequences (a_k) and (b_j) of complex numbers satisfy $a_k = \sum_j \binom{j}{k} b_j$ for all k, then this simply means that the two formal power series $A(x) := \sum_k a_k x^k$ and $B(x) := \sum_j b_j x^j$ satisfy $A(x) = B(x+1)$. Hence, $B(x) = A(x-1)$, and by comparing coefficients, we obtain $b_j = \sum_k (-1)^{k-j} \binom{k}{j} a_k$ for all j. Thus,

$$h_j = \sum_k (-1)^{k-j} \binom{k}{j} f_k(\mathcal{P}^*) \tag{4.4}$$

for all j, and the numbers h_j depend only on $\mathcal{P}^*$ and not on c. Thus, by considering the functional $-c$, we see that the following holds:

Theorem 4.1 (Dehn-Sommerville Relations [31, 72]). *If $\mathcal{P}$ is a D-dimensional simplicial convex polytope, then for all j,*

$$h_j(\mathcal{P}) = h_{D-j}(\mathcal{P}). \tag{4.5}$$

Note that the special case $j = 0$ of (4.5) together with (4.4) yields the *Euler-Poincaré Formula* for simplicial polytopes:

$$f_0(\mathcal{P}) - f_1(\mathcal{P}) + \ldots + (-1)^{D-1} f_{D-1}(\mathcal{P}) + (-1)^D = 1$$

Observe that if we substitute into (4.4) the face numbers $f_k(\mathcal{P}) = f_{D-1-k}(\mathcal{P}^*)$ and use the Dehn-Sommerville Relations, we arrive at equation (4.2) above.

Alternatively, the numbers $h_j(\mathcal{P})$ can be interpreted via so-called *shellings* of $\mathcal{P}$. This is the original approach of McMullen [52], who introduced the h-vector as the major tool in his proof of the UBT (until then the Upper Bound Conjecture). Vaguely speaking, a shelling of a simplicial polytope $\mathcal{P}$ is a way of building up the boundary of the polytope in a nice fashion. Specifically, it is an ordering $\mathcal{F}_1, \ldots, \mathcal{F}_m$ of the facets of $\mathcal{P}$ such that for each $1 < i \leq m$, the intersection of $\mathcal{F}_i$ with the union $\bigcup_{r<i} \mathcal{F}_r$ of the previous facets is "well-behaved": the intersection should be nonempty and "pure $(D-2)$-dimensional", i.e., for every $r < i$ there should be some $s < i$ such that $\mathcal{F}_r \cap \mathcal{F}_i \subseteq \mathcal{F}_s \cap \mathcal{F}_i$ and the latter is a $(D-2)$-dimensional face of $\mathcal{P}$.

Given such a shelling, one can define $h_j(\mathcal{P})$ as the number of facets $\mathcal{F}_i$ such that exactly j of the $(D-2)$-dimensional faces of $\mathcal{F}_i$ are already contained in some earlier $\mathcal{F}_r$. (Observe that since $\mathcal{F}_i$ is a $(D-1)$-dimensional simplex and hence has a total of D faces of dimension $D-2$, the interesting range for j is again between 0 and D.)

Since the vertices of $\mathcal{P}^*$ are in bijection with the facets of $\mathcal{P}$, any generic linear functional c in the polar set-up induces an ordering on the facets of $\mathcal{P}$, and as it turns out, this is a shelling of $\mathcal{P}$ and the two definitions of $h_j(\mathcal{P})$ agree. For an introduction to shellings of polytopes (including a definition for general polytopes, or still more generally, for polytopal complexes) and an overview of their applications and many references, see Chapter 8 of Ziegler [91], which also contains an account of McMullen's proof of the

Theorem 4.2 (Upper Bound Theorem [52]). *The h-vector of a simplicial d-dimensional polytope $\mathcal{P}$ on n vertices satisfies*

$$h_j(\mathcal{P}) \leq \min\left\{ \binom{n-D-1+j}{n-D-1}, \binom{n-j-1}{n-D-1} \right\} \tag{4.6}$$

for all j. Moreover, if equality is attained for some $\lfloor D/2 \rfloor \leq j \leq D$, then $\mathcal{P}$ is neighborly, *i.e., every set of at most $\lfloor D/2 \rfloor$ vertices span a face of $\mathcal{P}$, and then (4.6) holds with equality for all j.*

In order to see what h-vectors have to do with the probabilities $f_k(\mu, \boldsymbol{o})$, we need furthermore the following form of duality.

4.1.2 Gale Duality

Gale Duality for Vector Configurations. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbf{R}^r$ be (not necessarily distinct) vectors that linearly span $\mathbf{R}^r$. Consider the $\boldsymbol{v}_i$'s as columns of an $(r \times n)$-matrix

$$A := \begin{bmatrix} | & | & & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \\ | & | & & | \end{bmatrix}.$$

Suppose H is a oriented hyperplane $\{\boldsymbol{x} \in \mathbf{R}^r : \langle \boldsymbol{\nu}, \boldsymbol{x} \rangle = 0\}$ through the origin. We can encode the way H partitions the vectors by writing down the sequence $(\langle \boldsymbol{\nu}, \boldsymbol{v}_1 \rangle, \ldots, \langle \boldsymbol{\nu}, \boldsymbol{v}_n \rangle) \in \mathbf{R}^n$ of scalar products: $\boldsymbol{v}_i$ lies in H^+, on H, or in H^-, respectively, depending on whether $\langle \boldsymbol{\nu}, \boldsymbol{v}_i \rangle$ is positive, zero, or negative. (We could also decide to only remember the sign $\in \{+, 0, -\}$ of each of these scalar products, which would lead to the notion of *oriented matroids* [17], but let us stick with their actual values.) For any vector $\boldsymbol{\nu} \in \mathbf{R}^d$ (the zero vector $\boldsymbol{\nu} = \mathbf{0}$ is allowed, too), the vector $(\langle \boldsymbol{\nu}, \boldsymbol{v}_1 \rangle, \ldots, \langle \boldsymbol{\nu}, \boldsymbol{v}_n \rangle) \in \mathbf{R}^n$ is called a *linear valuation* of the $\boldsymbol{v}_i$'s.

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r \in \mathbf{R}^n$ denote the rows of A,

$$A = \begin{bmatrix} \text{------} & \boldsymbol{a}_1 & \text{------} \\ & \vdots & \\ \text{------} & \boldsymbol{a}_r & \text{------} \end{bmatrix}.$$

The set of all linear valuations of the $\boldsymbol{v}_i$'s is precisely the r-dimensional subspace of $\mathbf{R}^n$ spanned by these rows, or in other words, the image $\operatorname{im} A^T$ of the transpose of A.

Now, choose a basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n-r}$ of the orthogonal complement of $\operatorname{im} A^T$ in $\mathbf{R}^n$, i.e., of the kernel $\ker A$, and let B be the matrix which has these vectors as rows,

$$B := \begin{bmatrix} \text{------} & \boldsymbol{b}_1 & \text{------} \\ & \vdots & \\ \text{----} & \boldsymbol{b}_{n-r} & \text{----} \end{bmatrix}.$$

If we denote the columns of B by $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$,

$$B = \begin{bmatrix} | & | & & | \\ \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \\ | & | & & | \end{bmatrix},$$

then it is easy to see that the collection of vectors $\boldsymbol{w}_i \in \mathbf{R}^{n-r}$ is unique up to a bijective linear transformation of $\mathbf{R}^{n-r}$, and they, in turn, determine the $\boldsymbol{v}_i$'s up to a linear change of coordinates of $\mathbf{R}^r$.

The map that maps $\{\boldsymbol{v}_i : 1 \leq i \leq n\}$ to $\{\boldsymbol{w}_i : 1 \leq i \leq n\}$ and vice versa is called the *Gale transform*. Observe that this is a transformation not of individual vectors, but of whole sets (or rather multisets, since there can be repetitions) of vectors, also sometimes called *vector configurations*. The two vector configurations $\{\boldsymbol{v}_i : 1 \leq i \leq n\}$ and $\{\boldsymbol{w}_i : 1 \leq i \leq n\}$ are called *Gale duals* of each other.

By definition, we have $\operatorname{im} A^T = \ker B$ and $\ker A = \operatorname{im} B^T$. Thus, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n$ is a linear valuation of the $\boldsymbol{v}_i$'s iff $\boldsymbol{\alpha}$ is a *linear dependence* of the $\boldsymbol{w}_i$'s, i.e., $\sum_{i=1}^n \alpha_i \boldsymbol{w}_i = \mathbf{0}$, and vice versa.

Gale Diagrams of Point Configurations. The standard trick to translate affine notions into linear notions is to embed $\mathbf{R}^D$ as the hyperplane $\mathbf{R}^D \times \{1\}$ into $\mathbf{R}^{D+1}$. Then a point $\boldsymbol{q} \in \mathbf{R}^D$ corresponds to $(\boldsymbol{q}, 1) \in \mathbf{R}^{D+1}$, and an affine hyperplane $\mathrm{H} = \{\boldsymbol{q} \in \mathbf{R}^D : \langle \boldsymbol{\nu}, \boldsymbol{q} \rangle = t\}$ in $\mathbf{R}^D$ can be interpreted as the intersection of $\mathbf{R}^D \times \{1\}$ with the linear hyperplane $\{\boldsymbol{x} \in \mathbf{R}^{D+1} : \langle (\boldsymbol{\nu}, -t), \boldsymbol{x} \rangle = 0\}$, see Figure 4.1.

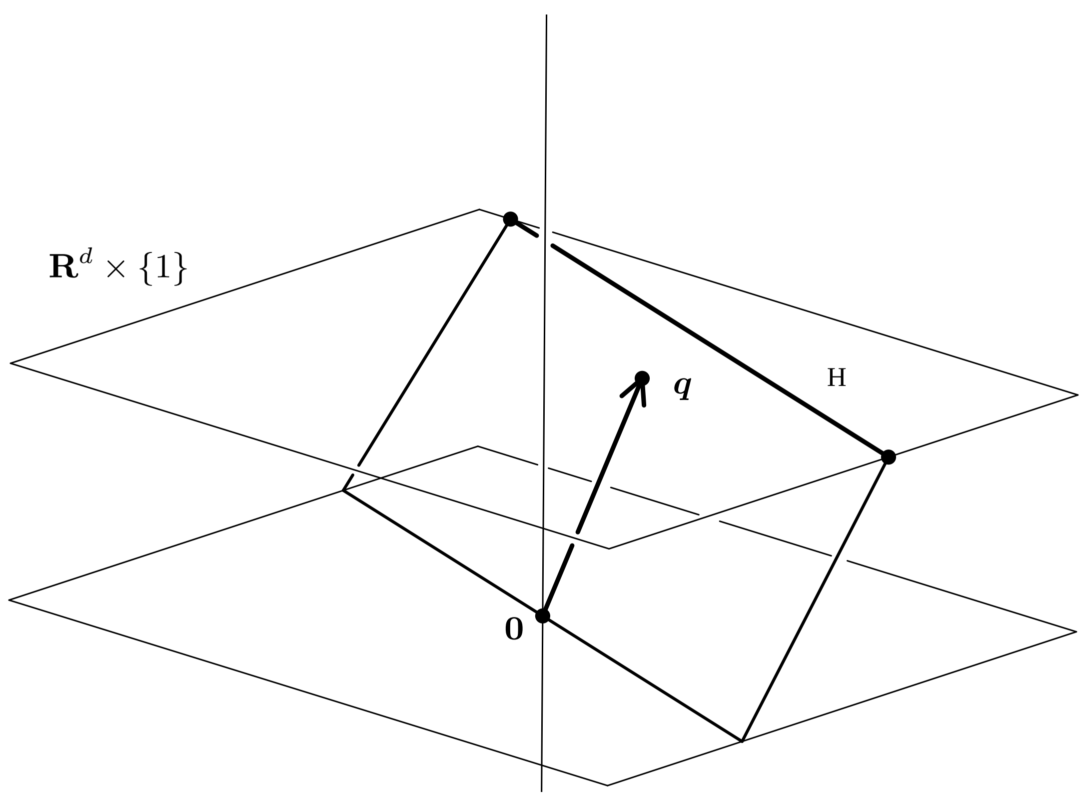


Figure 4.1: *Translating affine to linear notions.*

Given a multiset of points $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n \in \mathbf{R}^D$ that affinely span $\mathbf{R}^D$, the Gale dual of the vector configuration $(\boldsymbol{q}_1, 1), \ldots, (\boldsymbol{q}_n, 1) \in \mathbf{R}^{D+1}$ is a vector configuration $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ in $\mathbf{R}^{n-D-1}$ and is called the (linear) *Gale diagram* of the $\boldsymbol{q}_i$'s.

Observe that the Gale diagram of a point configuration has a special property: The vector $(0, \ldots, 0, 1) \in \mathbf{R}^{D+1}$ yields the linear valuation $(1, 1, \ldots, 1) \in \mathbf{R}^n$ of the vectors $(\boldsymbol{q}_i, 1)$. This translates to the linear dependence $\sum_{i=1}^n \boldsymbol{w}_i = \mathbf{0}$, i.e., the origin $\mathbf{0} \in \mathbf{R}^{n-D-1}$ is the center of gravity of the $\boldsymbol{w}_i$'s.

Let us now apply this to polytopes. Let $\mathcal{P}$ be the convex hull of $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n \in \mathbf{R}^D$. We do not assume that all $\boldsymbol{q}_i$ are vertices of the polytope $\mathcal{P}$, but we do assume that $\mathcal{P}$ is D-dimensional, i.e., that the $\boldsymbol{q}_i$'s affinely span $\mathbf{R}^D$. Let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n \in \mathbf{R}^{n-D-1}$ be the Gale diagram of the $\boldsymbol{p}_i$'s.

Suppose $I \subseteq \{1 \ldots n\}$ and the points $\boldsymbol{q}_i$, $i \in I$, lie on a *supporting* hyperplane H of $\mathcal{P}$, i.e., $\boldsymbol{q}_i \in \mathrm{H}$ for $i \in I$ and $\mathcal{P}$ is contained in one of the closed halfspaces bounded by H. If H is given as $\{\boldsymbol{q} \in \mathbf{R}^D : \langle \boldsymbol{\nu}, \boldsymbol{q} \rangle = t\}$, we get the linear valuation $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i := \langle \boldsymbol{\nu}, \boldsymbol{q}_i \rangle$ of the vectors $(\boldsymbol{q}_i, 1)$. This translates into a linear dependence $\sum_{i=1}^n \alpha_i \boldsymbol{w}_i$ for the dual configuration.

Moreover, since H is a supporting hyperplane, we may assume that $\alpha_i = 0$ for $i \in I$ and $\alpha_i \geq 0$ for $i \in \{1 \ldots n\} \backslash I$. Thus, the origin $\mathbf{0} \in \mathbf{R}^{n-D-1}$ lies in the convex hull $\operatorname{conv}\{\boldsymbol{w}_i : i \notin I\}$, as witnessed by the convex combination $\mathbf{0} = \sum_{i \notin I} \lambda_i \boldsymbol{w}_i$, where $\lambda_i = \frac{\alpha_i}{\sum_i \alpha_i}$. The remaining assertions of the following lemma are derived similarly, see [86].

Lemma 4.3. *Let $\mathcal{P}$ be the convex hull of points $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ that affinely span $\mathbf{R}^D$, and let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n \in \mathbf{R}^{n-D-1}$ be the Gale diagram of of the $\boldsymbol{q}_i$'s.*

1. *For $I \subseteq \{1 \ldots n\}$, the points $\boldsymbol{q}_i$, $i \in I$, lie on a supporting hyperplane of $\mathcal{P}$ iff the origin is contained in $\operatorname{conv}\{\boldsymbol{w}_i : i \notin I\}$*

2. *$\mathcal{P}$ is simplicial iff the origin does not lie in the convex hull of any subset of the $\boldsymbol{w}_i$'s of size less than $n - D$.*

3. *If $\mathcal{P}$ is simplicial, then for any $I \subseteq \{1 \ldots n\}$ of cardinality $|I| \leq D$, the set $\{\boldsymbol{q}_i : i \in I\}$ is the vertex set of an $(|I|-1)$-dimensional face of $\mathcal{P}$ iff the origin is contained in the interior of the convex hull of $\{\boldsymbol{w}_i : i \notin I\}$. In particular, $\boldsymbol{q}_i$ is a vertex of $\mathcal{P}$ iff the origin lies in the interior of $\{\boldsymbol{w}_l : l \neq i\}$.*

Now we are very close to the question considered at the outset of this chapter.

Definition 4.4. Let S be a set of n points in $\mathbf{R}^d$ and $\boldsymbol{o}$ a point such that $\boldsymbol{o} \in \operatorname{conv} S$, but $\boldsymbol{o}$ is not contained in the convex hull of fewer than $d+1$ points from S. Let us say in this case that $\boldsymbol{o}$ is *in general position* w.r.t. S. (this is satisfied if $S \dot{\cup} \{\boldsymbol{o}\}$ is in general position, but it is a weaker assumption). For integer k, we define

$$f_k(S, \boldsymbol{o}) := |\{X \subseteq S : |X| = d + 1 + k, \boldsymbol{o} \in \operatorname{int}(\operatorname{conv} X)\}|.$$

Observe that we could as well write $\boldsymbol{o} \in \operatorname{conv} X$, since by assumption, $\boldsymbol{o}$ cannot be contained in the boundary of $\operatorname{conv} X$. Moreover, since $\boldsymbol{o} \in \operatorname{int}(\operatorname{conv} X)$ is invariant under sufficiently small perturbations of X, we can perturb S, so as to bring $S \dot{\cup} \{\boldsymbol{o}\}$ into general position, without affecting $f_k(S, \boldsymbol{o})$.

With this notation, Lemma 4.3 says that if S is the Gale diagram of the vertex set of a D-dimensional simplicial convex polytope and if $\mathbf{0}$ is the origin in $\mathbf{R}^{n-D-1}$, then

$$f_k(S, \mathbf{0}) = f_{D-k-1}(\mathcal{P}) = f_k(\mathcal{P}^*),$$

where $\mathcal{P}^*$ is the simple polytope polar to $\mathcal{P}$.

Conversely, let $S = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\} \subseteq \mathbf{R}^d$, and let $\boldsymbol{o}$ be a point that lies in $\operatorname{conv} S$ and is in general position w.r.t. S. Note that this implies that S affinely spans $\mathbf{R}^d$ and that $\boldsymbol{o}$ lies in the interior of $\operatorname{conv} S$. It follows that there is a convex combination $\boldsymbol{o} = \sum_{i=1}^{n} \lambda_i \boldsymbol{p}_i$ with all $\lambda_i > 0$. Furthermore, up to a suitable translation, we may assume that $\boldsymbol{o}$ is the origin of $\mathbf{R}^d$ (i.e., we can interpret $\boldsymbol{p}_i \in S$ as the vector $\boldsymbol{p}_i - \boldsymbol{o}$). Under that assumption, if $\boldsymbol{o}$ lies in (the interior of) $\operatorname{conv}\{\boldsymbol{p}_i : i \in I\}$ for some $I \subseteq \{1 \ldots n\}$ and if $\alpha_i > 0$, $i \in I$, then $\boldsymbol{o}$ also lies in the (interior of) convex hull of $\{\alpha_i \boldsymbol{p}_i \colon i \in I\}$. Thus, we may rescale the elements of S at our leisure without affecting $f_k(S, \boldsymbol{o})$, and by rescaling each $\boldsymbol{p}_i$ by the factor of $\frac{1}{\lambda_i}$, with the λ_i above, we may assume that $\boldsymbol{o}$ is the center of gravity of S, i.e., $\boldsymbol{o} = \sum_{i=1}^{n} \boldsymbol{p}_i$. Therefore, if $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbf{R}^{n-d}$ is the Gale dual of the *vector* configuration $\boldsymbol{p}_i$, $1 \leq i \leq n$, then all the $\boldsymbol{v}_i$ lie on a common hyperplane $\mathrm{H} = \{\boldsymbol{x} \in \mathbf{R}^{n-d} : \langle \boldsymbol{\nu}, \boldsymbol{x} \rangle = 1\}$, and we can interpret the $\boldsymbol{v}_i$'s as points $\boldsymbol{q}_i \in \mathrm{H} \cong \mathbf{R}^{n-d-1}$. Then the $\boldsymbol{p}_i$'s are the Gale diagram of the $\boldsymbol{q}_i$'s, so Lemma 4.3 applies again. Summarizing, we arrive at the following theorem, which was proved in [86]:

Theorem 4.5. *1. If $\mathcal{P}$ is a D-dimensional simplicial polytope on $f_0(\mathcal{P}) = n$ vertices, then there exist a point set $S \subseteq \mathbf{R}^d$, $d := n - D - 1$, and a point $\boldsymbol{o} \in \mathbf{R}^d$ such that*

$$f_k(S, \boldsymbol{o}) = f_{D-k-1}(\mathcal{P}) = f_k(\mathcal{P}^*) \tag{4.7}$$

for all k. By a sufficiently small perturbation, if necessary, we may assume that $S \dot{\cup} \{\boldsymbol{o}\}$ is in general position.

2. *Conversely, if S is a set of n points in $\mathbf{R}^d$ that is non-degenerate w.r.t. a point $\boldsymbol{o} \in \operatorname{conv} S$, then there is a simplicial convex D-polytope, $D = n-d-1$, on $m \leq n$ vertices such that (4.7) holds for all k. In particular, the number $m = f_0(\mathcal{P})$ of vertices of $\mathcal{P}$ equals the number of points $\boldsymbol{p} \in S$ that can be strictly separated from $S \setminus \{p\}$ by a hyperplane through $\boldsymbol{o}$.*

In analogy to (4.2) and (4.4), we could now define the h-vector of S and $\boldsymbol{o}$ by $h_j(S, \boldsymbol{o}) := \sum_k (-1)^{k-j} \binom{k}{j} f_k(S, \boldsymbol{o})$. While this is a perfectly respectable definition, the reasons why h-vectors are useful for the study of polytopes is that they can be interpreted geometrically (via shellings of the polytope or linear objective functions on the polar polytope, as we saw above). In what follows, we describe such a geometric interpretation in the dual set-up, given by Welzl [86]. The basic approach is to first analyze the probability that a given line intersects the convex hull of n random points, which then translates to the above setting by the projection onto the hyperplane orthogonal to the line.

4.1.3 h-Vectors in the Dual Setting, and h-Functions

Let σ be an oriented $(d-1)$-dimensional simplex in $\mathbf{R}^d$. An oriented line $\ell \subseteq \mathbf{R}^d$ is said to *enter* σ if ℓ intersects the relative interior of σ in a single point and is directed from the positive to the negative side of σ. If ℓ is directed from the negative to the positive side, we say that it *leaves* σ.

Definition 4.6 (The h-vector of a point set and a line [86]). Consider a set S of n points in $\mathbf{R}^d$, and an oriented line ℓ which is disjoint from the convex hull of any subset of S of size less than d. We say in that case that ℓ is in general position w.r.t. S. For integer j, we define $h_j(S, \ell)$ as the number of j-facets of S which are entered by ℓ, and we call $h(S, \ell) := (h_0(S, \ell), \ldots, h_{n-d}(S, \ell))$ the *h-vector* of S and ℓ.

In order to define the corresponding concept for probability distributions, we need the notion of a *μ-random simplex*. Recall that orienting a $(d-1)$-dimensional simplex σ in $\mathbf{R}^d$ simply means to specify one of the halfspaces bounded by the hyperplane $\operatorname{aff} \sigma$ as positive. In dimension $d \geq 2$, if the simplex is spanned by affinely independent points $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d \in \mathbf{R}^d$, then an orientation is given in a natural way by the order in which the points come: simply define the positive halfspace as

$$\mathrm{H}^+([\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d]) := \Big\{\boldsymbol{q} \in \mathbf{R}^d : \det \begin{bmatrix} 1 & \cdots & 1 & 1 \\ \boldsymbol{p}_1 & & \boldsymbol{p}_d & \boldsymbol{q} \end{bmatrix} > 0\Big\}.$$

We denote the resulting oriented simplex by $[\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d]$. In dimension $d = 1$, we need an additional sign $\varepsilon \in \{+, -\}$ and set $\varepsilon[p]$ to be the oriented simplex with $\mathrm{H}^+(\varepsilon[p]) := \{q \in \mathbf{R}^1 : \varepsilon \cdot (q - p) > 0\}$.

Equipped with this notation, we define, for $d \geq 2$, a *μ-random (oriented, $(d-1)$-dimensional) simplex* as the oriented simplex $[P_1, \ldots, P_d]$ spanned by independent μ-random points $P_1, \ldots, P_d$ (by assumption on μ, the points are a.s. affinely independent). In dimension $d = 1$, we choose an additional independent random sign ε uniformly from $\{-1, +1\}$ and obtain the random simplex $\varepsilon[P_1]$.

In analogy to Definition 4.6, we would now like to define, for a real number $0 \leq y \leq 1$, $h_{\mu,\ell}(y)$ as the probability that a μ-random simplex σ is a *y-facet* of μ, i.e., $\mu(\mathrm{H}^+(\sigma)) = y$, and is entered by ℓ. Unfortunately, that probability will be zero for every y. This technical nuisance is remedied by first defining a distribution function and then taking the derivative.

Definition 4.7 (h-functions [84]). For a continuous probability distribution and an oriented line ℓ in $\mathbf{R}^d$, the function $H_{\mu,\ell} : [0, 1] \to [0, 1]$ is given by

$$H_{\mu,\ell}(y) := \Pr[\ell \text{ enters } \sigma \text{ and } \mu(\mathrm{H}^+(\sigma)) \leq y], \tag{4.8}$$

where σ is a μ-random oriented $(d-1)$-dimensional simplex. (Note that the map $(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d) \mapsto \mu(\mathrm{H}^+([\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d]))$ is continuous, hence measurable, on the open set of affinely independent d-tuples $(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d) \in \mathbf{R}^{d \times d}$ such that ℓ enters $[\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d]$.)

Clearly, $H_{\mu,\ell}$ is monotone, from which it follows that its derivative, the *h-function*

$$h_{\mu,\ell}(y) := \frac{\mathrm{d}}{\mathrm{d}y} H_{\mu,\ell}(y) \tag{4.9}$$

of μ and ℓ is defined almost everywhere (a.e., for short, i.e., the set of $y \in [0, 1]$ for which it is not defined has Lebesgue measure zero) and nonnegative. We note that h is, in fact, defined everywhere and continuous, as we will see in Theorem 4.13 below.

Furthermore, in the set-up of the above definitions and for integer k, let

$$f_k(S, \ell) := |\{X \subseteq S : |X| = d + k, \text{ and } \ell \text{ intersects } \operatorname{conv} X\}|,$$

and

$$f_k(\mu, \ell) := \Pr[\ell \text{ intersects } \operatorname{conv}\{P_1, \ldots, P_{d+k}\}],$$

where $P_1, P_2, P_3, \ldots$ are independent μ-distributed random points. In analogy to (4.3), we have for either orientation of ℓ,

$$f_k(S, \ell) = \sum_j \binom{j}{k} h_j(S, \ell) \tag{4.10}$$

for all k, as was shown in [86]. The continuous counterpart was proved in [84] and reads

$$f_k(\mu, \ell) = 2\binom{d+k}{d} \int_0^1 y^k h_{\mu,\ell}(y)\mathrm{d}y. \tag{4.11}$$

As before, it follows from (4.10) that the h-vector $h(S, \ell)$ is uniquely determined by $(f_0(S, \ell), \ldots, f_{n-d}(S, \ell))$ via $h_j(S, \ell) = \sum_k (-1)^{j-k}\binom{k}{j} f_k(S, \ell)$. Similarly, in the continuous case, the function $h_{\mu,\ell}$ can be shown to be uniquely determined by the sequence $f_k(\mu, \ell)$, $k \in \mathbf{N}_0$, which up to constant factors depending only on d and k is just the sequence of its *moments*. Roughly speaking, if the k^{th} moment $\int_{\mathbf{R}} y^k f(y)\mathrm{d}y$ of an integrable function f exists, then it is (up to constant factors) the k^{th} derivative at zero of the Fourier transform $\hat{f}(x) = \int_{\mathbf{R}} e^{ixy} f(y)\mathrm{d}y$. Under suitable niceness assumptions on f, the Taylor series of $\hat{f}$ converges, so the moments determine $\hat{f}$, which in turn determines f. We refer to [33] for a proof of these facts in a more general context. See also Remark 4.14 below.

Since $f_k(S, \ell)$, respectively $f_k(\mu, \ell)$ are independent of the orientation of ℓ, it follows that the same holds for $h(S, \ell)$, respectively $h_{\mu,\ell}$, which proves the following

Theorem 4.8 (Dehn-Sommerville Equations [86], [84]). *For S, μ, and ℓ in R^d as above, $|S| = n$, we have*

$$h_j(S, \ell) = h_{n-d-j}(S, \ell)$$

for all j, and

$$h_{\mu,\ell}(y) = h_{\mu,\ell}(1 - y)$$

for all $0 \leq y \leq 1$.

We are now ready to relate h-functions to the probabilities $f_k(\mu, \boldsymbol{o})$ which we considered at the beginning of this chapter.

Projections and Liftings. Consider a line $\tilde{\ell}$ in $\mathbf{R}^{d+1}$ (we mark objects in $\mathbf{R}^{d+1}$ with a tilde to tell them apart from those in $\mathbf{R}^d$). We identify $\mathbf{R}^d$ with the orthogonal complement $\tilde{\ell}^\perp$. Let π be the orthogonal projection $\mathbf{R}^{d+1} \to \tilde{\ell}^\perp \cong \mathbf{R}^d$.

If $\tilde{\mu}$ is a probability distribution in $\mathbf{R}^{d+1}$, then we call the probability distribution $\mu := \pi(\tilde{\mu})$ in $\mathbf{R}^d$ together with the point $\boldsymbol{o} := \pi(\tilde{\ell}) \in \mathbf{R}^d$ the *projection* of $\tilde{\mu}$ and $\tilde{\ell}$ and write $\pi(\tilde{\mu}, \tilde{\ell})$ for the pair $(\mu, \boldsymbol{o})$. Analogously, if $\tilde{S}$ is a point set in $\mathbf{R}^{d+1}$ and $S := \pi(\tilde{S}) \subseteq \mathbf{R}^d$, then we call the pair $(S, \boldsymbol{o})$ the projection of $\tilde{S}$ and $\tilde{\ell}$ and denote it by $\pi(\tilde{S}, \tilde{\ell})$. (Note that μ, S, and $\boldsymbol{o}$ are only defined up to an affine change of coordinates of $\mathbf{R}^d$, but all notions we will study are invariant under such transformations.) Observe that if $\tilde{\mu}$ is continuous, then so is μ. Similarly, if $\tilde{\ell}$ is in general position w.r.t. $\tilde{S}$ then $\boldsymbol{o}$ is in general position w.r.t. S.

Conversely, given $\boldsymbol{o} \in \mathbf{R}^d$ and a probability distribution μ (respectively, a point set S) in $\mathbf{R}^d$, a *lifting* of μ and $\boldsymbol{o}$ (respectively, of S and $\boldsymbol{o}$) is any line $\tilde{\ell} \subseteq \mathbf{R}^{d+1}$ together with a probability distribution $\tilde{\mu}$ (respectively, a point set $\tilde{S}$) in $\mathbf{R}^{d+1}$ such that $(\mu, \boldsymbol{o}) = \pi(\tilde{\mu}, \tilde{\ell})$ (respectively, $(S, \boldsymbol{o}) = \pi(\tilde{S}, \tilde{\ell})$).

If μ is continuous and $\boldsymbol{o}$ is in general position w.r.t. S, then there are suitable liftings with the same properties: Let $\tilde{\ell} = \{\boldsymbol{o}\} \times \mathbf{R} \subseteq \mathbf{R}^{d+1}$. Choose your favorite continuous probability distribution ν on $\mathbf{R}$ and take $\tilde{\mu}$ as the product measure $\mu \times \nu$ on $\mathbf{R}^d \times \mathbf{R} = \mathbf{R}^{d+1}$; respectively, pick independent ν-random numbers $t_{\boldsymbol{p}}$, $\boldsymbol{p} \in S$ and set $\tilde{S} := \{(\boldsymbol{p}, t_{\boldsymbol{p}}) \in \mathbf{R}^{d+1} : \boldsymbol{p} \in S\}$. We will only consider such "generic" liftings in what follows.

Assume now that $(\mu, \boldsymbol{o}) = \pi(\tilde{\mu}, \tilde{\ell})$, respectively, $(S, \boldsymbol{o}) = \pi(\tilde{S}, \tilde{\ell})$. We have

$$f_k(S, \boldsymbol{o}) = f_k(\tilde{S}, \tilde{\ell}).$$

and

$$f_k(\mu, \boldsymbol{o}) = f_k(\tilde{\mu}, \tilde{\ell})$$

for all $k \geq 0$. Therefore, by (4.10) and (4.11), respectively, we get for either orientation of $\tilde{\ell}$ that

$$f_k(S, \boldsymbol{o}) = \sum_j \binom{j}{k} h_j(\tilde{S}, \tilde{\ell}). \tag{4.12}$$

and

$$f_k(\mu, \boldsymbol{o}) = 2\binom{d+1+k}{d+1} \int_0^1 y^k h_{\tilde{\mu},\tilde{\ell}}(y)\mathrm{d}y. \tag{4.13}$$

Thus, $h(\tilde{S}, \tilde{\ell})$, respectively $h_{\tilde{\mu},\tilde{\ell}}$ depend only on S, respectively μ, and $\boldsymbol{o}$.

Definition 4.9. For a finite point set S, respectively a distribution μ, and a point $\boldsymbol{o}$ in $\mathbf{R}^d$ as above, we define

$$h_j(S, \boldsymbol{o}) := h_j(\tilde{S}, \tilde{\ell})$$

and

$$h_{\mu,\boldsymbol{o}}(y) := h_{\tilde{\mu},\tilde{\ell}}(y),$$

for arbitrary liftings $(\tilde{S}, \tilde{\ell})$ and $(\tilde{\mu}, \tilde{\ell})$ of S, respectively μ, and $\boldsymbol{o}$.

By (4.13), a pointwise upper bound for the h-function of a probability distribution and a line implies an upper bound for the probabilities $f_k(\mu, \boldsymbol{o})$ considered at the beginning of this chapter. The proof of the the upper bound proceeds by induction on the dimension and uses the following notions, which we will also need in Section 4.3.

4.1.4 The Upper Bound Theorem, Discrete and Continuous

Let $\ell \subseteq \mathbf{R}^d$ be an oriented line, $\boldsymbol{o}$ a point on ℓ, and σ an oriented $(d-1)$-dimensional simplex. We say that ℓ *enters* σ *before (respectively, after)* $\boldsymbol{o}$ if ℓ enters σ and $\boldsymbol{o} \in \mathrm{H}^-(\sigma)$ (respectively, $\boldsymbol{o} \in \mathrm{H}^+(\sigma)$).

Definition 4.10 (h^* and $^*h$). Let ℓ be an oriented line in $\mathbf{R}^d$ and $\boldsymbol{o} \in \ell$.

1. Suppose that $S \subseteq \mathbf{R}^d$, $|S| = n$, and that $\boldsymbol{o}$ is not contained in the convex hull of fewer than $d+1$ points from S and that ℓ is not intersected by the convex hull of fewer than d points from S. Then we define $h_j^* = h_j^*(S, \ell, \boldsymbol{o})$ and $^*h_j = {}^*h_j(S, \ell, \boldsymbol{o})$, as the number of j-facets of S that are entered by ℓ before, respectively, after, $\boldsymbol{o}$.

2. Similarly, for a continuous probability distribution μ in $\mathbf{R}^d$, we set

$$H^*(y) = H^*_{\mu,\ell,\boldsymbol{o}}(y) := \Pr[\ell \text{ enters } \sigma \text{ before } \boldsymbol{o} \text{ and } \mu(\mathrm{H}^+(\sigma)) \leq y].$$

   for $0 \leq y \leq 1$. As before, the derivative

$$h^*(y) = h^*_{\mu,\ell,\boldsymbol{o}}(y) := \frac{\mathrm{d}}{\mathrm{d}y} H^*_{\mu,\ell,\boldsymbol{o}}(y)$$

   is defined a.e. and nonnegative. The functions $^*H$ and $^*h$ are defined analogously, with "before" replaced by "after".

Note that by our assumptions about general position,

$$h_j(S,\ell) = h_j^*(S,\ell,\boldsymbol{o}) + {}^*h_j(S,\ell,\boldsymbol{o})$$

and

$$h_{\mu,\ell}(y) = h^*_{\mu,\ell,\boldsymbol{o}}(y) + {}^*h_{\mu,\ell,\boldsymbol{o}}(y).$$

Again, the moments of h^* and ${}^*h$ can be interpreted geometrically. We say that ℓ *passes into* (respectively, *exits from*) a compact convex set $\mathcal{C}$ *before* $\boldsymbol{o}$ if, while walking along ℓ in the direction in which it is oriented, we encounter the first (respectively, last) point of intersection of ℓ and $\mathcal{C}$ before $\boldsymbol{o}$.

Consider a finite set X such that ℓ enters $\operatorname{conv} X$ before $\boldsymbol{o}$. Either ℓ also exists from $\operatorname{conv} X$ before $\boldsymbol{o}$, or $\boldsymbol{o} \in \operatorname{conv} X$. For lack of a better notation, let us define

$$s_k(S,\ell,\boldsymbol{o}) := |\{X \subseteq S : |X| = d+1+k,\ \ell \text{ passes into } \operatorname{conv} X \text{ before } \boldsymbol{o}\}|$$

and

$$t_k(S,\ell,\boldsymbol{o}) := |\{X \subseteq S : |X| = d+1+k,\ \ell \text{ exits from } \operatorname{conv} X \text{ before } \boldsymbol{o}\}|.$$

Analogously,

$$s_k(\mu,\ell,\boldsymbol{o}) := \Pr[\ell \text{ passes into } \operatorname{conv}\{P_1,\dots,P_{d+1+k}\} \text{ before } \boldsymbol{o}\}$$

and

$$t_k(\mu,\ell,\boldsymbol{o}) := \Pr[\ell \text{ exits from } \operatorname{conv}\{P_1,\dots,P_{d+1+k}\} \text{ before } \boldsymbol{o}\},$$

where the P_i's are independent μ-random points. Then,

$$f_k(S,\boldsymbol{o}) = s_k(S,\ell,\boldsymbol{o}) - t_k(S,\ell,\boldsymbol{o}) \tag{4.14}$$

and

$$f_k(\mu,\boldsymbol{o}) = s_k(\mu,\ell,\boldsymbol{o}) - t_k(\mu,\ell,\boldsymbol{o}). \tag{4.15}$$

Observe that since the left-hand sides are independent of ℓ, so are the right.

The last link is provided by the following lemma. For h-vectors, it was proved in [86], and for h-functions in [84].

Lemma 4.11. *For all $k \geq 0$,*

$$s_k(S,\ell,\boldsymbol{o}) = \sum_j \binom{n-d-j}{k+1} h_j^*(S,\ell,\boldsymbol{o})$$

and

$$t_k(S,\ell,\boldsymbol{o}) = \sum_j \binom{j}{k+1} h_j^*(S,\ell,\boldsymbol{o}).$$

Similarly, in the continuous case,

$$s_k(\mu,\ell,\boldsymbol{o}) = 2\binom{d+1+k}{d} \int_0^1 (1-y)^{k+1} h^*(y)\mathrm{d}y$$

and

$$t_k(\mu,\ell,\boldsymbol{o}) = 2\binom{d+1+k}{d} \int_0^1 y^{k+1} h^*(y)\mathrm{d}y.$$

By substituting this into (4.14) and (4.15), respectively, and applying telescopic summation and integration by parts, respectively, we obtain

$$f_k(S,\boldsymbol{o}) = \sum_j \binom{j}{k} \left(\sum_{i=0}^{j} h_i^*(S,\ell,\boldsymbol{o}) - h_{n-d-i}^*(S,\ell,\boldsymbol{o}) \right) \tag{4.16}$$

and

$$f_k(\mu,\boldsymbol{o}) = 2\binom{d+1+k}{d} \int_0^1 y^k \left((d+1) \int_0^y h^*(x) - h^*(1-x)\,\mathrm{d}x \right) \mathrm{d}y. \tag{4.17}$$

By comparing (4.16) with (4.12), we conclude (see [86]):

Theorem 4.12. *If S is a set of n points in $\mathbf{R}^d$ and $\boldsymbol{o}$ is a generic point w.r.t. S, then for any choice of a generic line ℓ through $\boldsymbol{o}$, we have*

$$h_j(S,\boldsymbol{o}) = \sum_{i=0}^{j} \left(h_i^*(S,\ell,\boldsymbol{o}) - h_{n-d-i}^*(S,\ell,\boldsymbol{o}) \right)$$

for all j.

Similarly, (4.17) and (4.13) together with the uniqueness of moments imply (see [84])

Theorem 4.13. *Let μ be a continuous probability distribution and $\boldsymbol{o}$ be a point in $\mathbf{R}^d$. Then for any line ℓ through $\boldsymbol{o}$,*

$$h_{\mu,\boldsymbol{o}}(y) = (d+1) \int_0^y \left(h_{\mu,\ell,\boldsymbol{o}}^*(x) - h_{\mu,\ell,\boldsymbol{o}}^*(1-x) \right) \mathrm{d}x \tag{4.18}$$

for $0 \le y \le 1$. In particular, h is continuous and differentiable a.e.

Remark 4.14. In the continuous setting, there is a technical issue which deserves a brief comment. Namely, even though a monotone function F is differentiable a.e. and its derivative is a Lebesgue integrable function, F need not be the integral of its derivative. For instance, even for non-vanishing monotone F, the derivative might be zero a.e. (see the well-known example of the *Cantor function* in Section 1.5 of [39]. Strictly speaking, the formulae in Lemma 4.11 and (4.11) for the moments of h and h^* are correct only if we know that such problems do not arise for H or H^*. Yet we apply these formulae in order to conclude continuity of h, for instance, which seems to be begging the question.

The way to navigate around these difficulties is to define the moments as Lebesgue-Stieltjes integrals with respect to the "distribution functions" H or H^*. The above line of reasoning, properly rephrased, then establishes certain identities for these distribution functions, from which it can be concluded that they and their derivatives behave "nicely", i.e., that the above-mentioned pathologies do not occur. See [84] for the details.

Based on these findings, one can now give an alternative proof of the UBT by induction on the dimension. This gives the following tight upper bounds on the entries of the h-vector, respectively the values of the h-function, see [86], respectively [84].

Theorem 4.15 (Discrete Upper Bound Theorem (UBT)[86]). *Let S be a set of n points in $\mathbf{R}^d$, and let $\boldsymbol{o} \in \mathbf{R}^d$ be generic w.r.t. S. Then*

$$h_j(S, \boldsymbol{o}) \leq \min\left\{ \binom{j+d}{d}, \binom{n-d-1-j}{d} \right\}$$

Moreover, equality is attained for all j if and only if every hyperplane through $\boldsymbol{o}$ and disjoint from S has at least $\lfloor \frac{n-d+1}{2} \rfloor$ points of S on either side. More precisely, if there is a hyperplane $\mathrm{H} \ni \boldsymbol{o}$ disjoint from S such that $|\mathrm{H}^- \cap S| = a$, say, then for $0 \leq j \leq \frac{n-d-1}{2}$,

$$h_j(S, \boldsymbol{o}) \leq \binom{j+d}{d} - \binom{j-a+d}{d}.$$

Theorem 4.16 (Continuous Upper Bound Theorem (CUBT) [84]). *If $\boldsymbol{o} \in \mathbf{R}^d$ and μ is a continuous probability distribution in $\mathbf{R}^d$, then*

$$h_{\mu,\boldsymbol{o}}(y) \leq \frac{d+1}{2} \min\{y^d, (1-y)^d\}$$

Moreover, equality is attained for all y if and only if μ is balanced about $\boldsymbol{o}$, i.e., if every hyperplane through $\boldsymbol{o}$ equipartitions μ. More precisely, if there is a hyperplane $\mathrm{H} \ni \boldsymbol{o}$ such that $\mu(\mathrm{H}^-) = a$, say, then for $0 \le y \le 1/2$,

$$h(y) \le \begin{cases} \frac{d+1}{2} y^d & \text{if } 0 \le y \le a, \text{ and} \\ \frac{d+1}{2}\left(y^d - (y-a)^d\right) & \text{if } a \le y \le \frac{1}{2}. \end{cases}$$

Remark 4.17. It can be shown that μ is balanced about $\boldsymbol{o}$ if and only if its radial projection $\breve{\mu}$ onto the unit sphere centered at $\boldsymbol{o}$ is symmetric about $\boldsymbol{o}$, i.e. invariant under reflection about $\boldsymbol{o}$. In dimension $d \le 2$, this is rather trivial; for $d \ge 3$, we refer to Schneider [67], Corollary 3.4.

Corollary 4.18.

$$f_k(S, \boldsymbol{o}) \le \sum_{j=0}^{\lfloor \frac{n-d-1}{2} \rfloor} \binom{j}{k}\binom{j+d}{d} + \sum_{j=0}^{\lceil \frac{n-d-2}{2} \rceil} \binom{n-d-1-j}{k}\binom{j+d}{d}.$$

Equality is achieved if and only if every hyperplane through $\boldsymbol{o}$ has at least $\lfloor \frac{n-d+1}{2} \rfloor$ points of S on either side.

$$f_k(\mu, \boldsymbol{o}) \le \frac{\sum_{i=0}^{k} \binom{d+k}{i}}{2^{d+k}}.$$

Equality is achieved if and only if μ is balanced about $\boldsymbol{o}$.

We will further investigate f_k in Section 4.4.

4.2 Approximate Point Masses

This section is a somewhat technical interlude that lays the groundwork for Section 4.3. The goal is to prove Lemma 4.22, which enables us to analyze individual values of a Lebesgue integrable function $f : [0,1] \to \mathbf{R}$ in terms of integrals of the form $\int_0^1 x^j (1-x)^k f(x) \mathrm{d}x$ (provided these integrals exist). Note that for $k = 0$, these are precisely the moments of f.

We will need the following facts (see, for instance, [65]): For real numbers $\alpha, \beta \ge 0$,

$$\int_0^1 x^\alpha (1-x)^\beta dx = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!},$$

where the *generalized factorials* are defined by

$$\alpha! := \int_0^\infty x^\alpha e^{-x}\, dx.$$

(The integral on the right, which is also often denoted by $\Gamma(\alpha+1)$, converges for $\alpha > -1$.) For natural numbers this definition agrees with the usual inductive one. Also, the familiar relation $(\alpha+1)! = (\alpha+1)\,\alpha!$ still holds. Moreover, *Stirling's formula* provides a useful asymptotic estimate:

$$\alpha! \sim \frac{\alpha^\alpha}{e^\alpha}\sqrt{2\pi\alpha}$$

as $\alpha \to \infty$, where $\phi \sim \psi$ means that $\lim \frac{\phi}{\psi} = 1$.

Definition 4.19. For $\alpha, \beta \geq 0$, define

$$\delta_{\alpha,\beta}(x) := \frac{(\alpha+\beta+1)!}{\alpha!\,\beta!} x^\alpha (1-x)^\beta$$

for $0 \leq x \leq 1$.

By definition, $\delta_{\alpha,\beta} \geq 0$ and $\int_0^1 \delta_{\alpha,\beta}(x)dx = 1$ for all $\alpha, \beta \geq 0$. Thus, each $\delta_{\alpha,\beta}$ is a probability density on the unit interval. Moreover, intuitively speaking, if $\alpha, \beta \to \infty$ and if the fractions $\frac{\alpha}{\alpha+\beta}$ converge to some number y in the unit interval, then the distributions $\delta_{\alpha,\beta}$ become more and more concentrated around y, see Figure 4.2.

To make this idea precise, we need some preparations. Fix $y \in (0,1)$. For $t > 0$, let $\alpha = \alpha(y,t) := ty$ and $\beta = \beta(y,t) := t(1-y)$. Given $x \in (0,1)$ and $r := x - y$, let us write $x = (1 + \frac{r}{y})y$ and $(1-x) = (1 - \frac{r}{1-y})(1-y)$. Thus,

$$\delta_{\alpha,\beta}(x) = \underbrace{\frac{(t+1)\,t!}{\alpha!\,\beta!}\frac{\alpha^\alpha\beta^\beta}{t^t}}_{(*)} \Bigg(\underbrace{\left(1+\frac{r}{y}\right)^y \left(1-\frac{r}{1-y}\right)^{1-y}}_{(**)} \Bigg)^t. \tag{4.19}$$

For a real number $p \geq 1$, the function $x \mapsto x^p$ is convex for $x \in [0,\infty)$. Hence, $1 + pr \leq (1+r)^p$ for all $r \in [-1,\infty)$. Applying this with $p = \frac{1}{y}$ and $p = \frac{1}{1-y}$, respectively, we see that $(**) \leq (1+r)(1-r) = 1 - r^2$. Moreover, by Stirling's formula,

$$(*) \sim \frac{(t+1)\sqrt{2\pi t}}{\sqrt{2\pi\alpha}\sqrt{2\pi\beta}} \sim \frac{\sqrt{t}}{\sqrt{2\pi\, y(1-y)}} \tag{4.20}$$

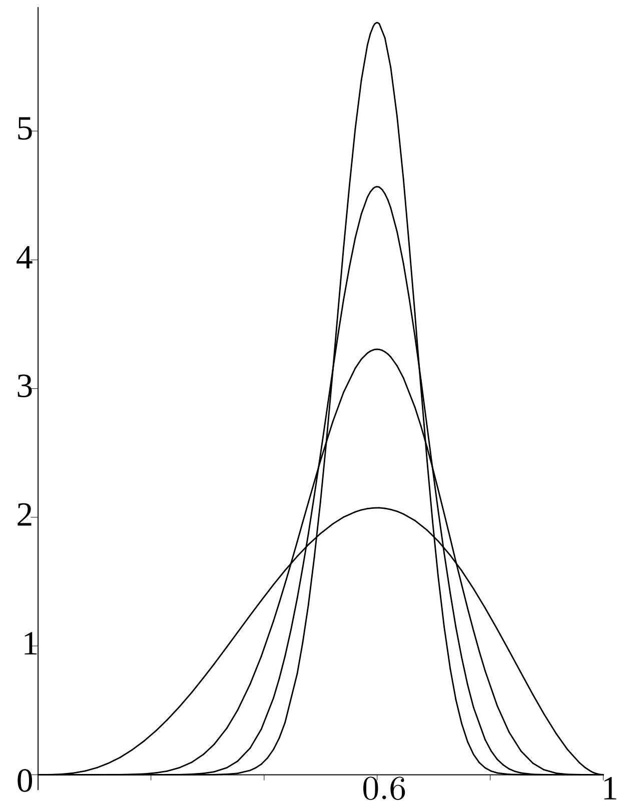


Figure 4.2: *The functions $\delta_{\alpha,\beta}$ for $(\alpha, \beta) = (3, 2), (9, 6), (18, 12), (30, 20)$.*

as $t \to \infty$.

We will restrict ourselves to showing that for this choice of the parameters α, β, the distributions $\delta_{\alpha,\beta}$ converge to the point mass at y (in a sense made precise below). The crucial ingredient is the following special case of the Lebesgue Differentiation Theorem (see [39], Chapter 3):

Fact 4.20. *Let f be a Lebesgue integrable function on the interval $[0, 1]$, that is, a measurable function such that $\|f\|_1 := \int_0^1 |f(x)|\,dx < \infty$. Then almost every $y \in [0, 1]$ is a* Lebesgue point *of f, in the following sense: For every $\varepsilon > 0$, there is some $\rho = \rho(\varepsilon, y) > 0$ such that, for all $r < \rho$,*

$$\frac{1}{r} \int_{|x-y|<r} |f(x) - f(y)|\,dx < \varepsilon. \tag{4.21}$$

(That is, if we average the difference $|f(x) - f(y)|$ over a small interval of radius r around y, the result tends to zero as $r \to 0$.)

Lemma 4.21. *Suppose that $y \in (0,1)$ is a Lebesgue point of an integrable function f. If $\alpha = ty$ and $\beta = t(1-y)$ as above, then*

$$\int_0^1 |f(y) - f(x)|\delta_{\alpha,\beta}(x)\,dx \to 0 \tag{4.22}$$

as $t \to \infty$.

Proof. Fix some $\varepsilon > 0$, and let $\rho > 0$ be such as asserted in Fact 4.20. We denote the integral on the left-hand side of (4.22) by I; we decompose I into a number of integrals which we can handle separately: First, set $s := \lceil \log_2(\rho\sqrt{t})\rceil - 1$ (i.e., $\rho/2 \le 2^s/\sqrt{t} < \rho$). Next, let $r_{-1} := 0$, $r_i := 2^i/\sqrt{t}$, for $i \in \{0 \dots s\}$, and $r_{s+1} := \infty$. Finally, define $A_i := \{x \in (0,1) : r_{i-1} \le |x - y| < r_i \text{ for } 0 \le i \le s+1\}$. Then $I = I_0 + I_1 + \ldots + I_{s+1}$, where

$$I_i := \int_{A_i} |f(y) - f(x)|\delta_{\alpha,\beta}(x)\,dx. \tag{4.23}$$

From (4.20) we infer that for sufficiently large t, the quantity $(**)$ from Equation (4.19) is at most $C\sqrt{t}$ (where C and the meaning of "sufficiently large" depend only on y). Hence,

$$\delta_{\alpha,\beta}(x) \le \begin{cases} C\sqrt{t}(1 - r_{i-1})^t, & x \in A_i, i \in \{0 \dots s\}, \text{ and} \\ C\sqrt{t}(1 - \rho^2/4)^t, & x \in A_{s+1}. \end{cases} \tag{4.24}$$

Thus, by (4.21),

$$I_0 \le C\sqrt{t} \underbrace{\int_{A_0} |f(x) - f(y)|\,dx}_{\le \varepsilon r_0 = \varepsilon/\sqrt{t}} \le C\varepsilon. \tag{4.25}$$

Similarly, for $i \in \{1 \dots s\}$,

$$I_i \le C\sqrt{t}\left(1 - \frac{2^{2(i-1)}}{t}\right)^t \frac{2^i}{\sqrt{t}}\varepsilon \le C\varepsilon\, 2^i e^{-2^{2(i-1)}}. \tag{4.26}$$

Finally,

$$I_{s+1} \le C\sqrt{t}\left(1 - \frac{\rho^2}{4}\right)^t 2\|f\|_1. \tag{4.27}$$

Since $\rho > 0$ and $\|f\|_1 < \infty$, this last term tends to zero as $t \to \infty$. In particular, it will be less than $C\varepsilon$ if t is sufficiently large (in terms of f, y and

ε). Then, (4.25), (4.26), and (4.27) together yield

$$I \le C\varepsilon \left(2 + \sum_{i=1}^{s} 2^i e^{-2^{2(i-1)}} \right). \tag{4.28}$$

The series $\sum_{i=1}^{\infty} 2^i e^{-2^{2(i-1)}}$ converges. Therefore, since C depends only on y and since ε is arbitrarily small, we conclude that $I \to 0$ as $t \to \infty$. $\square$

Since $\int_0^1 \delta_{\alpha,\beta}(x)\,dx = 1$, we have $\left| f(y) - \int_0^1 f(x)\delta_{\alpha,\beta}(x)\,dx \right| \le \int_0^1 |f(y) - f(x)|\delta_{\alpha,\beta}(x)\,dx$. Thus, it follows from Lemma 4.21 that for every Lebesgue point y of f, $\int_0^1 f(x)\delta_{\alpha,\beta}(x)\,dx \to f(y)$ as $t \to \infty$. For a geometric interpretation of these approximating integrals, we will now replace α and β by suitable integers.

Lemma 4.22. *For every Lebesgue point $y \in (0,1)$ of f (in particular, for almost every y), there exist sequences $(j(\nu))_{\nu\in\mathbf{N}}$ and $(k(\nu))_{\nu\in\mathbf{N}}$ of positive integers, such that $\frac{j(\nu)}{j(\nu)+k(\nu)} \to y$, $j(\nu), k(\nu) \to \infty$, and*

$$\int_0^1 f(x)\delta_{j(\nu),k(\nu)}(x)dx \to f(y)$$

as $\nu \to \infty$.

Proof. Fix y. For each $\nu \in \mathbf{N}$, choose a large *integer* $t = t(\nu)$ such that $t(\nu) \to \infty$ and that the distance between $\alpha = ty$ and the nearest integer is less than ν^{-1}; let $j = j(\nu)$ be that nearest integer, and define $k = k(\nu) := t - j$. Clearly, $\lim \frac{j}{j+k} = y$.

We have to show that $\int_0^1 |f(x) - f(y)|\delta_{j(\nu),k(\nu)}(x)dx \to 0$ as $\nu \to \infty$. As before, we decompose this integral into two parts, which we handle separately. For this purpose, fix some parameter $r > 0$ such that $y - r > 0$ and $y + r < 1$. On the one hand, let $A = \{x \in (0,1) : |x - y| > r\}$. Then, for sufficiently large ν, we have $|x - \frac{j}{j+k}| > r/2$ for all $x \in A$. Therefore, by (4.19) and (4.20), $\delta_{j,k}(x) \le (1 - r^2/4)^t (t+1)\sqrt{t}$. This latter expression converges to zero as ν (and hence t) tends to infinity. Therefore,

$$\int_A |f(x) - f(y)|\delta_{j,k}(x)\,dx \to 0$$

as $\nu \to \infty$. On the other hand, consider the set $(0,1) \setminus A = [y-r, y+r]$. On this compact interval, the ratio

$$\frac{\delta_{j,k}(x)}{\delta_{\alpha,\beta}(x)} = \frac{\alpha!}{j!}\frac{\beta!}{k!} x^{j-\alpha}(1-x)^{k-\beta}$$

converges *uniformly* to 1 as $\nu \to \infty$, by Stirling's formula and by choice of α, β, j, and k. Hence, for large ν, $\delta_{j,k}(x) \le 2\delta_{\alpha,\beta}(x)$ for all $x \in [y-r, y+r]$, and so

$$\int_{y-r}^{y+r} |f(x) - f(y)|\delta_{j,k}(x)\,dx \le 2\int_{y-r}^{y+r} |f(x) - f(y)|\delta_{\alpha,\beta}(x)\,dx \to 0$$

as $\nu \to \infty$, by Lemma 4.21. □

4.3 The Generalized Lower Bound Theorem

The UBT tells us the maximum number of facets of any D-dimensional polytope with n vertices. What about the minimum? For general polytopes, this question is again answered by the UBT in its polar form: A D-dimensional polytope with m facets can have at most $c_D(m) = \binom{m-\lceil\frac{D+1}{2}\rceil}{\lceil\frac{D-1}{2}\rceil} + \binom{m-\lfloor\frac{D+1}{2}\rfloor}{\lfloor\frac{D-1}{2}\rfloor}$ vertices, and this bound is attained by the polar-to-cyclic polytope $\mathcal{C}_D^*(m)$. Read the other way around, a D-polytope with n vertices can have as few as $c_D^{-1}(n)$ facets, where $c_D^{-1}(n) := \min\{m : c_D(m) \ge n\}$, which for fixed D is approximately the $\lfloor D/2 \rfloor^{\text{th}}$ root of n.

The polar-to-cyclic polytopes are simple. What if we consider only simplicial polytopes? Here is a class of simplicial polytopes with few facets:

Stacked Polytopes. We consider polytopes that are obtained by glueing simplices along facets, see Figure 4.3. More formally, a D-polytope $\mathcal{P}$ is a *stacked*

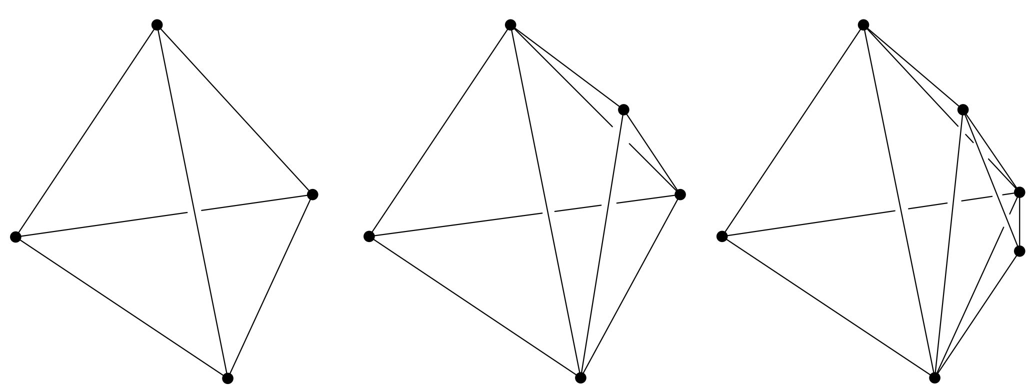


Figure 4.3: *Two stacking operations.*

polytope if there is a sequence $\mathcal{P}_1, \ldots, \mathcal{P}_s$ of simplicial D-polytopes such that $\mathcal{P}_1$ is a D-simplex and each $\mathcal{P}_{i+1}$ arises as the convex hull of $\mathcal{P}_i \cup \{\boldsymbol{q}\}$, where $\boldsymbol{q}$ is a point that is *just beyond* some facet $\mathcal{F}$ of $\mathcal{P}_i$. Here, a point $\boldsymbol{q}$ is just beyond a facet $\mathcal{F}$ of a polytope $\mathcal{Q}$ if $\boldsymbol{q} \in \mathrm{H}^+(\mathcal{F})$ and $\boldsymbol{q} \in \mathrm{H}^-(\mathcal{F}')$ for all other facets $\mathcal{F}'$ of $\mathcal{Q}$, where we assume that each facet-defining hyperplane is oriented in such a way that the polytope lies in the closed negative halfspace.

For the polar polytope, this can be interpreted as follows: Glueing a simplex to a facet $\mathcal{F}$ of a simplicial polytope $\mathcal{P}$, corresponds to cutting off a vertex v of $\mathcal{P}^*$ by a hyperplane that intersects only the edges incident to v; for instance, by the hyperplane spanned by the midpoints of these edges, i.e., we remove v from the vertex set of $\mathcal{P}$ and add the d midpoints as new vertices.

It is easy to analyze the face numbers of stacked polytopes by induction on s: If $\mathcal{P}_s$ is obtained by stacking s simplices, then

$$f_k(\mathcal{P}_s) = \binom{D}{k+1} + s \cdot \binom{D}{k}.$$

Further, the h-vector of $\mathcal{P}_s$ is

$$h(\mathcal{P}_s) = (1, s, s, \ldots, s, 1),$$

i.e., $h_j(\mathcal{P}_s) = 1$ for $j = 0, D$ and $h_j(\mathcal{P}_s) = s$ for $1 \leq j \leq D-1$, of which we can convince ourselves as follows: Let us interpret the glueing as cutting off a vertex v from the polar polytope as described above. How does this change the h-vector? We may assume that the h-vector of the old polytope is defined by a linear objective function that is maximized by v and almost orthogonal to the cutting hyperplane (but slightly tilted, so as to still be generic w.r.t. the new vertices). Thus, by cutting off v, we remove one vertex with in-degree D, and by inserting the new vertices, we add one vertex of in-degree j, for every $1 \leq j \leq D$ (the new vertices form a $(D-1)$-dimensional simplex, and have one additional incoming edge each).

Note that for any simplicial D-polytope $\mathcal{P}$,

$$h_1(\mathcal{P}) = f_0(\mathcal{P}) - D.$$

The following theorem implies that stacked polytopes simultaneously minimize the number of k-faces, for every k, among simplicial polytopes of a given dimension and with a prescribed number of vertices.

Theorem 4.23 (Generalized Lower Bound Theorem (GLBT) [73]). *Let $\mathcal{P}$ be a simplicial D-polytope. Then the h-vector of $\mathcal{P}$ satisfies*

$$h_j(\mathcal{P}) \geq h_{j-1}(\mathcal{P})$$

for $1 \leq j \leq \lfloor \frac{D}{2} \rfloor$.

The theorem, which was conjectured by McMullen and Walkup [56], bears its name because it generalizes the so-called *Lower Bound Theorem (LBT)*, which was conjectured by Brückner in 1909 and proved by Barnette [13]. The LBT states that for any simplicial D-polytope, $f_1 \geq D \cdot f_0 - \binom{d+1}{2}$, which can be rewritten as $h_2 - h_1 \geq 0$.

For the GLBT, the only known proofs in fact establish it as part of the still more powerful *g-Theorem*, which states that a certain set of combinatorial conditions completely characterize the integer vectors that can appear as h-vectors of simplicial polytopes. These conditions were formulated by McMullen [53], and their sufficiency was proved by Billera and Lee and [16], while their necessity was shown by Stanley [73], using sophisticated tools from algebraic geometry. Later, McMullen [54, 55] found a simpler and more geometric proof of the necessity part, but even that proof is too involved to be discussed here.

The name "g-Theorem", at any rate, refers to the name that the differences $h_j - h_{j-1}$ have been given:

Definition 4.24 (g-vectors). The *g-vector* $g(\mathcal{P}) = (g_0(\mathcal{P}), \ldots, g_{\lfloor D/2 \rfloor}(\mathcal{P}))$ of a simplicial D-polytope $\mathcal{P}$ is defined by

$$g_j = g_j(\mathcal{P}) := h_j(\mathcal{P}) - h_{j-1}(\mathcal{P}),$$

for $0 \leq j \leq \lfloor D/2 \rfloor$, where we set $h_{-1} := 0$. Similarly, for a set S of n points in $\mathbf{R}^d$ and a point $\boldsymbol{o} \in \mathbf{R}^d$ that is generic w.r.t. S, the g-vector is given by

$$g_j = g_j(S, \boldsymbol{o}) := h_j(S, \boldsymbol{o}) - h_{j-1}(S, \boldsymbol{o}),$$

for $0 \leq j \leq \lfloor (n-d-1)/2 \rfloor$. (Observe that it is sufficient to consider this range of j, by the Dehn-Sommerville relations.)

With this notation, the GLBT states that

$$g_j(\mathcal{P}) \geq 0$$

for $1 \leq j \leq \lfloor D/2 \rfloor$, respectively

$$g_j(S, \boldsymbol{o}) \geq 0$$

for $0 \leq \lfloor (n-d-1)/2 \rfloor$.

By Theorem 4.12, we know that for any generic oriented line ℓ through $\boldsymbol{o}$,

$$g_j(S, \boldsymbol{o}) = h_j^*(S, \ell, \boldsymbol{o}) - h_{n-d-j}^*(S, \ell, \boldsymbol{o}).$$

Thus, by invoking the GLBT for the polytope that arises as the Gale dual of S w.r.t. $\boldsymbol{o}$ as origin, we see that $h_j^*(S, \ell, \boldsymbol{o}) \geq h_{n-d-j}^*(S, \ell, \boldsymbol{o}) = {}^*h_j(S, \ell, \boldsymbol{o})$ for $0 \leq j \leq (n-d-1)/2$. Since this applies to any generic point $\boldsymbol{o}$ on ℓ, we can interpret the GLBT "dynamically", see [86], as saying that for such j, we can never leave more j-facets of S than we have already entered as we move along an oriented line, starting from a point outside of $\operatorname{conv} S$.

Dual-to-stacked configurations. Here is what happens in the dual during a stacking operation: Let S be a set of n points in $\mathbf{R}^d$ and $\boldsymbol{o} \in \operatorname{conv} S$ a generic point. Pick points $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{d+1} \in S$ that span a d-dimensional simplex which contains $\boldsymbol{o}$ in its interior. This corresponds to picking a facet $\mathcal{F}$ of the Gale dual poytope $\mathcal{P}$ (or a vertex of $\mathcal{P}^*$). Consider a lifting $\tilde{S}, \tilde{\ell}$ of $S, \boldsymbol{o}$ such that the d-simplex $\tilde{\sigma}$ spanned by the lifted points $\tilde{\boldsymbol{p}}_1, \ldots, \tilde{\boldsymbol{p}}_{d+1}$ become the "topmost" facet, i.e., such that $\tilde{\sigma}$ with suitable orientation is the unique $(n-d)$-facet of $\tilde{S}$ which is entered by $\tilde{\ell}$. In other words, if we consider the points of intersection of $\tilde{\ell}$ with the d-simplices spanned by points from $\tilde{S}$ in the order in which they appear along $\tilde{\ell}$, then $\tilde{\boldsymbol{b}} := \tilde{\sigma} \cap \tilde{\ell}$ is the last such intersection. Let $\tilde{\boldsymbol{a}}$ be the second-to-last of these intersections, and pick two further points $\tilde{\boldsymbol{o}}$ and $\tilde{\boldsymbol{q}}$ on $\tilde{\ell}$ such that the points $\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{o}}, \tilde{\boldsymbol{q}}, \tilde{\boldsymbol{b}}$ appear in this order along $\tilde{\ell}$. Then the Gale dual of $\tilde{S} \dot{\cup} \{\tilde{\boldsymbol{q}}\} \subset \mathbf{R}^{d+1}$ w.r.t. $\tilde{\boldsymbol{o}}$ as origin is (combinatorially equivalent to) the polytope obtained by glueing a simplex to the facet $\mathcal{F}$ of $\mathcal{P}$. A dual-to-stacked configuration is a set S that arises through a sequence of such operations from a multiset of $D+1$ points in $\mathbf{R}^0$ (which is the Gale dual of a D-simplex).

In the remainder of this section, we prove a continuous analogue of the GLBT. The proof will by no means be an independent one, but rather a straightforward derivation from the discrete version, using Lemma 4.22.

Definition 4.25 ($\boldsymbol{g}$-functions). The *g-function* $g_{\mu,\boldsymbol{o}}$ of a continuous probability measure μ and a point $\boldsymbol{o}$ in $\mathbf{R}^d$ is defined, for a.e. $0 \leq y \leq 1$, by

$$g_{\mu,\boldsymbol{o}}(y) := \frac{1}{d+1} \cdot \frac{\mathrm{d}}{\mathrm{d}y} h_{\mu,\boldsymbol{o}}(y).$$

Thus, by Theorem 4.13, we have

$$g_{\mu,\boldsymbol{o}}(y) = h_{\mu,\ell,\boldsymbol{o}}^*(y) - h_{\mu,\ell,\boldsymbol{o}}^*(1-y)$$

for any generic oriented line ℓ through $\boldsymbol{o}$.

Theorem 4.26 (Continuous Generalized Lower Bound Theorem (CGLBT)). *The g-function of a continuous probability distribution μ and a point $\boldsymbol{o}$ in $\mathbf{R}^d$ satisfies*

$$g_{\mu,\boldsymbol{o}}(y) \geq 0$$

for a.e. $0 \leq y \leq 1/2$.

We will need the following simple but useful fact (see [84]).

Lemma 4.27 (Counting Permutations). *Suppose $\boldsymbol{X} = (X_1, \ldots, X_n)$ is a vector of independently and identically distributed random variables which take values in some set N, and let A and B be measurable sets of n-tuples $(x_i)_{i=1}^n$ of elements in N. Assume furthermore that there exist integers $l, m \geq 0$ such that for every $\boldsymbol{a} = (a_i) \in A$ and every $\boldsymbol{b} = (b_i) \in B$, there are exactly l permutations π such that $\boldsymbol{a}_\pi := (a_{\pi(i)}) \in B$, and exactly m permutations π' for which $\boldsymbol{b}_{\pi'} \in A$.*

Then

$$l \cdot \Pr[\boldsymbol{X} \in A] = m \cdot \Pr[\boldsymbol{X} \in B].$$

Lemma 4.28. *Consider a continuous probability distribution μ, a point $\boldsymbol{o}$, and an oriented line ℓ through $\boldsymbol{o}$ in $\mathbf{R}^d$. Let $P_1, P_2, P_3, \ldots,$ be independent μ-random points, and for nonnegative integer n, let $S_n := \{P_1, \ldots, P_n\}$. Then, for all integers $j, k \geq 0$,*

$$\int_0^1 \delta_{j,k}(x) h^*_{\mu,\ell,\boldsymbol{o}}(x)\mathrm{d}x = \frac{j+k+1}{\binom{j+k+d}{d}} \mathbf{E}[h^*_j(S_{j+k+d}, \ell, \boldsymbol{o})]$$

and

$$\int_0^1 \delta_{j,k}(1-x) h^*_{\mu,\ell,\boldsymbol{o}}(x)\mathrm{d}x = \frac{j+k+1}{\binom{j+k+d}{d}} \mathbf{E}[h^*_k(S_{j+k+d}, \ell, \boldsymbol{o})],$$

where "$\mathbf{E}$" denotes the expected value.

Proof. The transformation theorem for image measures yields

$$\int_0^1 x^j (1-x)^k h^*_{\mu,\ell,\boldsymbol{o}}(x)\mathrm{d}x$$
$$= \underset{B_{\ell,\boldsymbol{o}}}{\int \cdots \int} \mu(\mathrm{H}^+([\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d]))^j \mu(\mathrm{H}^-([\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d]))^k \mathrm{d}\mu(\boldsymbol{p}_d) \ldots \mathrm{d}\mu(\boldsymbol{p}_1),$$

where $B_{\ell,\boldsymbol{o}} := \{(\boldsymbol{p}_i)_{i=1}^d \in \mathbf{R}^{d\times d} : \ell \text{ enters } [p_1,\ldots,p_d] \text{ before } \boldsymbol{o}\}$, and this can be further rewritten as

$$\int\limits_{C^{j,k}_{\ell,\boldsymbol{o}}} \cdots \int \mathrm{d}\mu(\boldsymbol{p}_{d+j+k}) \ldots \mathrm{d}\mu(\boldsymbol{p}_{d+1}) \mathrm{d}\mu(\boldsymbol{p}_d) \ldots \mathrm{d}\mu(\boldsymbol{p}_1)$$

where $C^{j,k}_{\ell,\boldsymbol{o}}$ is the set of all point tuples $(\boldsymbol{p}_i)_{i=1}^{d+j+k} \in \mathbf{R}^{d\times(d+j+k)}$ such that $(\boldsymbol{p}_i)_{i=1}^d \in B^d_{\ell,\boldsymbol{o}}$ and $\boldsymbol{p}_{d+1},\ldots,\boldsymbol{p}_{d+j} \in \mathrm{H}^+([\boldsymbol{p}_1,\ldots,\boldsymbol{p}_d])$ and $\boldsymbol{p}_{d+j+1},\ldots,\boldsymbol{p}_{d+j+k} \in \mathrm{H}^-([\boldsymbol{p}_1,\ldots,\boldsymbol{p}_d])$. This last integral, in turn, is just the probability

$$\Pr[(P_1,\ldots,P_{d+j+k}) \in C^{j,k}_{\ell,\boldsymbol{o}}]$$

where $P_1,\ldots,P_{d+j+k}$ are independent μ-random points.

It remains to observe that for a random permutation Π of $\{1,\ldots,d+j+k\}$ and a given point tuple $(\boldsymbol{p}_i)_{i=1}^{d+j+k} \in \mathbf{R}^{d\times(d+j+k)}$, we have

$$\Pr[(\boldsymbol{p}_{\Pi(i)})_{i=1}^{d+j+k} \in C^{j,k}_{\ell,\boldsymbol{o}}] = \frac{h_j^*(\{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_{d+j+k}\},\ell,\boldsymbol{o})}{2\binom{d+j+k}{d}\binom{j+k}{j}}.$$

Thus, the first part of Lemma 4.28 follows by applying the Counting Permutations Lemma to each of the sets

$$C^{j,k}_{\ell,\boldsymbol{o}}(m) := \{(\boldsymbol{p}_i)_{i=1}^{d+j+k} \in C^{j,k}_{\ell,\boldsymbol{o}} : h_j^*(\{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_{d+j+k}\},\ell,\boldsymbol{o}) = m\},$$

for integer m, and summing over all m. The second part is proved analogously. □

Proof of Theorem 4.26. Let $P_1, P_2, P_3, \ldots$ be independent μ-random points. For a Lebesgue point y of g with $0 < y < 1/2$, Lemmas 4.22 and 4.28 provide us with integer sequences $(j(\nu))_{\nu\in\mathbf{N}}$ and $(k(\nu))_{\nu\in\mathbf{N}}$ such that

$$\begin{aligned} g_{\mu,\boldsymbol{o}}(y) &= \lim_{\nu\to\infty} \int_0^1 \delta_{j(\nu),k(\nu)}(1-x) g_{\mu,\boldsymbol{o}}(x)\mathrm{d}x \\ &= \lim_{\nu\to\infty} \frac{j(\nu)+k(\nu)+1}{\binom{j(\nu)+k(\nu)+d}{d}} \mathbf{E}[g_j(S_{j(\nu)+k(\nu)+d},\boldsymbol{o})]. \end{aligned}$$

Moreover, $y < 1/2$ and $|\frac{j(\nu)}{j(\nu)+k(\nu)} - y| < 1/\nu$ by construction, so for large ν, we have $j(\nu) \le (j(\nu)+k(\nu)-1)/2$, and therefore $g_j(S_{d+j(\nu)+k(\nu)},\boldsymbol{o}) \ge 0$, by the GLBT. It follows that the expectation is nonnegative, too, and hence so is $g_{\mu,\boldsymbol{o}}(y)$. Since this holds for every Lebesgue point, the proof is complete. □

4.4 The First Selection Lemma

In Section 4.1, we analyzed the number of d-dimensional simplices spanned by an n-point set $S \subset \mathbf{R}^d$, which contain a given generic point $\boldsymbol{o}$ (in their interior). This number is at most

$$\binom{\lfloor \frac{n+d}{2} \rfloor}{d+1} + \binom{\lceil \frac{n+d}{2} \rceil}{d+1} \sim \frac{1}{2^d}\binom{n}{d+1}.$$

In this section, we consider what happens if S (respectively, a continuous probability distribution μ) is given but we are allowed to choose $\boldsymbol{o}$. For instance, can we find a point $\boldsymbol{o}$ which is contained in (the interior of) "many", i.e., of a positive fraction of, all simplices? The assertion that this is always possible is known as the *First Selection Lemma* and was proved by Boros and Füredi [21] for the planar case, and generalized to general dimension by Bárány [10]. In fact, Boros and Füredi actually showed that if $\boldsymbol{o}$ is a so-called *centerpoint* of a set S of n points in general position in the plane, then $\boldsymbol{o}$ is contained in the interior of at least $\frac{2}{9}\binom{n}{3}$ S-triangles (and the constant $\frac{2}{9}$ is best possible). We use h-vectors to show that centerpoints, a notion which we review below, work in any dimension, and the same method, with h-functions instead of h-vectors, establishes a "Continuous First Selection Lemma".

Depth and Centerpoints. If $S \subseteq \mathbf{R}^1$ is a multiset of n real numbers, then any number $c \in \mathbf{R}^1$ such that both, $|\{p \in S : p \leq c\}| \geq n/2$ and $|\{p \in S : p \geq c\}| \geq n/2$, is called a *median* of S. Centerpoints are a generalization of this concept to higher dimensions.

Definition 4.29 (Depth). Let S be a finite set of n points in $\mathbf{R}^d$. We make no general position assumptions (in fact, we can even allow multisets, i.e., repetitions of the points). The *depth in* S of a point $\boldsymbol{p} \in \mathbf{R}^d$ is defined as the minimum number of points (with multiplicity) from S in any halfspace containing $\boldsymbol{p}$,

$$\mathrm{depth}_S(\boldsymbol{p}) := \min\{|S \cap H| : H \text{ a halfspace, } \boldsymbol{p} \in H\}.$$

Similarly, for a (not necessarily continuous) probability distribution μ, the *depth of* $\boldsymbol{p}$ *in* μ is defined as

$$\mathrm{depth}_\mu(\boldsymbol{p}) := \min\{\mu(H) : H \text{ a halfspace, } \boldsymbol{p} \in H\}.$$

(At first sight, we should consider the *infimum*, since there are infinitely many halfspaces, but it is not hard to see that it is attained.)

Theorem 4.30 (Centerpoint Theorem). *Let S be a (multi)set of n points in $\mathbf{R}^d$. Then there exits a* centerpoint *of S, i.e. a point $\boldsymbol{c} \in \mathbf{R}^d$ (not necessarily in S) such that* $\operatorname{depth}_S(\boldsymbol{c}) \geq \lceil \frac{n}{d+1} \rceil$.

Similarly, for every probability distribution μ in $\mathbf{R}^d$, there exists a centerpoint $\boldsymbol{c}$ in the sense that $\operatorname{depth}_\mu(\boldsymbol{c}) \geq \frac{1}{d+1}$.

Observe that the factor $\frac{1}{d+1}$ is optimal (for instance, if S is the vertex set of a d-dimensional simplex, then there is no point of depth greater than 1 w.r.t. S).

Theorem 4.31 (First Selection Lemma). *For any pair of integers $d \geq 1$ and $k \geq 0$, there is a constant $s(d, k) > 0$ such that the following holds:*

1. *If S is a set of n points in general position $\mathbf{R}^d$ and if $\boldsymbol{c}$ is any centerpoint of S, then the number $f_k(S, \boldsymbol{c})$ of $(d + 1 + k)$-element subsets of S that contain $\boldsymbol{c}$ in the interior of their convex hull satisfies*

$$f_k(S, \boldsymbol{c}) \geq s(d, k) \cdot \binom{n}{d+1+k} - O(n^{d+k}). \tag{4.29}$$

2. *Moreover, for any n-point set $S \subseteq \mathbf{R}^d$ (not necessarily in general position), there exists a centerpoint $\boldsymbol{c}$ of S such that*

$$\bar{f}_k(S, \boldsymbol{c}) \geq s(d, k) \cdot \binom{n}{d+1+k} - O(n^{d+k}), \tag{4.30}$$

   where $\bar{f}_k(S, \boldsymbol{c}) := |\{X \subseteq S : |X| = d + 1 + k, \boldsymbol{c} \in \operatorname{conv} X\}|$, *i.e., we also count subsets that contain $\boldsymbol{c}$ on the boundary of their convex hull.*

Theorem 4.32 (Continuous First Selection Lemma). *If $\boldsymbol{c}$ is a centerpoint of a continuous probability distribution μ in $\mathbf{R}^d$, then*

$$f_k(\mu, \boldsymbol{c}) \geq s(d, k),$$

with the same constant $s(d, k)$ as in Theorem 4.31.

We first prove the continuous version of the First Selection Lemma, which follows quite easily from the second part of the following lemma. The discrete version will require no new insights, but a bit more work to avoid the difficulty that a centerpoint need not be generic w.r.t. S.

Lemma 4.33. *The h-vector and the h-function with respect to a point attain the maximum given by the (C)UBT up to the depth of the point:*

1. *Let S be a finite point set in $\mathbf{R}^d$, and let $\boldsymbol{o}$ be a generic point w.r.t. S. Suppose every hyperplane disjoint from S and containing $\boldsymbol{o}$ has at least a points of S on either side (in particular, this holds if $\operatorname{depth}_S(\boldsymbol{o}) = a$). Then $h_j(S,\boldsymbol{o}) = \binom{j+d}{d}$ for $0 \le j \le a-1$.*

2. *If μ is a continuous probability distribution and $\boldsymbol{o}$ a point in $\mathbf{R}^d$ with $\operatorname{depth}_\mu(\boldsymbol{o}) = a$, then $h_{\mu,\boldsymbol{o}}(y) = \frac{d+1}{2}y^d$ for $0 \le y \le a$.*

Proof. We prove the continuous case. The proof in the discrete setting is perfectly analogous. Pick any directed line $\ell \subset \mathbf{R}^d$ through $\boldsymbol{o}$. Observe that for $0 < y < a$, we have $H^*_{\mu,\ell,\boldsymbol{o}}(1-y) = {}^*H_{\mu,\ell,\boldsymbol{o}}(y) = 0$, hence $h^*_{\mu,\ell,\boldsymbol{o}}(1-y) = {}^*h_{\mu,\ell,\boldsymbol{o}}(y) = 0$ for a.e. such y. It follows that

$$g_{\mu,\boldsymbol{o}}(y) = h^*_{\mu,\ell,\boldsymbol{o}}(y) = h_{\mu,\ell}(y) \tag{4.31}$$

for a.e. $0 < y < a$. It remains to observe that $\boldsymbol{o}$ also has depth at least a in the orthogonal projection $\overline{\mu}$ of μ onto the orthogonal complement $\ell^\perp \equiv \mathbf{R}^{d-1}$. By induction, we can conclude that $h_{\mu,\ell}(y) = h_{\overline{\mu},\boldsymbol{o}}(y) = \frac{d}{2}y^{d-1}$ for $0 \le y \le a$, and so, by Theorem 4.13 and (4.31),

$$h_{\mu,\boldsymbol{o}}(y) = d+1\int_0^y \frac{d}{2}x^{d-1}\mathrm{d}x = \frac{d+1}{2}y^d$$

for $0 \le y \le a$, as desired. □

Proof of the Continuous First Selection Lemma. By the Dehn-Sommerville Equation and (4.13),

$$\begin{aligned}\frac{1}{2\binom{d+1+k}{d+1}} f_k(\mu,\boldsymbol{o}) &= \int_0^1 y^k h_{\mu,\boldsymbol{o}}(y)\mathrm{d}y \\ &= \int_0^{1/2} \big(y^k + (1-y)^k\big) h_{\mu,\boldsymbol{o}}(y)\mathrm{d}y.\end{aligned}$$

Thus, by the preceding lemma and the CGLBT, if $\boldsymbol{o}$ has depth a in μ then

$$\begin{aligned}&\frac{f_k(\mu,\boldsymbol{o})}{(d+1)\binom{d+1+k}{d+1}} \\ &\ge \int_0^a \big(y^k + (1-y)^k\big)y^d\mathrm{d}y + \int_a^{1/2}\big(y^k + (1-y)^k\big)a^d\mathrm{d}y \\ &= \frac{a^{d+1+k}}{d+1+k} + \sum_{i=0}^{k}(-1)^i\frac{a^{d+1+i}}{d+1+i} + \frac{a^d\big((1-a)^{k+1} - a^{k+1}\big)}{k+1}\end{aligned}$$

Hence, if $\boldsymbol{o}$ is a centerpoint of μ, i.e., $a \geq \frac{1}{d+1}$, then

$$f_k(\mu, \boldsymbol{o}) \;\geq\; s(d,k) \;:= \tag{4.32}$$
$$\tbinom{d+1+k}{d+1}\left(\tfrac{(d+1)^{-d-k}}{d+1+k} + \textstyle\sum_{i=0}^{k}(-1)^i\binom{k}{i}\tfrac{(d+1)^{-d-i}}{d+1+i} + \tfrac{d^{k+1}-1}{(d+1)^{d+k}}\right) \;>\; 0.$$

□

For instance, for the case $k = 0$, we see that the probability that the convex hull of independent μ-random points $P_1, \ldots, P_{d+1}$ contains a given centerpoint of μ is at least

$$\frac{d-1}{(d+1)^d} + \frac{2}{(d+1)^{d+1}}. \tag{4.33}$$

On the other hand, if μ is a distribution for which no point has depth larger than $\frac{1}{d+1}$ (for instance, the uniform distribution on the union of $d+1$ small balls centered at the vertices of some d-dimensional simplex), then the second part of the CUBT shows that the constant $s(d, 0)$ in the First Selection Lemma cannot be chosen larger than

$$2(d+1)\left(\int_0^{1/2} y^d \mathrm{d}y - \int_0^{\frac{1}{2}-\frac{1}{d+1}} y^d \mathrm{d}y\right) = \frac{1}{2^d}\left(1 - \left(\frac{d-1}{d+1}\right)^{d+1}\right),$$

which, for large d, is approximately $(1-e^{-2})/2^d$ and still quite far from the lower bound (4.33).

As mentioned above, in the discrete setting, we cannot apply our method right away, since a centerpoint of a point set need not be generic (i.e., need not be disjoint from all convex hulls of less than $d+1$ points from S), even if the point set is in general position, see Figure 4.4.

However, we can always find a generic point that is almost a centerpoint:

Observation 4.34. *Let $\boldsymbol{c}$ be a centerpoint of a set S of n points in general position in $\mathbf{R}^d$. Then any point $\boldsymbol{o}$ sufficiently close to $\boldsymbol{c}$ has depth at least $\lceil \frac{n}{d+1} \rceil - d$ in S.*

Proof. By general position, any $d+1$ points from S span a d-dimensional simplex, and each such simplex contains a maximal inscribed d-dimensional ball. Let $r = r(S) > 0$ be the minimal radius of any such ball. Note that if two parallel hyperplanes are at distance less than $2r$ from each other, then the closed strip between them contains at least d points from S (else it would contain a simplex and its inscribed ball). Assume then that $\boldsymbol{o}$ is at distance less

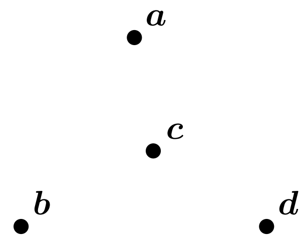


Figure 4.4: *For this set of 4 points in the plane, the point $\boldsymbol{c} \in S$ is the only centerpoint.*

than $2r$ from $\boldsymbol{c}$, and let H be any closed hyperplane through $\boldsymbol{o}$. The parallel translate H_1 of H through $\boldsymbol{c}$ is at distance less than $2r$ from H, hence $|\overline{\mathrm{H}^+} \cap S| \geq |\overline{\mathrm{H}_1^+} \cap S| - |\overline{\mathrm{H}_1^+} \cap \mathrm{H}^- \cap S| \geq \lceil \frac{n}{d+1} \rceil - d$. ☐

Proof of the First Selection Lemma. First assume that S is in general position. Let $\boldsymbol{c}$ be a centerpoint of S, and let $\boldsymbol{o}$ be a point sufficiently close to $\boldsymbol{c}$ and generic w.r.t. S. We also assume that $\boldsymbol{c}$ and $\boldsymbol{o}$ are not strictly separated by any hyperplane spanned by points from S. Since $\operatorname{depth}_S(\boldsymbol{o}) = a \geq \frac{n}{d+1} - d$, the same argument as in the proof of the continuous version leads to

$$\begin{aligned} f_k(S, \boldsymbol{o}) &\geq \sum_{j=0}^{a-1} \left(\tbinom{j}{k} + \tbinom{n-d-1-j}{k}\right)\tbinom{j+d}{d} + \tbinom{a-1+d}{d} \sum_{j=a}^{n-d-1-a} \tbinom{j}{k} \\ &= s(d,k)\binom{n}{d+k+1} - O(n^{d+k}), \end{aligned}$$

with the constant $s(d,k)$ as defined in (4.32). Moreover, by choice of $\boldsymbol{o}$, we have that $\boldsymbol{c} \in \operatorname{conv} X$ for all $X \subseteq S$ with $\boldsymbol{o} \in \operatorname{conv} X$, hence $\bar{f}_k(S, \boldsymbol{c}) \geq f_k(S, \boldsymbol{o})$. To conclude the proof for the case of general position, remains to note that by the following lemma (taken from Chapter 9 of [51]), $\boldsymbol{c}$ does not lie on the boundary of $\operatorname{conv} X$ for more than $O(n^{d+k})$ subsets $X \subseteq S$ of cardinality $d + 1 + k$.

Lemma 4.35. *If S is a set of n points in general position in $\mathbf{R}^d$, then no point $\boldsymbol{c} \in \mathbf{R}^d$ is contained in more than dn^{d-1} hyperplanes spanned by S.*

If $S = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\} \subseteq \mathbf{R}^d$ is not in general position, then we choose point sequences $(\boldsymbol{p}_i^\nu)_{\nu \in \mathbf{N}}$, $1 \leq i \leq n$, such that $\boldsymbol{p}_i^\nu \to \boldsymbol{p}_i$ as $\nu \to \infty$ and that each

$S^\nu := \{\boldsymbol{p}_1^\nu, \ldots, \boldsymbol{p}_n^\nu\}$ is in general position (for instance, we can take each $\boldsymbol{p}_i^\nu$ independently uniformly at random from the open ball of radius $1/\nu$ centered at $\boldsymbol{p}_i$). Moreover, if we take a centerpoint $\boldsymbol{c}^\nu$ of each S^ν, then since all $\boldsymbol{c}^\nu$ are contained in some big compact set, by passing to a suitable subsequence, if necessary, we may assume that they converge to some point $\boldsymbol{c}$. We know that each $\boldsymbol{c}^\nu$ is contained in the convex hull of at least $N = s(d,0)kd\binom{n}{d+k+1} - O(n^{d+k})$ subsets of S^ν of cardinality $d+k+1$, i.e., there is some collection $\mathcal{I}^\nu$ of $(d+1+k)$-element index sets $I \subseteq \{1 \ldots n\}$ such that $\boldsymbol{c}^\nu \in \operatorname{conv}\{\boldsymbol{p}_i^\nu : i \in I\}$ for all $I \in \mathcal{I}^\nu$, and $|\mathcal{I}^\nu| = N$. Since there are only finitely many such collections of index sets, one of them, call it $\mathcal{I}$, must appear as $\mathcal{I} = \mathcal{I}^\nu$ for infinitely many ν. Therefore, $\boldsymbol{c} \in \operatorname{conv}\{\boldsymbol{p}_i : i \in I\}$ for all $I \in \mathcal{I}$, which completes the proof of the First Selection Lemma. $\square$

Chapter 5

Self-Embracing Distributions

In the Educational Times of April, 1864, Question 1491, James Joseph Sylvester [77] formulated what became known as his *Four-Point Problem* (quoted after Pfiefer [63]):

> Show that he chance of four points forming the apices of a reentrant quadrilateral is $1/4$ if they be taken at random in an indefinite plane, but $1/4 + e^2 + x^2$, where e is a finite constant and x a variable quantity, if they be limited by an area of any magnitude and of any form.

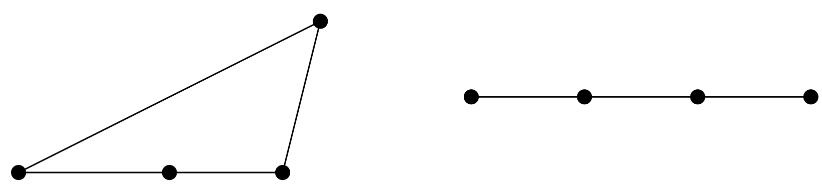
Here, "reentrant quadrilateral" means "not a convex quadrilateral", i.e., four points form a reentrant quadrilateral iff one of them is contained in the convex hull of the other three.

Various solutions came in, some of them by well-known mathematicians, and most of them different (see [63] for a detailed account). To give but a few examples, for the case of four points taken at random in the entire plane, Cayley and Sylvester asserted that the probability in question was $1/4$, while according to DeMorgan, it was $1/2$. Woolhouse, in turn, suggested, that the answer was $35/12\pi^2$, by computing the probability for points drawn at random from a disk and letting the radius tend to infinity (the value is, in fact, the same for all disks, independently of the radius, which makes the taking of the limit particularly easy).

With the advantage of hindsight and the classes on twentieth century probability and measure theory that we have taken, we know that the reason for these discordant results is that in order to speak meaningfully about random points, we have to specify a probability distribution. (Moreover, there is no uniform probability measure on the whole plane.)

In more rigorous terms, the Four-Point Problem concerns the probability $\triangle(\mu)$, for a given probability distribution μ in $\mathbf{R}^2$, that among four random points i.i.d. $\sim \mu$, there is one is contained in the convex hull of the other three? Equivalently, we can consider the complementary probability $\square(\mu) := 1 - \triangle(\mu)$ that the four points are in convex position.

In order to avoid the nuisance of uncivil configurations like the following, we will assume that μ is what we called continuous, i.e., that every line has μ-measure zero.



The Four-Point Problem is closely related to the question we investigated in the previous chapter. The simplest interesting instance of that question was: What is the probability

$$f_0(\mu, \boldsymbol{o}) = \Pr[\boldsymbol{o} \in \operatorname{conv}\{P_1, P_2, P_3\}]$$

that a given point $\boldsymbol{o}$ is contained in the convex hull of three random points P_1, P_2, P_3 i.i.d. $\sim \mu$. If instead of a point specified in advance, we consider a fourth independent random point P_4, we arrive at the probability

$$\Pr[P_4 \in \operatorname{conv}\{P_1, P_2, P_3\}],$$

which is just $\frac{1}{4}\triangle(\mu)$.

When the dependence on the underlying distribution became evident, Sylvester reformulated his problem more carefully and asked: Which distributions minimize, respectively maximize, $\square(\mu)$? Despite the phrase “an area of any magnitude and any form” in the original formulation, investigations focussed on the case that μ is the uniform distribution on bounded *convex* set $K \subseteq \mathbf{R}^2$ (where we assume that the interior of K is nonempty, to ensure that μ is continuous). For this class of distributions, the problem was completely solved by Blaschke [18] (see also [19], §24, for a textbook exposition), who showed that

$$\frac{2}{3} \leq \Box(\mu) \leq 1 - \frac{35}{12\pi^2} \approx 0.704. \tag{5.1}$$

Both bounds are tight; the lower bound is attained iff K is a triangle, and the upper bound iff K is an ellipse.

If one drops the convexity assumption then $\sup_\mu \Box(\mu) = 1$; indeed, $\Box(\mu) = 1$ if μ is the uniform distribution on the circle (or any other strictly convex Jordan curve in the plane). But even if we exclude as too degenerate distributions that are concentrated on sets without interior, and even if we further restrict our attention to the case that μ is the uniform distribution on a bounded open subset $V \subseteq \mathbf{R}^2$, $\Box(\mu)$ can be arbitrarily close to 1. This is the case, for instance, if V is a sufficiently thin open annulus, see [66].

The problem of determining the infimum

$$\Box_* := \inf_\mu \Box(\mu),$$

for general continuous probability distributions, on the other hand, is much more intricate. Unlike the probabilities $f_0(\mu, \boldsymbol{o})$, for which exact bounds and a complete characterization of the extreme cases are available even for the generalization of the problem to an arbitrary number of random points in any dimension, the exact value of $\Box_*$ is still unknown, and this chapter will be mostly concerned with narrowing the gap of our knowledge.

The "continuous" Four-Point Problem can again be equivalently recast in terms of finite point sets, as was pointed out by Scheinerman and Wilf [66]: For a finite set S of points in general position in the plane, let $\Box(S)$ denote the number of 4-element subsets of S that are in convex position, and set $\Box(n) := \min\{\Box(S) : |S| = n\}$. It is not hard to show that the sequence $\Box(n)/\binom{n}{4}$ is non-decreasing (and obviously bounded by 1), and as we will see below, its limit is precisely

$$\Box_* = \lim_{n\to\infty} \frac{\Box(n)}{\binom{n}{4}}.$$

In this discrete context, the problem is also known as that of determining the *rectilinear crossing number* of complete graphs.

We review the concept of (rectilinear) crossing numbers of graphs, some basic facts and previously known bounds, and the connection to the Four-Point Problem in Section 5.1, and from then on work mainly in the technically more convenient discrete setting.

In Section 5.2, we refine the techniques from Chapter 4 to derive a first lower bound for $\square_*$. While this bound will be further improved in the subsequent section, the proof technique and the combinatorial encoding of planar point sets by means of so-called *staircases of encounters* might be of interest in their own right.

In Section 5.3, we describe how to express $\square(S)$ in terms of the numbers $e_j(S)$ of j-edges of S. Together with the lower bounds for the numbers $E_j(S)$ of $(\leq j)$-edges of S which we will derive in Chapter 6, we obtain a lower bound which comes quite close to the known upper bounds.

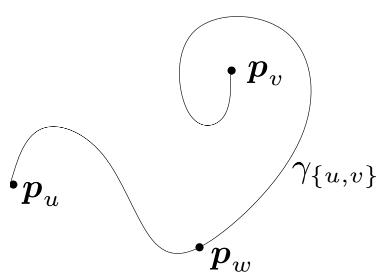
This technique also works for a generalization (one among several) of the Four Point Problem to a larger number of points as well as to higher dimensions, which we discuss in Section 5.4.

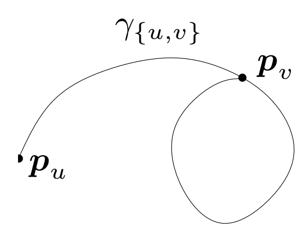
5.1 Crossing Numbers

Consider an abstract graph $G = (V, E)$. A *drawing* of G is a mapping that assigns to each vertex $v \in V$ a point $\boldsymbol{p}_v \in \mathbf{R}^2$, and to every edge $e \in E$ a Jordan arc φ_e (i.e., the image of the closed unit interval under a continuous injective map) such that the following conditions are satisfied:

1. Different vertices are mapped to different points, $\boldsymbol{p}_u \neq \boldsymbol{p}_v$ if $u \neq v$.

2. The arc γ_e associated with an edge $e = \{u, v\}$ has $\boldsymbol{p}_u$ and $\boldsymbol{p}_v$ as its endpoints and contains no $\boldsymbol{p}_w$, $w \in V$, in its relative interior. Thus, the following two situations are forbidden:





3. The relative interiors of any two arcs only intersect in a finite number of points. Such a point of intersection is called a *crossing* in the drawing.

When speaking about the *number of crossings* in a given drawing, we count the crossings with multiplicity: if a crossing $\boldsymbol{q}$ is contained in the relative interiors of s arcs, then it is counted $\binom{s}{2}$ times. The *crossing number* of the graph G, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings in any drawing of G.

It is usually assumed that in a drawing, no three arcs cross in a common point, that there are no crossings between arcs with a common endpoint, and that any two arcs cross in at most one point. These are sensible assumptions when considering the crossing number, since they can be ensured by local modifications that do not increase the number of crossings, see Figure 5.1, and we will also assume this in what follows. (We note that things become more subtle when one considers the so-called *pairwise crossing number* which counts the number of pairs of arcs that cross.)

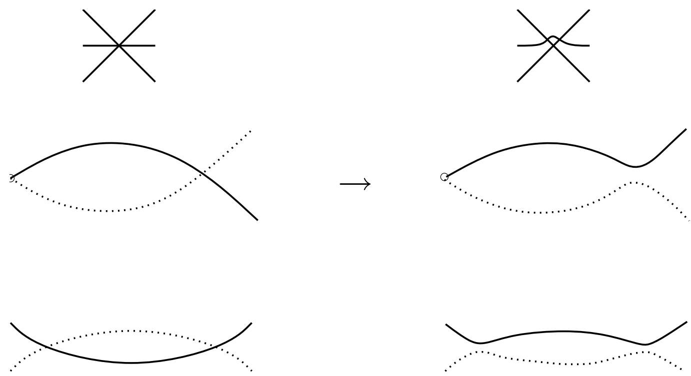


Figure 5.1: *Local modifications that do not increase the number of crossings.*

If all the arcs in a drawing are line segments then the drawing is called *rectilinear* or *straight-edge*, and the *rectilinear crossing number* $\overline{\mathrm{cr}}(G)$ of a graph G is defined as the minimum number of crossings in any straight-edge drawing of G.

It is a well-known theorem (proved by Steinitz [74], and independently by Wagner [83], and by Fáry [38]) that if a graph G is *planar*, i.e., $\mathrm{cr}(G) = 0$, then there exists also a crossing-free straight-edge drawing of G, i.e., $\overline{\mathrm{cr}}(G) = 0$. Bienstock and Dean [15] showed that the relation $\overline{\mathrm{cr}}(G) = \mathrm{cr}(G)$ holds more generally whenever $\mathrm{cr}(G) \leq 3$. On the other hand, they also exhibited an infinite family of graphs whose crossing number is 4 but whose rectilinear crossing number is arbitrarily large. Thus, if $\mathrm{cr}(G) \geq 4$, then $\overline{\mathrm{cr}}(G)$ cannot even be bounded in terms of $\mathrm{cr}(G)$.

Another example that matters become more complicated once we leave the realm of planar graphs behind is the following: While planarity of a graph can be tested in linear time (see Hopcroft and Tarjan [45]), it is $\mathcal{NP}$-complete to decide whether $\mathrm{cr}(G) \leq k$ for a given graph G and integer k (see Garey and Johnson [40]). The hardness part of the proof carries over to the rectilinear crossing number, but it appears to be still unknown whether the problem of determining the latter is in $\mathcal{NP}$.

These computational hardness results give some indication that the crossing number and its rectilinear variant are intricate and subtle graph paramters. A striking symptom of just how much so is that neither of them is fully understood even for specific and very basic classes of examples, such as complete graphs or complete bipartite graphs, which we will discuss below.

We refrain from attempting to survey the numerous applications which crossing numbers have found in discrete and computational geometry. Instead, as good starting points for exploring, we recommend the survey articles by Pach [61] and by Pach and Tóth [62], Chapter 4 in Matoušek's book [51], and the online bibliography by Vrťo [81].

Turán's Brick Factory Problem. The question of determining the crossing number of complete bipartite graphs, posed by Turán, actually marks the appearance of the notion of crossing numbers on the mathematical stage. In a letter dated February, 1968, Turán wrote about his experience in a labour camp during the Second World War (quoted after Guy [42]):

> "In 1944 our labour combattation had the extreme luck to work—thanks to some very rich comrades—in a brick factory near Budapest. Our work was to bring out bricks from the ovens where they were made and carry them on small vehicles which run on rails in some of several open stores which happened to be empty. Since one could never be sure which store will be available, each oven was connected by rail with each store. Since we had to settle a fixed amount of loaded cars daily it was in our interest to finish it as soon as possible. After being loaded in the (rather warm) ovens the vehicles run smoothly with not much effort; the only trouble arose at the crossing of two rails. Here the cars jumped out, the bricks fell down; a lot of extra work and loss of time arose. Having this experience a number of times it occurred to me why on earth did they build the rail system so uneconomically; minimizing the number of crossings the production could be made much more economical."

Thus, the problem of determining the crossing number of the complete bipartite graph $K_{n,m}$ (the minimum number of crossings needed to connect n ovens to m stores) became known as *Turán's Brick Factory Problem*.

Solutions were submitted by Zarankiewicz [89, 90] and Urbanik [79]. Both asserted that

$$(?) \qquad \mathrm{cr}(K_{n,m}) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor. \tag{5.2}$$

For a while, this was called Zarankiewicz's Theorem, but then it was found that the proof of the lower bound contained a fatal fallacy, and (5.2) is now referred to as *Zarankiewicz's Conjecture*; see Guy's survey [42] for a detailed account. We note that the conjecture has been verified for $\min\{m, n\} \leq 6$ by Kleitman [48] and for $m = 7, n \leq 10$ by Woodall [88].

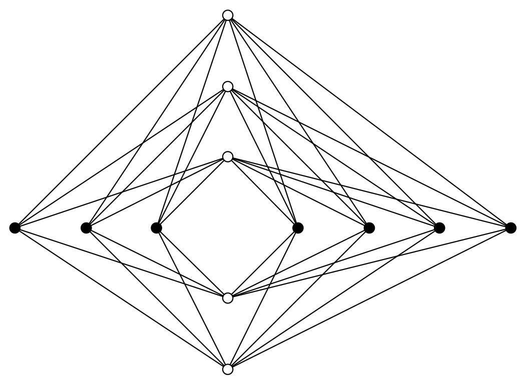


Figure 5.2: *The upper bound construction for* $\mathrm{cr}(K_{n,m})$ *and* $\overline{cr}(K_{n,m})$.

While the lower bound part of (5.2) remains unresolved, the construction that establishes the upper bound is very simple and produces even a rectilinear drawing, see Figure 5.2: Place m points on the x-axis, $\lfloor m/2 \rfloor$ of them to the left and $\lceil m/2 \rceil$ of them to the right of the origin, and n points on the y-axis, $\lfloor n/2 \rfloor$ below and $\lceil n/2 \rceil$ above the origin. Connect every point on the x-axis to every point on the y-axis by a straight segment.

Cylindrical Drawings. The crossing number of complete bipartite graphs is also relevant for the crossing number of complete graphs, the second class of examples mentioned above. We begin the discussion of this connection with the description of so-called *cylindrical drawings* of $K_{n,m}$: These are drawings where the m vertices of one vertex class are placed on a circle C_0, the n vertices of the other class lie on a circle C_1 properly enclosing C_0, and each vertex from the inner circle C_0 is joined to every vertex on the outer circle by an arc whose relative interior lies in the open annulus bounded by C_0 and

C_1. We can interpret this as a drawing of $K_{n,m}$ on the surface of a cylinder Z, with m vertices on the bottom "rim" C_0 and n vertices on the top "rim" C_1. Alternatively, we can picture Z as the strip $\mathbf{R} \times [0,1]$ modulo the equivalence relation $(x,y) \sim (x+k,y)$ for all $k \in \mathbf{Z}$, with $C_0 = [0,1] \times \{0\}/\sim$ and $C_1 = [0,1] \times \{1\}/\sim$.

We describe a particular construction of that kind, due to Anthony Hill and reproduced in Guy, Jenkyns, and Schaer [44]: Assume for simplicity that $m = n$. For $0 \leq i,j \leq n-1$, let $\boldsymbol{p}_i := (0, i/n) \in C_0$ and $\boldsymbol{q}_i := (1, i/n) \in C_1$, and let $\gamma_{i,j}$ be the arc $\{\boldsymbol{p}_i + t \cdot (1, \frac{j}{n}) : 0 \leq t \leq 1\}/\sim$ on Z, which joins $\boldsymbol{p}_i$ and $\boldsymbol{q}_{i+j \mod n}$, see Figure 5.3.

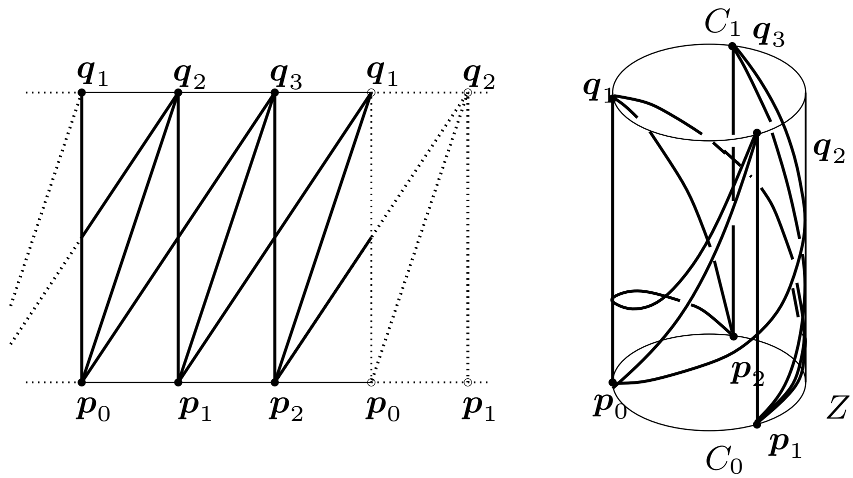


Figure 5.3: *Two ways of picturing Hill's construction.*

Let us call j the "slope" of $\gamma_{i,j}$. Two arcs with the same slope do not cross, so we can blame each crossing in the drawing on the line with larger slope. Thus, $\gamma_{i,j}$ will be blamed precisely for the crossings with the arcs $\gamma_{a,b}$ with $i < a \leq a+b < i+j$. There are $j-1$ of these with $b = 0$, and $\binom{j-1}{2}$ of them with $b > 0$. Thus, we obtain a total of

$$n \sum_{j=1}^{n-1} \left(\binom{i-1}{2} + i - 1 \right) = \frac{1}{6} n^2 (n-1)(n-2) \tag{5.3}$$

crossings. A similar construction for $K_{n,n+1}$ results in $\frac{1}{6}(n+1)n(n-1)^2$ crossings, see [44]. We note that (5.3) is optimal for for cylindrical drawings of $K_{n,n}$, as proved by Richter and Thomassen [64].

Complete Graphs. Given a cylindrical drawing of $K_{n,m}$, we can complete it to a drawing of K_{n+m} by inserting the missing edges in the disks bounded by C_0 and C_1, at the cost of $\binom{m}{4} + \binom{n}{4}$ additional crossings. (The drawing on the cylinder can be transformed into a drawing in the plane by first projecting centrally on a circumscribed sphere, and then using stereographic projection.)

If this is applied to $K_{\frac{n}{2},\frac{n}{2}}$, n even, with the cylindrical drawings described above, then the resulting drawing contains $\frac{1}{64}n(n-2)^2(n-4)$ crossings. For odd n, we first draw K_{n+1} in the above fashion and then remove an arbitrary vertex and all incident edges. The bounds obtained for both parity cases can be summarized as

$$\operatorname{cr}(K_n) \leq \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor. \qquad (5.4)$$

This construction was described by Blažek and Koman [20] (who also gave another construction with the same number of crossings) and independently by Guy [42]. Moon [57] gave another very simple construction for drawings of complete graphs, which gives the same asymptotic upper bound for $\operatorname{cr}(K_n)$ as (5.4): He showed that if we choose n points independently and uniformly at random from the unit sphere and join any two of them by a minor great circle arc, then the expected number of crossings in the resulting drawing is $\frac{3}{8}\binom{n}{4}$.

It is conjectured that this is optimal, and that moreover, (5.4) gives the true value of $\operatorname{cr}(K_n)$.

In asymptotic form, this would actually follow from Zarankiewicz's conjecture (5.2), as first pointed out by Kainen [46]. The reasoning is a typical example of a basic double counting argument about crossing numbers: Consider an optimal drawing of K_{2n}. There are $\binom{2n}{n}$ ordered partitions of the vertex set into two color classes of equal size, and each of these partitions induces an ordered copy of $K_{n,n}$, drawn in the plane. Each of these copies contains at least $\operatorname{cr}(K_{n,n})$ crossings.

How often do we count a given crossing? By our assumption that any two edges cross at most once, once the drawing is prescribed, we can identify a crossing with the set of 4 endpoints of the crossing edges. Given these 4 endpoints, there are 4 possibilities of assigning two of them to one color class and two to the other such that the crossing survives: for each edge, we have to choose one endpoint in the first color class. For the remaining vertices, there are $\binom{2n-4}{n-2}$ ways to distribute them into the two color classes. Altogether, there are $4\binom{2n-4}{n-2}$ copies of $K_{n,n}$ that contain the given crossing. Hence,

$$\operatorname{cr}(K_{2n}) \geq \frac{\binom{2n}{n}}{4\binom{2n-4}{n-2}} \operatorname{cr}(K_{n,n}),$$

and by dividing both sides by $\binom{2n}{4}$, we obtain

$$\frac{\mathrm{cr}(K_{2n})}{\binom{2n}{4}} \geq \frac{3}{2}\frac{\mathrm{cr}(K_{n,n})}{\binom{n}{2}^2}. \tag{5.5}$$

Similar arguments show that

$$\frac{\mathrm{cr}(K_{n+1})}{\binom{n+1}{4}} \geq \frac{\mathrm{cr}(K_n)}{\binom{n}{4}} \tag{5.6}$$

and

$$\frac{\mathrm{cr}(K_{n+1,n+1})}{\binom{n+1}{2}^2} \geq \frac{\mathrm{cr}(K_{n,n})}{\binom{n}{2}^2}. \tag{5.7}$$

Thus, the sequences in (5.6) and (5.7) are nondecreasing, and obviously bounded from above by 1. Hence, they converge to certain limits cr_* and $\mathrm{cr}_{*,*}$, respectively, which determine the respective crossing numbers up to lower-order terms. Moreover, by (5.5),

$$\mathrm{cr}_* \geq \frac{3}{2}\,\mathrm{cr}_{*,*}$$

If Zarankiewicz's conjecture is correct, it implies that $\mathrm{cr}_{*,*} = 1/4$, and hence $\mathrm{cr}_* = 3/8$. In particular, this would mean that the above-mentioned constructions give asymptotically optimal drawings of K_n.

Note, however, that neither of these constructions produces rectilinear drawings. The above double-counting arguments carry over verbatim to rectilinear drawings, so the analogues of (5.5), (5.6), and (5.7) for the rectilinear crossing number hold as well. We denote the corresponding limits of the renormalized rectilinear crossing number of complete graphs and complete bipartite graphs by $\overline{\mathrm{cr}}_*$ and $\overline{\mathrm{cr}}_{*,*}$, respectively. As we will see below, $\overline{\mathrm{cr}}_* = \square_*$, and we will show in Section 5.3 that is strictly larger than $3/8 = 0.375$. There appears to be no manifest conjecture as to what the true value of $\overline{\mathrm{cr}}_*$ might be, or what might an optimal straight-edge drawing of K_n might look like. The best construction to date is due to Aichholzer, Aurenhammer, and Krasser [3] and yields $(\square_* =)\overline{\mathrm{cr}}_* < 0.38074$.

Observe that a rectilinear drawing of K_n is completely determined by the placement of its vertices, and that the third condition for drawings (that two arcs share at most a finite number of points) implies that the resulting point set is in general position. Moreover, among the 6 edges spanned by any 4-element subset of the vertices, there is precisely one crossing (between the diagonals) if the corresponding 4 points are in convex position, and no crossing otherwise, see Figure 5.4.

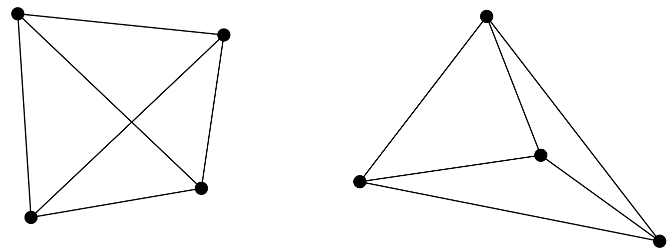


Figure 5.4: *One crossing or no crossing.*

Thus, with the notation introduced at the beginning of this chapter, we have $\overline{\text{cr}}(K_n)$ equals the minimum number $\square(n)$ of convex 4-element subsets in any set of n points in general position in the plane.

The Connection to the Four Point Problem. As Scheinerman and Wilf observed, the limit $\overline{\text{cr}}_* = \lim_{n\to\infty} \overline{\text{cr}}(K_n)/\binom{n}{4} = \lim_{n\to\infty} \square(n)/\binom{n}{4}$ coincides with the infimum for Sylvester's Four-Point Problem,

$$\square_* = \inf_{\mu} \square(\mu) = \lim_{n\to\infty} \frac{\square(n)}{\binom{n}{4}}. \tag{5.8}$$

To see why this is, consider first an arbitrary continuous probability distribution μ, and n points $P_1, \ldots, P_n$ i.i.d. $\sim \mu$. Clearly,

$$\square(\mu)\binom{n}{4} = \mathbf{E}[\square(\{P_1, \ldots, P_n\})] \geq \square(n),$$

and therefore, $\square(\mu) \geq \square(n)/\binom{n}{4}$ for all n, which shows the "$\geq$" part of (5.8). For the other direction, consider an n-point set S which achieves $\square(n) = \square(S)$. For each point $\boldsymbol{p} \in S$, let $B(\boldsymbol{p})$ be a small disk of radius $\varepsilon > 0$ centered at $\boldsymbol{p}$, and let μ_n be the uniform distribution on the union of these disks. If we take four random points P_1, P_2, P_3, P_4 i.i.d. $\sim \mu$, then the probability that some two of them lie in the same disk $B(p)$ is at most $6/n$. On the other hand, if the P_i's lie in pairwise distinct disks, say $P_i \in B(\boldsymbol{p}_i)$, $1 \leq i \leq 4$, and if ε is chosen sufficiently small, then the P_i's are in convex position if and only if the $\boldsymbol{p}_i$'s are. For every ordered 4-element subset $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4\} \subseteq S$, we have $\Pr[P_i \in B(\boldsymbol{p}_i) \text{ for } 1 \leq i \leq 4] = 1/n^4$, and there are $4! \cdot \square(S) = 24\square(n)$ such ordered 4-element subsets in convex position. Hence we obtain a sequence of probability distributions μ_n with

$$\square(\mu_n) \leq \frac{24\square(n)}{n^4} + \frac{6}{n}$$

for all n, and letting $n \to \infty$, we obtain the "$\leq$" part of (5.8).

Small Cases, and Constructions of Rectilinear Drawings. Table 5.1 summarizes what is known about $\mathrm{cr}(K_n)$ and $\Box(n) = \overline{\mathrm{cr}}(K_n)$ for small values of n (see [43, 71, 22, 3]).

n	≤ 4	5	6	7	8	9	10	11	12
$\mathrm{cr}(K_n)$	0	1	3	9	18	36	60	≤ 102	≤ 153
$\Box(n)$	0	1	3	9	19	36	62	102	153

Table 5.1: *Crossing numbers for complete graphs on few vertices.*

By monotonicity of the sequence $\Box(n)/\binom{n}{4}$, every lower bound for $\Box(n_0)$ (respectively, $\mathrm{cr}(K_{n_0})$) for some small integer n_0 yields a lower bound for $\Box_*$ (respectively, cr_*). The best estimate obtained in this fashion is (confer [3])

$$\Box_* > 0.31151.$$

As mentioned above, the best upper bound to date is due to Aichholzer, Aurenhammer, and Krasser [3]. The construction is based on a computer-generated rectilinear drawing of K_{36} with few crossings. In this drawing, every vertex of K_{36} is replaced by a tiny cloud of points arranged along a convex curve very close to a halving line. Upon doing the calculations, this yields

$$\Box_* < 0.38074.$$

We conclude this section by mentioning a construction due to Singer [71]. While it gives a worse upper bound for $\Box_*$ than the construction of Aichholzer, Aurenhammer, and Krasser, Singer's construction is somewhat more "conceptual". Essentially the same construction was independently found by Edelsbrunner and Welzl [36] to give a lower bound of $\Omega(n \log n)$ for the maximum number of halving edges of a set of n points in the plane.

Example 5.1 (The Tripod Construction). Suppose we are given a rectilinear drawing of K_n with few crossings, i.e., a set $S \subset \mathbf{R}^2$ in general position with few convex quadrilaterals. By applying a suitable affine transformation, we may assume that S is very "flat", i.e., that the slopes of all lines spanned by S are less than some small $\varepsilon > 0$ in absolute value. Now take three copies S_1, S_2, and S_3 of S, the second and the third copy rotated by 120 and 240 degrees, respectively, and place them on three rays emanating from the origin

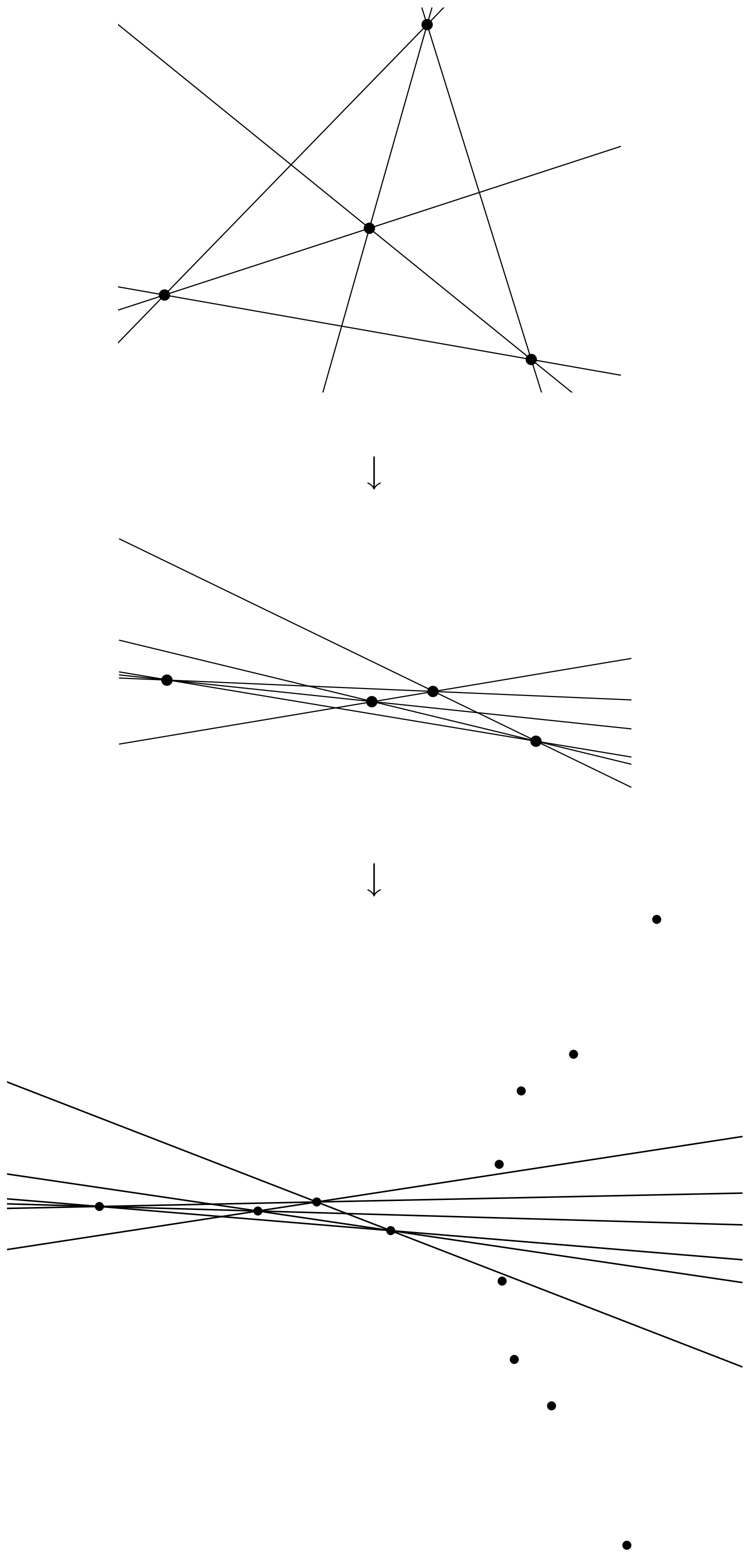


Figure 5.5: *Singer's construction.*

such that every line spanned by S_i has one of the other two copies on either side, see Figure 5.5

This yields a set S' of $3n$ points in general position, and systematic counting shows that

$$\Box(S') = 3 \cdot \Box(S) + 3n\binom{n}{3} + 3\binom{n}{2}^2.$$

By applying this recursively, one obtains $\Box_* \le 5/13 \approx 0.38462$.

For further refinements of this construction, which yield $\Box_* \le 6467/16848 \approx 0.38384$, see Brodsky, Durocher, and Gethner [23].

5.2 Staircases of Encounters

In this section, we develop a new approach towards a lower bound for $\Box_*$. It is based on the method of h-vectors described in Chapter 4, with suitable refinements. The aim is to show the following:

Theorem 5.2.

$$\Box_* \ge (53 + 5\sqrt{13})/216 > 0.3288$$

A Warm-up. We first outline how to obtain a lower bound of

$$\Box_* \ge 1/4$$

by a straightforward application of the Upper Bound Theorem 4.15.

We recall the conclusions of the theorem in the special context we are considering. Let S be a set of n points in general position in $\mathbf{R}^2$. For a point $\boldsymbol{p} \in S$, the number

$$f_0(S \setminus \boldsymbol{p}, \boldsymbol{p}) = \{X \subseteq S \setminus \boldsymbol{p} : |X| = 3, \boldsymbol{p} \in \operatorname{conv} X\}$$

(where "$S \setminus \boldsymbol{p}$" is short for "$S \setminus \{\boldsymbol{p}\}$") can be expressed in terms of the h-vector of $\boldsymbol{p}$ relative to $S \setminus \boldsymbol{p}$,

$$f_0(S \setminus \boldsymbol{p}) = \sum_{j=0}^{n-4} h_j(S \setminus \boldsymbol{p}, \boldsymbol{p}). \tag{5.9}$$

The Upper Bound Theorem tells us that the entries of the h-vector can be bounded in terms of the depth of $\boldsymbol{p}$ in $S \setminus \boldsymbol{p}$: if there is a line ℓ through p such that one of the open halfplanes defined by ℓ contains only $a \le \lfloor n/2 \rfloor - 2$ points from $S \setminus \boldsymbol{p}$, then for $0 \le j \le \lfloor n/2 \rfloor - 2$,

$$h_j(S \setminus \boldsymbol{p}, \boldsymbol{p}) \le \binom{j+2}{2} - \binom{j-a+2}{2}. \tag{5.10}$$

Furthermore, the Dehn-Sommerville Equations read

$$h_j(S \setminus \boldsymbol{p}, \boldsymbol{p}) = h_{n-4-j}(S \setminus \boldsymbol{p}, \boldsymbol{p}). \tag{5.11}$$

Combining (5.9), (5.10), and (5.11), we obtain

$$\begin{aligned} f_0(S \setminus \boldsymbol{p}, \boldsymbol{p}) &\leq \sum_{j=0}^{\lfloor n/2 \rfloor - 2} \left(\binom{j+2}{2} - \binom{j-a+2}{2} \right) + \sum_{j=0}^{\lceil n/2 \rceil - 3} \left(\binom{j+2}{2} - \binom{j-a+2}{2} \right) \\ &= \binom{\lfloor n/2 \rfloor + 1}{3} - \binom{\lfloor n/2 \rfloor + 1 - a}{3} + \binom{\lceil n/2 \rceil}{3} - \binom{\lceil n/2 \rceil - a}{3}. \end{aligned}$$

Observe that this last expression is monotonically increasing in a. Thus, in order to bound $f_0(S \setminus \boldsymbol{p}, \boldsymbol{p})$, it suffices to bound a.

Up to a suitable rotation, we may assume that no two points of S have the same x-coordinate. Thus, if we order the points in S as $\boldsymbol{p}_0, \boldsymbol{p}_1, \ldots, \boldsymbol{p}_{n-1}$ according to their x-coordinate, the vertical line through $\boldsymbol{p}_a$ has a points on one side and $n-1-a$ points on the other.

Now we use this to estimate the number of concave (i.e., non-convex) 4-element subsets of S, which we can write as

$$\begin{aligned} \triangle(S) &= \sum_{\boldsymbol{p} \in S} f_0(S \setminus \boldsymbol{p}, \boldsymbol{p}) \\ &\leq n\left(\binom{\lfloor n/2 \rfloor + 1}{3} + \binom{\lceil n/2 \rceil}{3} \right) - \sum_{a=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\binom{\lfloor n/2 \rfloor + 1 - a}{3} + \binom{\lceil n/2 \rceil - a}{3} \right) \\ &\qquad - \sum_{a=0}^{\lceil \frac{n-3}{2} \rceil} \left(\binom{\lfloor n/2 \rfloor + 1 - a}{3} + \binom{\lceil n/2 \rceil - a}{3} \right) \\ &= n\left(\binom{\lfloor n/2 \rfloor + 1}{3} + \binom{\lceil n/2 \rceil}{3} \right) - 2\left(\binom{\lfloor n/2 \rfloor + 2}{4} + \binom{\lceil n/2 \rceil + 1}{4} \right) \\ &= \tfrac{3}{4}\binom{n}{4} + O(n^3). \end{aligned}$$

Since this holds for every n-point set S, we obtain $\square(n) \geq \frac{1}{4}\binom{n}{4} - O(n^3)$, i.e., $\square_* \geq 1/4$, as advertised.

We note that both steps in the above reasoning are essentially tight. On the one hand, given a, it is not difficult (details omitted) to construct a set S of n points in general position and a point $\boldsymbol{p} \in S$ such that

1. there is a line through $\boldsymbol{p}$ that contains a points on one side, and

2. $h_j(S \setminus \boldsymbol{p}, \boldsymbol{p})$ attains the upper bound (5.10) for all $j \leq n/2 - 2$.

On the other hand, here is a recursive construction of a set S_k of $2k+3$ points in general position that contains 3 points of depth 0 and 2 points of depth a, for $1 \leq a \leq k$: Let S_0 be the vertex set of an arbitrary triangle which contains the origin $\mathbf{0}$ in its interior. Assume now that we have constructed P_k, that $\mathbf{0}$ does not lie on any line spanned by two points of S_k and that any line through $\mathbf{0}$ contains at least $k+1$ points from S_k on either side. Choose any line through $\mathbf{0}$ which avoids S_k. This line determines two open halfplanes, one of which contains exactly $k+1$ points of S_k. For a suitably chosen $\varepsilon > 0$, let $\boldsymbol{p}, \boldsymbol{p}'$ be two new points in that halfplane such that $\boldsymbol{p}\mathbf{0}\boldsymbol{p}'$ is an isosceles triangle of height ε^2 whose base pp' is parallel to the chosen line and of length ε. It is easy to see that for sufficiently small ε, $S_{k+1} := S_k \cup \{p, p'\}$ has again the desired properties.

We remark that it is not hard to show that these "shallow" sets of $n = 2k+3$ points contain $\frac{1}{2}\binom{n}{4} + O(n^3)$ convex quadrilaterals.

"Global" versus "local". We now refine our analysis. The basic idea is to exploit a certain trade-off, to be made precise below, between the "global" number of all crossings on the one hand and the "local" number of crossings involving a specific point on the other hand. The main technical tool will be a slightly different encoding of the g-vector of a point by means of so-called "staircases of encounters."

Let P be a set of n points in general position in the plane. For $\boldsymbol{p} \in P$, we define

$$\square(\boldsymbol{p}, P \setminus \boldsymbol{p}) := |\{T \in \tbinom{P \setminus \boldsymbol{p}}{3} : T \cup \boldsymbol{p} \text{ is in convex position}\}|. \tag{5.12}$$

As a first step, observe that we can express $\square(P)$ as the sum

$$\square(P) = \frac{1}{4} \sum_{p \in P} \square(p, P \setminus p). \tag{5.13}$$

We now introduce the key ingredient of our proof of Theorem 5.2, which combines "global" and "local" considerations:

Definition 5.3. Let S be a set of $n+1$ points in general position in the Euclidean plane, and let $\boldsymbol{q} \in S$ We define

$$\Lambda(\boldsymbol{q}, S \setminus \boldsymbol{q}) := \max\left\{ \frac{\square(S \setminus \boldsymbol{q})}{\binom{n}{4}}, \frac{\square(\boldsymbol{q}, S \setminus \boldsymbol{q})}{\binom{n}{3}} \right\}. \tag{5.14}$$

Moreover, let

$$\Lambda(n) := \min_S \max_{\boldsymbol{q}\in S} \Lambda(\boldsymbol{q}, S \setminus \boldsymbol{q}),$$

where the minimum is taken over all sets S of $n+1$ points in general position and the maximum over all $\boldsymbol{q} \in S$. Similarly,

$$\Lambda^{\mathsf{x}}(n) := \min_S \max_{\boldsymbol{v}} \Lambda(\boldsymbol{v}, S \setminus \boldsymbol{v}),$$

with the minimum taken over all sets S of $n+1$ points in general position, and the maximum over all *vertices* $\boldsymbol{v}$ of $\operatorname{conv} S$ ("x" for "extreme point").

Lemma 5.4.

$$\square_* = \liminf_{n\to\infty} \Lambda^{\mathsf{x}}(n) = \liminf_{n\to\infty} \Lambda(n).$$

Proof. Observe that for all n,

$$\square(n) \le \Lambda^{\mathsf{x}}(n) \le \Lambda(n).$$

It follows that $\square_* \le \liminf \Lambda^{\mathsf{x}}(n) \le \liminf \Lambda(n) =: c$, and it suffices to show that conversely, $c \le \square_*$. To this end, fix $\varepsilon > 0$, and choose $n_0 \in \mathbf{N}$ such that $\Lambda(n) \ge c - \varepsilon$ for all $n \ge n_0$.

Claim A. Suppose $|S| = n+1 > n_0$. Then there exists a point $\boldsymbol{p} \in S$ such that $\square(\boldsymbol{p}, S \setminus \boldsymbol{p}) \ge (c-\varepsilon)\binom{n-1}{3}$.

To see this, let $\boldsymbol{q} \in S$ such that $\Lambda(\boldsymbol{q}, S\setminus\boldsymbol{q}) \ge c-\varepsilon$. If $\square(\boldsymbol{q}, S\setminus\boldsymbol{q}) \ge (c-\varepsilon)\binom{n}{3}$, then $\boldsymbol{q}$ is the point we are looking for. Otherwise, $\square(S \setminus \boldsymbol{q}) \ge (c-\varepsilon)\binom{n}{4}$, by definition of $\Lambda(\boldsymbol{q}, S \setminus \boldsymbol{q})$. Thus, by applying (5.13) to the set $P = S \setminus \boldsymbol{q}$, we see that there is some $\boldsymbol{p} \in S \setminus \boldsymbol{q}$ for which

$$\square(\boldsymbol{p}, S \setminus \{\boldsymbol{p}, \boldsymbol{q}\}) \ge \frac{4}{n}\square(S \setminus \boldsymbol{q}) \ge (c-\varepsilon)\binom{n-1}{3}.$$

Since $\square(\boldsymbol{p}, S \setminus \boldsymbol{p}) \ge \square(\boldsymbol{p}, S \setminus \{\boldsymbol{p}, \boldsymbol{q}\})$, this proves Claim A.

Claim B. For all $n \ge n_0$,

$$\square(n) \ge (c-\varepsilon)\binom{n-1-n_0}{4}$$

We proceed by induction on n. The claim is clearly true for $n = n_0$. Moreover, if $n > n_0$, let P be a set of n points achieving $\square(P) = \square(n)$. By Claim A, there is some point $p \in P$ with $\square(p, P \setminus p) \ge (c-\varepsilon)\binom{n-2}{3}$. By induction,

$\square(P \setminus p) \geq (c-\varepsilon)\binom{n-2-n_0}{4}$. Together with the quadrilaterals in which p participates, this yields

$$\square(P) \geq (c-\varepsilon)\left(\binom{n-2}{3} + \binom{n-2-n_0}{4}\right) \geq (c-\varepsilon)\binom{n-1-n_0}{4}$$

quadrilaterals in P, which proves Claim B.

Finally, since $\lim_{n\to\infty} \binom{n-1-n_0}{4}/\binom{n}{4} = 1$, it follows that $\square_* \geq c-\varepsilon$, and because this holds for all $\varepsilon > 0$, the proof is complete. $\square$

Staircases of Encounters. In view of Lemma 5.4, our goal is to estimate $\Lambda(n)$ or $\Lambda^{\times}(n)$. We focus on the latter, and now develop the necessary tools.

Let S be a set of $n+1$ points in general position in the plane. Fix a vertex $\boldsymbol{v}$ of the convex hull of S, and set $P = S \setminus \boldsymbol{v}$.

Consider a point $\boldsymbol{p} \in P$, and let ℓ be the line through $\boldsymbol{p}$ and $\boldsymbol{v}$, oriented from $\boldsymbol{p}$ towards $\boldsymbol{v}$. Let L be the set of points of P that lie to the left of ℓ, and R the set of those that lie to the right.

For $k := |L|$, the grid

$$\beta(\boldsymbol{p}) := \beta(\boldsymbol{v}, P, \boldsymbol{p}) := \{0 \ldots k-1\} \times \{0 \ldots n-2-k\}$$

will be referred to as the *box* of p. We “fill” this box, i.e. we define a subset $\lambda(\boldsymbol{p}) \subseteq \beta(\boldsymbol{p})$, in the following fashion: Enumerate the points in L in the order $\boldsymbol{q}_0, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_{k-1}$ in which they are first encountered when we rotate ℓ clockwise, and set

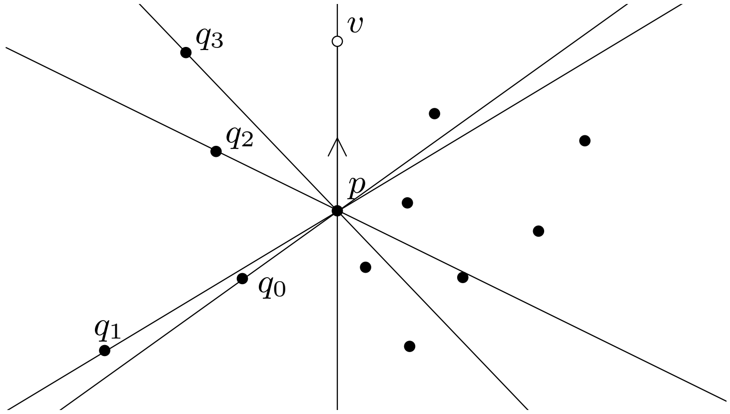
$$\lambda(\boldsymbol{p}) := \lambda(\boldsymbol{v}, P, \boldsymbol{p}) \;\; := \{(a,b) \in \beta(\boldsymbol{v}, P, \boldsymbol{p}) : b < |\mathrm{H}^-(\boldsymbol{q}_a, \boldsymbol{p}) \cap R|\}. \quad (5.15)$$

Here, $\mathrm{H}^-(\boldsymbol{q}, \boldsymbol{p})$ denotes the open halfspace to the right of the oriented line from $\boldsymbol{q}$ through $\boldsymbol{p}$. See also Figure 5.6. We call $\lambda(p)$ the *staircase of encounters* of p.

We proceed to relate these staircases of encounters to the object of our investigation, $\Lambda^{\times}(\boldsymbol{v}, P)$. For $\boldsymbol{p} \in P$ and $0 \leq i \leq n-2$, let $\delta_i(\boldsymbol{p})$ be the number of entries of $\lambda(\boldsymbol{p})$ on the ith diagonal, i.e.

$$\delta_i(\boldsymbol{p}) = \delta_i(\boldsymbol{v}, P, \boldsymbol{p}) := |\{(a,b) \in \lambda(\boldsymbol{v}, P, \boldsymbol{p}) : a+b = i\}|. \quad (5.16)$$

The upcoming Lemma 5.5 and Corollary 5.7 express $\Lambda^{\times}(\boldsymbol{v}, P)$ in terms of the numbers $\delta_i(\boldsymbol{p})$. The first describes the connection of the $\delta_i(\boldsymbol{p})$'s with $\square(v, P)$. Note that for $\boldsymbol{p} \in P$, the sum $\sum_i \delta_i(p)$ counts the number of pairs $\{\boldsymbol{q}, \boldsymbol{r}\} \subset P \setminus \boldsymbol{p}$ such that $\boldsymbol{p} \in \mathrm{conv}\{\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{v}\}$. Therefore:



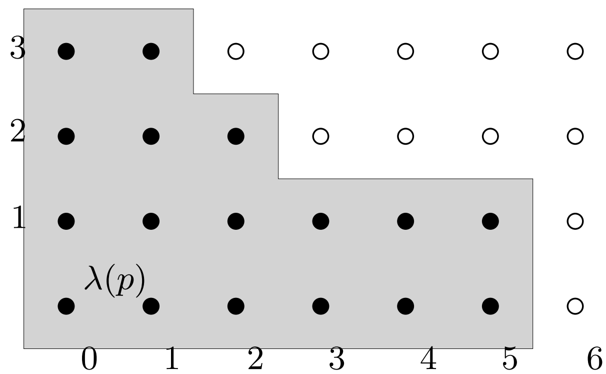


Figure 5.6: *Staircase of Encounters.*

Lemma 5.5. *The number of triples $T \subseteq P$ that form a convex quadrilateral with $\boldsymbol{v}$ is*

$$\square(\boldsymbol{v}, P) = \binom{n}{3} - \sum_{\boldsymbol{p} \in P} \sum_{i} \delta_i(\boldsymbol{p}). \tag{5.17}$$

In order to relate the $\delta_i(\boldsymbol{p})$'s to $\square(P)$, we have to work a little more. Recall that for $\boldsymbol{p} \in P$, $f_0(\boldsymbol{p}, P \setminus \boldsymbol{p})$ denotes the number of triples $T \subseteq P \setminus \boldsymbol{p}$ that contain $\boldsymbol{p}$ in their convex hull. Observe that

$$\square(P) = \binom{n}{4} - \sum_{\boldsymbol{p} \in P} f_0(\boldsymbol{p}, P \setminus \boldsymbol{p}). \tag{5.18}$$

Lemma 5.6. *For $\boldsymbol{p} \in P$, we can express the g-vector (see Definition 4.24) of $\boldsymbol{p}$ relative to $P \setminus \boldsymbol{p}$ as*

$$g_i(P \setminus \boldsymbol{p}) = \delta_i(\boldsymbol{v}, P, \boldsymbol{p}) - \delta_{n-3-i}(\boldsymbol{v}, P, \boldsymbol{p}). \tag{5.19}$$

Corollary 5.7. *Lemma 5.6 implies*

$$f_0(\boldsymbol{p}, P \setminus \boldsymbol{p}) = \sum_i (n-3-2i)\delta_i(p).$$

Thus, by (5.18), we get

$$\square(P) = \binom{n}{4} - \sum_{p\in P}\sum_{i=0}^{n-3}(n-3-2i)\delta_i(p). \tag{5.20}$$

Proof of Lemma 5.6. Let ℓ be the line through $\boldsymbol{p}$ and $\boldsymbol{v}$, oriented from $\boldsymbol{p}$ towards $\boldsymbol{v}$, and let ℓ_L and ℓ_R be two parallel translates of ℓ to the left and right of ℓ, respectively. As in the definition of $\delta_i(\boldsymbol{p})$, we write L for the set of points from $P \setminus \boldsymbol{p}$ that lie to the left of ℓ, and R for the set of those to the right.

The $\delta_i(\boldsymbol{p})$'s only depend on the circular ordering of the rays that emanate from $\boldsymbol{p}$ and pass through the points of $P \setminus \boldsymbol{p}$. The same holds for the numbers $f_k(P \setminus \boldsymbol{p}) = \{X \subseteq P \setminus \boldsymbol{p} : |X| = k+3, \boldsymbol{p} \in \operatorname{conv} X\}$, which we know determine the g-vector of $\boldsymbol{p}$ relative to $P \setminus \boldsymbol{p}$. Thus, by sliding the points along these rays if necessary, we may assume that all the points in L lie on ℓ_L and all points from R lie on ℓ_R, see Figure 5.7.

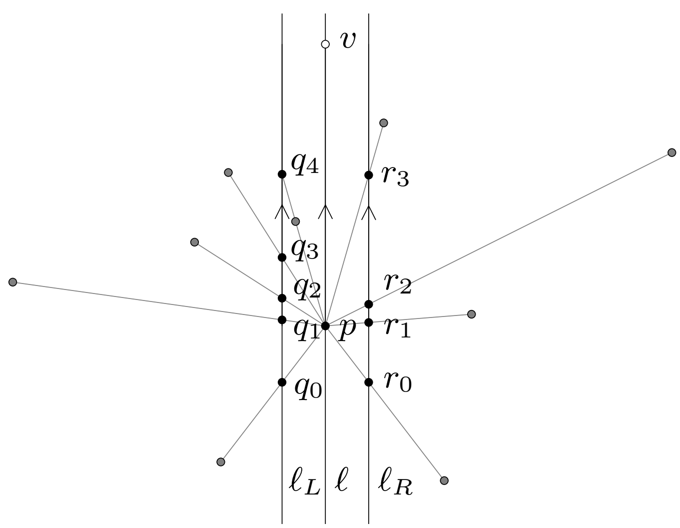


Figure 5.7: *Numberings of L and R.*

Consider the points in L and R in the order in which they appear along the lines ℓ_L and ℓ_R, respectively, $L = \{q_0, \ldots, q_{k-1}\}$ and $R = \{r_0, \ldots, r_{n-k-1}\}$. For the points in L, this agrees with the ordering in the definition of $\lambda(\boldsymbol{p})$.) Then each pair $(a, b) \in B(p)$ corresponds to the pair $(q_a, r_b) \in L \times R$.

The crucial observation is that since we assume that $P \setminus \boldsymbol{p}$ is concentrated on the two lines ℓ_L and ℓ_R, the oriented edge $[\boldsymbol{q}_a, \boldsymbol{r}_b]$ is a $(a+b)$-edge of $P \setminus \boldsymbol{p}$ that is entered by ℓ. Moreover, it is entered before $\boldsymbol{p}$ (see Definition 4.10) iff $(a, b) \in \lambda(\boldsymbol{p})$. Therefore, $\delta_i(\boldsymbol{v}, P, \boldsymbol{p})$ counts the the number of i-edges of $P \setminus \boldsymbol{p}$ that are entered by ℓ before $\boldsymbol{p}$. In other words,

$$\delta_i(\boldsymbol{v}, P, \boldsymbol{p}) = h^*(P \setminus \boldsymbol{p}, \ell, \boldsymbol{p}),$$

from which the lemma follows immediately. □

Let us rewrite these conclusions as follows: Define

$$\Gamma(\boldsymbol{v}, P) := \min\{\Gamma_1(\boldsymbol{v}, P), \Gamma_2(\boldsymbol{v}, P)\},$$

where

$$\Gamma_1(\boldsymbol{v}, P) := \frac{\sum_{i=0}^{n-3}(n-3-2i)\sum_{\boldsymbol{p}\in P}\delta_i(\boldsymbol{v}, P, \boldsymbol{p})}{\binom{n}{4}}$$

and

$$\Gamma_2(\boldsymbol{v}, P) := \frac{\sum_{i=0}^{n-3}\sum_{\boldsymbol{p}\in P}\delta_i(\boldsymbol{v}, P, \boldsymbol{p})}{\binom{n}{3}}.$$

With this notation, (5.17) and (5.20) just state that

$$\Lambda(\boldsymbol{v}, P) = 1 - \Gamma(\boldsymbol{v}, P). \tag{5.21}$$

A Lower Bound for General Staircases. A *staircase* is a set $\lambda \subseteq \mathbf{N}_0 \times \mathbf{N}_0$ of pairs of nonnegative integers such that $(a, b) \in \lambda$ and $0 \leq a' \leq a$ and $0 \leq b' \leq b$ imply $(a', b') \in \lambda$.

Let us look back at what we did so far: In order to analyze $\Lambda(\boldsymbol{v}, P)$, we associated a certain staircase $\lambda(\boldsymbol{p}) = \lambda(\boldsymbol{v}, P, \boldsymbol{p})$ with every point $\boldsymbol{p} \in P$. Then we counted the number of entries on the ith diagonal of each of these staircases, and, in (5.21), expressed $\Lambda(\boldsymbol{v}, P)$ in terms of the resulting numbers $\delta_i(\boldsymbol{p})$.

Let us now forget about the geometric context. For a staircase λ and integer i, let

$$\delta_i(\lambda) := \{(a, b) \in \lambda : a + b = i\}.$$

For $1 \leq k \leq n-3$, let $\beta_k := \{0 \ldots k\} \times \{0 \ldots n-3-k\}$, and consider a sequence $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-3})$ of staircases $\lambda_k \subseteq \beta_k$. Taking (5.21) as a starting point, we define

$$\Gamma_1(\boldsymbol{\lambda}) := \frac{\sum_i(n-3-2i)\sum_k \delta_i(\lambda_k)}{\binom{n}{4}}, \tag{5.22}$$

$$\Gamma_2(\boldsymbol{\lambda}) := \frac{\sum_i \sum_k \delta_i(\lambda_k)}{\binom{n}{3}}, \tag{5.23}$$

and

$$\Gamma(\boldsymbol{\lambda}) := \min\left\{\Gamma_1(\boldsymbol{\lambda}), \Gamma_2(\boldsymbol{\lambda})\right\}. \tag{5.24}$$

(Observe that $\binom{n}{3}\Gamma_2(\boldsymbol{\lambda}) = \sum_k |\lambda_k|$.) We proceed to prove an upper bound for $\Gamma(\boldsymbol{\lambda})$, which, by (5.21) and by Lemma 5.4, yields a lower bound for $\square_*$.

Observe that there is a certain trade-off between Γ_1 and Γ_2: On the one hand, Γ_2 is maximized if $\lambda_k = \beta_k$ for all k ("all boxes are full"). On the other hand, it is not hard to see (but we need not worry about that) that Γ_1 is maximized if $\lambda_k = \{(a,b) \in \beta_k : a + b \leq (n-3)/2\}$ ("all boxes are filled up to the middle diagonal"). Roughly speaking, we obtain the upper bound for $\Gamma(\boldsymbol{\lambda})$ by finding the "equilibrium" of Γ_1 and Γ_2.

As a first step, we observe that we can restrict our attention to staircases of a special shape. Let us say that λ_k results from *filling the box β_k up to the jth diagonal* if, for all $(a,b) \in \beta_k$,

$$a + b < j \Rightarrow (a,b) \in \lambda_k, \quad \text{and} \quad a + b > j \Rightarrow (a,b) \notin \lambda_k.$$

(Observe that we do not say anything about the elements of λ_k *on* the jth diagonal.)

Lemma 5.8. *Suppose that $M = \binom{n}{3}\Gamma_2(\boldsymbol{\lambda}) = \sum_i \sum_k \delta_i(\lambda_k)$ is prescribed. Under this constraint, $\Gamma_1(\boldsymbol{\lambda})$ is maximized iff each λ_k is obtained by filling β_k up to the jth diagonal, for a certain $j = j(M)$.*

Proof. Let j be maximal with the property that $\sum_k \sum_{i<j} \delta_i(\beta_k) \leq M$. Suppose that $(a,b) \in \beta_k \backslash \lambda_k$ and $(a',b') \in \lambda_{k'}$, for some k, k', such that $a+b < j$ and $a' + b' > j$. Then by removing (a',b') from $\lambda_{k'}$ and by adding (a,b) to λ, we increase Γ_1 while leaving Γ_2 invariant. The remaining elements of the staircases are distributed in an arbitrary fashion on the jth diagonals. $\square$

Thus, we may assume that all λ_k's are of this kind. The question remains, up to which diagonal the β_k's are filled.

Lemma 5.9. *Let $j = \lfloor \alpha(n-3) \rfloor$, for $\alpha \in [0,1]$, and suppose that each λ_k is obtained by filling β_k up to the jth diagonal. Then,*

$$\Gamma_1(\boldsymbol{\lambda}) = \underbrace{12\alpha^2(1 - 2\alpha + \alpha^2)}_{=:\, F_1(\alpha)} + O(1/n) \tag{5.25}$$

and

$$\Gamma_2(\boldsymbol{\lambda}) = \underbrace{\alpha^2(3-2\alpha)}_{=:\,F_2(\alpha)} + O(1/n). \tag{5.26}$$

The proof of Lemma 5.9 consists of straightforward calculations, which we defer to the end of this section.

Having the preceding lemma at our disposal, it is now easy to prove the desired estimate for $\Gamma(\boldsymbol{\lambda})$: For $\boldsymbol{\lambda}$ as in Lemma 5.9, we have

$$\Gamma(\boldsymbol{\lambda}) = \min\{F_1(\alpha), F_2(\alpha)\} + o(1).$$

Moreover, since we are interested in the limit behavior as $n \to \infty$, we can ignore the $o(1)$ error term. Thus, since we want to prove an upper bound for Γ, the question remains which α maximizes $\min\{F_1(\alpha), F_2(\alpha)\}$.

Let us first consider the interval $[\frac{1}{2}, 1]$: Here, F_1 is a monotonically decreasing function while F_2 is increasing. Moreover, $F_1(1/2) = 3/4 > 1/2 = F_2(1/2)$ and $F_1(1) = 0 < 1 = F_2(1)$, so $\max_{\alpha \in [1/2,1]} \min\{F_1(\alpha), F_2(\alpha)\}$ is attained at some α for which $F_1(\alpha) = F_2(\alpha)$. The roots of $F_1(\alpha) - F_2(\alpha) = 9\alpha^2 - 22\alpha^3 + 12\alpha^4$ are

$$0,\ 0,\ \frac{1}{12}(11+\sqrt{13}),\ \frac{1}{12}(11-\sqrt{13}).$$

Thus, the root we are looking for is $\alpha^* = (11-\sqrt{13})/12$. Moreover, by considering first and second derivatives at 0, we see that $F_1 \geq F_2$ on the interval $[0, \alpha^*]$. Therefore, since F_2 is increasing, α^* maximizes $\min\{F_1, F_2\}$ over the whole interval $[0,1]$, and $F_1(\alpha^*) = F_2(\alpha^*) = (163-5\sqrt{13})/216) < 0.6712$. We have proved:

Theorem 5.10. *For every sequence* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n-3}$ *of staircases* $\lambda_k \subseteq \{0 \ldots k\} \times \{0 \ldots n-3-k\}$, *we have*

$$\Gamma(\boldsymbol{\lambda}) \leq (163-5\sqrt{13})/216) + O(1/n).$$

Corollary 5.11. *In particular, for every set* S *of* $n+1$ *points in general position and for any vertex* v *of* $\mathrm{conv}(S)$,

$$\Lambda(v, S \setminus v) \geq (53+5\sqrt{13})/216 + O(1/n).$$

By Lemma 5.4, this also establishes Theorem 5.2.

Proof of Lemma 5.9. By symmetry, we have $\delta_i(\beta_k) = \delta_i(\beta_{n-3-k})$ for all i, k. Furthermore, for $k \leq (n-3)/2$,

$$\delta_i(\beta_k) = \begin{cases} i+1 & \text{if} \quad 0 \leq i \leq k, \\ k+1 & \text{if} \quad k < i < n-3-k, \\ n-2-i & \text{if} \quad n-3-k \leq i \leq n-3. \end{cases}$$

By assumption, $\delta_i(\lambda_k) = \delta_i(\beta_k)$ for $i < j$ and $\delta_i(\lambda_k) = 0$ for $i > j$. Therefore, for $0 \leq i \leq \min\{j, (n-3)/2\}$,

$$\begin{aligned} \sum_{k=0}^{n-3} \delta_i(\lambda_k) &= 2 \sum_{k=0}^{(n-3)/2} \delta_i(\lambda_k) + O(n) \\ &= 2\Big(\sum_{k<i} (k+1) + \sum_{k=i}^{\lfloor (n-3)/2 \rfloor} (i+1) \Big) + O(n) \\ &= (n-3)(i+1) - 2\binom{i+1}{2} + O(n). \end{aligned}$$

(Here, the $O(n)$ error term covers the fact that when $n-3$ is even, the middle term $\delta_i(\lambda_{(n-3)/2})$ appears once too often. This error term also takes care of the difference between $\lfloor (n-3)/2 \rfloor (i+1)$ and $(n-3)(i+1)/2$ in the third step. Similar simplifications will be made in what follows.) Thus, for $j = \lfloor \alpha n \rfloor \leq (n-3)/2$,

$$\begin{aligned} \sum_{i<j} \sum_k \delta_i(\lambda_k) &= (n-3)\binom{j+1}{2} - 2\binom{j+1}{3} + O(n^2) \\ &= \underbrace{(3\alpha^2 - 2\alpha^3)}_{= F_2(\alpha)} \binom{n}{3} + O(n^2) \end{aligned}$$

Furthermore, again by symmetry, $\delta_i(\beta_k) = \delta_{n-3-i}(\beta_k)$ for $o \leq i, k \leq n-3$.

Hence, for $j > (n-3)/2$, we get again

$$\begin{aligned}
\sum_{i<j}\sum_{k}\delta_i(\lambda_k) &= 2\sum_{i=0}^{(n-3)/2}\sum_{k}\delta_i(\lambda_k) - \sum_{i=0}^{n-3-j}\sum_{k}\delta_i(\lambda_k) + O(n^2) \\
&= (2F_2(1/2) - F_2(1-\alpha))\binom{n}{3} + O(n^2) \\
&= F_2(\alpha)\binom{n}{3} + O(n^2).
\end{aligned}$$

Here, the second to last step follows from the case $j \le (n-3)/2$, while the last one reflects the property $F_2(\alpha) = 2F_2(1/2) - F_2(1-\alpha)$, which is easily verified. Thus, we have proved (5.26).

For (5.25), we observe that, for $j \le (n-3)/2$,

$$\begin{aligned}
\sum_{i<j}\sum_{k} i\delta_i(\lambda_k) &= \sum_{i<j} i\left((n-3)(i+1) - 2\binom{i+1}{2} + O(n)\right) \\
&= 2(n-3)\binom{j+1}{3} - 6\binom{j+1}{4} + O(n^3) \\
&= \left(8\alpha^3 - 6\alpha^4\right)\binom{n}{4} + O(n^3).
\end{aligned}$$

Therefore, for $j \le (n-3)/2$,

$$\begin{aligned}
&\sum_{i<j}\sum_{k}(n-3-2i)\delta_i(\lambda_k) \\
&= (n-3)\left(3\alpha^2 - 2\alpha^3\right)\binom{n}{3} - 2\left(8\alpha^3 - 6\alpha^4\right)\binom{n}{4} + O(n^3) \\
&= \underbrace{\left(12\alpha^2 - 24\alpha^3 + 12\alpha^4\right)}_{=F_1(\alpha)}\binom{n}{4} + O(n^3).
\end{aligned}$$

This proves (5.25) for the case that $j \le (n-3)/2$. Finally, to establish it for $j > (n-3)/2$, we observe that for every k, we can rewrite $\sum_i (n-3-2i)\delta_i(p) = \sum_i (n-2-i)(\delta_i(\lambda_k) - \delta_{n-3-i}(\lambda_k))$. Thus, if $j > (n-3)/2$, then for i between $(n-3)/2$ and j, $\delta_i(\lambda_k) - \delta_{n-3-i}(\lambda_k) = 0$, so these terms cancel each other out. Therefore, up to an $o(1)$ error term, Γ_1 is the same whether the boxes β_k are filled up to the j-th or up to the $(n-3-j)$-th diagonal. This completes the proof because F_1 is also symmetric about $\frac{1}{2}$. $\square$

5.3 Convex Quadrilaterals and k-Sets

We now describe yet another approach to find a lower bound for the number $\square(S)$ of convex quadrilaterals of a finite point set S in general position in the plane. Our goal is to prove the following

Theorem 5.12. *Let S be a set of n points in the plane in general position. Then the number of convex quadrilaterals determined by S is at least*

$$(3/8+\varepsilon)\binom{n}{4}+O(n^3)>0.37501\binom{n}{4},$$

where $\varepsilon \approx 1.0887\cdot 10^{-5}$.

We note that a lower bound of $3/8\binom{n}{4}$ has been established independently by Ábrego and Fernández-Merchant [1], using methods similar to ours.

The small ε is significant because as noted in Section 5.1, the ordinary crossing number of K_n is at most $3/8\binom{n}{4}+O(n^3)$. Thus, while it is well-known that the ordinary crossing number and the rectilinear crossing number of complete graphs differ (the smallest n for which they differ is 8, see Table 5.1), our lower bound shows that the difference lies in the asymptotically relevant term.

The first ingredient for the proof of Theorem 5.12 is a lemma that expresses $\square(S)$ as a positive linear combination of the numbers $e_j(S)$ of j-facets of S (one might say, as the "second moment" of the distribution of j-facets).

Lemma 5.13. *For every set S of n points in the plane in general position,*

$$\square(S)=\sum_{j<\frac{n-2}{2}} e_j(S)\left(\frac{n-2}{2}-j\right)^2-\frac{3}{4}\binom{n}{3}.$$

The proof of this lemma is based on the following observation, which says, roughly speaking, that the "k^{th} moment" of the distribution of j-facets of a finite set in general position in dimension d gives, up to appropriate renormalization, the expected number of facets of the polytope spanned by a random $(d+k)$-element subset:

Observation 5.14. *Let S be a set of n points in general position in $\mathbf{R}^d$. Then, for all k,*

$$\sum_j \binom{j}{k} e_j(S) = \sum_{X \in \binom{S}{d+k}} f_{d-1}(\operatorname{conv}(X)).$$

Proof. By general position, we have $f_{d-1}(\operatorname{conv} X) = e_0(X) = e_k(X)$ for all $X \in \binom{S}{d+k}$. Thus, the right hand side of the above equation counts the number of pairs (X, σ), where $X \in \binom{S}{d+k}$ and σ is a k-facet of X.

The left-hand side is just a different way of counting these pairs. For each j-facet σ of S, there are $\binom{j}{k}$ possibilities to complete the d points spanning σ to a $(d+k)$-element subset which has σ as a k-facet: we have to choose the remaining k points from the positive side of σ. □

Proof of Lemma 5.13. Specializing the previous observation to $d = k = 2$, we obtain

$$\sum_{j=0}^{n-2} \binom{j}{2} e_j(S) = \sum_{X \in \binom{S}{4}} f_1(\operatorname{conv} X) = 3\triangle(S) + 4\square(S).$$

Moreover, we have

$$\square(S) + \triangle(S) = \binom{n}{4}.$$

Thus, we can substitute $\triangle(S) = \binom{n}{4} - \square(S)$ into the first equation and obtain

$$\sum_{j=0}^{n-2} \binom{j}{2} e_j(S) = \square(S) + 3\binom{n}{4}.$$

Next, we use that

$$\sum_{j=0}^{n-2} e_j(S) = 2\binom{n}{2}, \tag{5.27}$$

which implies that we can write

$$3\binom{n}{4} = \sum_{j=0}^{n-2} e_j(S) \frac{(n-2)(n-3)}{8}$$

to get

$$\begin{aligned}
\Box(S) &= \sum_{j=0}^{n-2}\left(\binom{j}{2} - \frac{(n-2)(n-3)}{8}\right) e_j(S) \\
&= \sum_{j=0}^{n-2}\underbrace{\left(\binom{j}{2} - \frac{(n-2)(n-4)}{8}\right)}_{=0 \text{ for } j=(n-2)/2} e_j(S) - \frac{n-2}{8}\sum_{j=0}^{n-2} e_j(S) \\
&= \sum_{j<\frac{n-2}{2}} \left(\frac{n-2}{2} - j\right)^2 e_j(S) + \frac{3}{4}\binom{n}{3},
\end{aligned}$$

where we use (5.27) and the fact that $e_j(S) = e_{n-2-j}(S)$. □

Having expressed $\Box(S)$ as a positive linear combination of the $e_j(S)$'s (up to a lower-order error term), we can substitute any lower bound for the numbers $e_j(S)$ to obtain a lower bound for $\Box(S)$.

It is not difficult to derive a sharp lower bound for each individual e_j:

Proposition 5.15. *If S is a set of n points in general position in the plane, then for all $j < \frac{n-2}{2}$,*

$$e_j(S) \geq 2j + 3.$$

For every $j \geq 0$ and $n \geq 2j+3$, there is a point set for which this bound is attained.

Proof. Take an arbitrary j-edge $[\boldsymbol{p}, \boldsymbol{q}]$ of S. Let ℓ be an oriented line parallel and very close to the right of $[\boldsymbol{p}, \boldsymbol{q}]$ such that ℓ is disjoint from S. Thus, there are $j+2$ points from S to the left of ℓ and $n-2-j \geq j$ to the right. By the Upper Bound Theorem 4.15, the number of j-edges of S that are intersected by ℓ is precisely $h_j(S,\ell) + h_{n-2-j}(S,\ell) = 2j+2$ (this fact about j-edges in the plane was already noted in [37]), and there is at least one additional j-edge, namely $[\boldsymbol{p}, \boldsymbol{q}]$. □

The following construction shows that the bound is sharp.

Example 5.16. Let S_0 be the vertex set of a regular $(2j+3)$-gon centered at the origin $\mathbf{0}$, and let S_1 be any set of $n-2j-3$ points very close to $\mathbf{0}$ such that the whole set $S := S_0 \dot\cup S_1$ is in general position.

Every line through any point in S_1 has at least $j+1$ points of S_0 on both sides, so the j-edges of S are the longest diagonals of S_0, of which there are $2j+3$.

Using the bound from Proposition 5.15 in the formula of Lemma 5.13, we get

$$\Box \geq \sum_{j<\frac{n-2}{2}} (2j+3)\left(\frac{n-2}{2}-j\right)^2 - \frac{3}{4}\binom{n}{3} = \frac{1}{4}\binom{n}{4} + O(n^3).$$

This lower bound for $\Box$ is weaker than than the estimate derived in the previous section. Its weakness rests mainly in the fact that the point set in Example 5.16 is highly attuned to the specific j at hand.

To obtain the stronger lower bound stated in Theorem 5.12, we do "integration by parts", i.e., we pass from j-facets to $(\leq j)$-facets. We substitute $e_j = E_j - E_{j-1}$ in Lemma 5.13 (with the notation $E_j = \sum_{i=0}^{j} e_i$ introduced in Chapter 2) and rearrange to get the following:

Lemma 5.17. *For every set S of n points in the plane in general position,*

$$\Box(S) = \sum_{j<\frac{n-2}{2}} (n-2j-3)E_j(S) - \frac{3}{4}\binom{n}{3} + r_n(S),$$

where

$$r_n(S) = \begin{cases} \frac{1}{4}E_{\frac{n-3}{2}}(S), & \text{if } n \text{ is odd, and} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Note that the last two terms in the above formula are $O(n^3)$.

To arrive at the conclusion of Theorem 5.12, we use the following two theorems. We will prove the first one Chapter 6:

Theorem 6.1. *Let S be a set of n points in general position in the plane. Then, for every $0 \leq j < \frac{n-2}{2}$, the number of $(\leq j)$-edges of S satisfies*

$$E_j(S) \geq 3\binom{j+2}{2}.$$

This bound is tight for $j < n/3$.

Plugging this into the formula in Lemma 5.17 yields

$$\Box(S) \geq 3/8\binom{n}{4} + O(n^3).$$

In order to obtain the tiny improvement over $3/8$, we will exploit the fact that while the lower bound $E_j \geq 3\binom{j+2}{2}$ is sharp for $j < n/3$, it is no longer tight for j close to $n/2$ (in particular, observe that for odd n, $E_{(n-3)/2} = \binom{n}{2} \sim 4\binom{(n-3)/2}{2}$). Specifically, we will use the following result of Welzl [85]:

Theorem 5.18. *Let S be a set of n points in the plane, and consider a (not necessarily contiguous) index set $K \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$. Then the total number of k-sets with $k \in K$ satisfies*

$$\sum_{k \in K} a_k(S) \leq 2n\sqrt{2\sum_{k\in K} k}.$$

In particular, let $m = \lfloor n/2 \rfloor$, and apply this theorem to the intervals of the form $\{j+2, j+3, \ldots, m\}$. Observing that e_i is precisely the number of $(i+1)$-sets, we obtain that for all $j \leq m-1$,

$$E_{m-1} - E_j \leq 2n\sqrt{2\sum_{i=j+2}^{m} i} = 2n\sqrt{m^2 + m - j^2 - 3j - 2},$$

and since $E_{m-1} \geq \binom{n}{2}$,

$$E_j \quad \geq \quad \binom{n}{2} - 2n\sqrt{m^2 + m - j^2 - 3j - 2}.$$

For $j \geq n/6$, we can simplify this to

$$E_j(S) \geq \binom{n}{2} - n^2\sqrt{1 - 4(j/n)^2} + O(n). \tag{5.28}$$

(The only reason for the assumption $j \geq n/6$ is that for smaller j, we would need the more cumbersome error term $O(n^{3/2})$, and we are only going to use (5.28) for j close to $n/2$, anyway.)

Combining the the estimate from Theorem 6.1 with (5.28), we see that

$$E_j \geq 3\binom{j+2}{2} + n^2 \max\left(0, \frac{1 - 3(j/n)^2}{2} - \sqrt{1 - 4(j/n)^2}\right) + O(n).$$

The "max" term is positive for $j/n \geq t_0 = \sqrt{(2\sqrt{13} - 5)/9} \approx 0.4956$, so we do gain when j is very near $n/2$. Substituting into Lemma 5.17, we get

$$\begin{aligned}
\Box(S) &= \sum_{j<\frac{n-2}{2}} (n-2j-3)E_j(S) + O(n^3) \\
&\geq \sum_{j<\frac{n-2}{2}} 3(n-2j-3)\binom{j+2}{2} + O(n^3) \\
&\quad + n^3 \sum_{t_0 n \leq j < \frac{n-2}{2}} (1-2(j/n)) \left(\frac{1-3(j/n)^2}{2} - \sqrt{1-4(j/n)^2} \right) \\
&= \frac{3}{8}\binom{n}{4} + n^4 \int_{t_0}^{1/2} (1-2t) \left(\frac{1-3t^2}{2} - \sqrt{1-4t^2} \right) dt + O(n^3).
\end{aligned}$$

Thus,

$$\Box \geq (3/8+\varepsilon)\binom{n}{4} + O(n^3),$$

with

$$\varepsilon = 24 \int_{t_0}^{1/2} (1-2t) \left(\frac{1-3t^2}{2} - \sqrt{1-4t^2} \right) dt \approx 1.0887 \cdot 10^{-5}.$$

This completes the proof of Theorem 5.12. We remark that in the set-up of Theorem 5.18, an asymptotically stronger bound of $O(n \left(|K| \sum_{k\in K} k\right)^{1/3})$ can be proved [6, 32]. This, in turn, can be used for a further tiny improvement in the ε. We omit the details.

5.4 Generalizations

For the Four-Point Problem, it does not matter whether we consider the probability $\Box(\mu)$ that four random points P_1, P_2, P_3, P_4 i.i.d. $\sim \mu$ are in convex position, or the complementary probability $\triangle(\mu) = 1 - \Box(\mu)$, or

$$\frac{1}{4}\triangle(\mu) = \Pr[P_4 \in \mathrm{conv}\{P_1, P_2, P_3\}, \qquad (5.29)$$

since any one of these three quantities determines the other two.

Depending on which viewpoint we take, however, different generalizations of the original problem suggest themselves.

If we focus on (5.29), then the following is a natural generalization of the problem to a larger number of random points: If $P_1, \ldots, P_n$, are i.i.d. according to a continuous probability distribution μ in the plane, what is

$$\Pr[P_n \in \operatorname{conv}\{P_1, \ldots, P_{n-1}\}]?$$

Equivalently, we can ask, what is

$$\Pr[P_n \notin \operatorname{conv}\{P_1, \ldots, P_{n-1}\}]?$$

In other words, to formulate the problem in more symmetric terms, what is the expected number of vertices of $\operatorname{conv}\{P_1, \ldots, P_n\}$?

In the plane, this is the same as the expected number of *edges* of the polygon $\operatorname{conv}\{P_1, \ldots, P_n\}$. This seems to lead the "right" formulation of the question for generalizations to higher dimensions: What is the expected number of facets of the polytope spanned by n independent μ-random points? Here, "right" just means that this variant of the problem lends itself to the approach developed in Section 5.3. Again, the question can be equivalently recast in terms of finite point sets, and we can apply Observation 5.14 to any number of points in any dimension. The bounds obtained in the general case are much weaker, though, since we have less precise knowledge about the distribution of j-facets in higher dimensions. We will discuss this in Chapter 6.

Another way of generalizing the problem to a larger number of random points $P_1, \ldots, P_n$ i.i.d. $\sim \mu$ is to ask: What is the probability $p(n, \mu)$ that $P_1, \ldots, P_n$ are in convex position?

Unfortunately, it is not clear how to extend the j-facet approach to this problem. The question can still be equivalently reformulated in terms of finite points sets, but trouble is that for a finite set $S \subseteq \mathbf{R}^2$ and $n \geq 5$, the numbers $e_j(S)$ apparently no longer determine the number of n-element subsets of S that are in convex position: Let us denote by $t = (S)$, $q = q(S)$ and $p = p(S)$ the number of 5-element subsets of S whose convex hull is a triangle, a quadrilateral, and a pentagon, respectively. For $|S| = n$, we we only get the two equations

$$t + q + p = \binom{n}{5}$$

and

$$\sum_{j=0}^{n-2} \binom{j}{3} e_j = 3t + 4q + 5p,$$

which are insufficient to express p in terms of the e_j's.

The probabilities $p(n, \mu)$ have been studied for uniform distributions on a convex body K, in which case we just denote them by $p(n, K)$. We conclude this chapter with a few notes on that topic.

Valtr [80] determined the exact probabilities for the case of triangles,

$$p(n, \text{triangle}) = \frac{2^n(3n-3)!}{(n-1)!^3(2n)!},$$

Observe that this is asymptotically equivalent to $(27/2e^2n^{-2})^n$ (where "e", for once, denotes the base of the natural logarithm). Moreover, every convex body K can be sandwiched between two triangles T_1 and T_2 such that the ratio $\operatorname{area}(T_2)/\operatorname{area}(T_1)$ is bounded by a constant (this is a consequence of John's Lemma, see [51], Section 13.4). Thus, there are universal constants $0 < c_1 < c_2 < \infty$ such that for every convex body $K \subseteq \mathbf{R}^2$,

$$c_1 \leq n^2 \sqrt[n]{p(n, K)} \leq c_2.$$

Bárány [11] showed that for every K, the limit $\lim_{n\to\infty} n^2 \sqrt[n]{p(n, K)}$ exists. Furthermore, he proved a "limit shape" result: if we condition upon the event that $P_1, \ldots, P_n$ are in convex position, then with high probability, their convex hull is very close (in the sense of the Hausdorff distance) to certain convex body $K_0 \subseteq K$.

For further information about the probabilities $p(n, K)$ and a number of related questions, we refer to the survey article by Schneider [68].

Chapter 6

Lower Bounds for ($\leq k$)-Sets

In the previous chapter, we considered the number of convex quadrilaterals in a finite point set in the plane. As we saw, this is just Sylvester's Four-Point Problem in a discrete guise.

As our main result, we derived Theorem 5.12, which gives a lower bound for the minimum number $\square(n)$ of convex quadrilaterals in any set of n points in general position in the plane.

The objective of this chapter will be to provide the missing ingredient for the proof of Theorem 5.12, namely the following estimate for the number $E_j = \sum_{i=0}^{j} e_i$ of $(\leq j)$-edges of a point set:

Theorem 6.1. *Let S be a set of n points in general position in the plane. For every $0 \leq j < \frac{n-2}{2}$,*

$$E_j(S) \geq 3\binom{j+2}{2}.$$

This bound is tight for $j < n/3$.

Since in the plane, the number e_j of j-edges equals the number a_{j+1} of $(j+1)$-sets, Theorem 6.1 can be equivalently stated as a bound for the number $A_k = \sum_{i=1}^{k} a_i$ of $(\leq k)$-sets:

Theorem 6.1'. *For every set S of n points in general position in the plane and every $1 \leq k < n/2$,*

$$A_k(S) \geq 3\binom{k+1}{2}$$

This bound is tight for $k \leq n/3$, as is shown by the "Tripod Construction" which we already encountered in Chapter 5 (Example 5.1).

Recall that the point set S produced by that construction was partitioned into three parts S_1, S_2, S_3 consisting of $n/3$ points each that were arranged very close to three rays through the origin. Before, we also cared about the internal structure of the parts S_i, but now we only need the property that every line spanned by two points from the same part S_i separates the remaining two parts.

It follows that for $1 \leq k \leq n/3$, every k-set of S contains the l points farthest from 0 in one S_i, for some $1 \leq l \leq k$, and the $(k-l)$ points farthest from 0 in another S_j. Hence the number of k-sets is $3k$ and the number of $(\leq k)$-sets equals $3\binom{k+1}{2}$.

The lower bound in Theorem 6.1' was first formulated by Edelsbrunner, Hasan, Seidel, and Chen [34]. Unfortunately, their proof contains a gap that seems to be fatal, as their method of estimating A_k would in fact lead to a linear upper bound for the number of k-sets.

Our proof will, however, follow the same basic approach via so-called *circular* or *allowable sequences*, which were introduced by Goodman and Pollack [41]. We review this notion in Section 6.1 and then prove a slightly more general estimate which implies Theorem 6.1.

In Section 6.2, we will discuss extensions of Theorem 6.1 to higher dimensions.

6.1 The Lower Bound in the Plane

We now proceed to prove the lower bound stated in Theorem 6.1.

Let $\mathbf{\Pi}$ be (a halfperiod of) a *circular sequence* of $\{1 \ldots n\}$. That is, $\mathbf{\Pi} = (\Pi_0, \ldots, \Pi_{\binom{n}{2}})$ is a sequence of permutations of $\{1 \ldots n\}$ such that Π_0 is the identity permutation $(1, 2, \ldots, n)$, $\Pi_{\binom{n}{2}}$ is the reverse permutation $(n, n-1, \ldots, 1)$, and any two consecutive permutations differ by exactly one transposition of two elements in adjacent positions.

Circular sequences, which were introduced by Goodman and Pollack [41], can be used to encode any planar point set. For our purposes and for simplicity, however, we only consider the case of a point set S in general position. Moreover, we will make the additional assumption that no two segments spanned by points from S are parallel (we can assume this without loss of generality, since it can be ensured by sufficiently small perturbations of the points, and this will not affect the number of convex quadrilaterals or the number of k-sets).

Let ℓ be a directed line which is not orthogonal to any of the lines spanned by points from S, and assume that $S = \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$, where the points are labeled according to the order in which their orthogonal projections appear along the line. Now suppose that we start rotating ℓ counterclockwise. Then the ordering of the projections changes whenever ℓ passes through a position where it is orthogonal to a segment $\boldsymbol{uv}$, with $\boldsymbol{u}, \boldsymbol{v} \in S$. When such a change occurs, $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent in the ordering, and the ordering changes by $\boldsymbol{u}$ and $\boldsymbol{v}$ being transposed. Thus, if we keep track of all permutations of the projections as the line ℓ is rotated by $180°$, we obtain a circular sequence $\boldsymbol{\Pi} = \boldsymbol{\Pi}(S)$. (The sequence also depends on the initial choice of ℓ, which for sake of definiteness, we can assume to be vertical and directed upwards).

Observe that if a circular sequence arises in this fashion from a point set, then the $(i-1)$-edges (and hence the i-sets) of the point set correspond to transpositions between elements in positions i and $i+1$, or in positions $n-i$ and $n-i+1$. These will be referred to as *i-critical* transpositions of the circular sequence.

For $k \leq n/2$, we consider the number of *$(\leq k)$-critical transpositions*, i.e., the number of transpositions that are i-critical for some $i \leq k$.

Theorem 6.2. *For any circular sequence $\boldsymbol{\Pi}$ on n elements and any $k < n/2$, the number of $(\leq k)$-critical transpositions is at least $3\binom{k+1}{2}$.*

If the sequence arises from a set S of n points in general position in the plane as the list of the combinatorially different orthogonal projections of S onto a rotating directed line, then the i-critical swaps are in one-to-one correspondence with the i-sets of S, and hence with the $(i-1)$-edges of S. Thus, the number $E_j = \sum_{i=0}^{j} e_i$ of $(\leq j)$-edges of S is at least $3\binom{j+2}{2}$, which will prove Theorem 6.1.

Proof. Fix k and let $m := n - 2k$. It will be convenient to label the points so that the starting permutation is

$$\Pi_0 = (a_k, a_{k-1}, \ldots, a_1, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_k).$$

We introduce some terminology. For $1 \leq i \leq k$, we say that an element x *exits* (respectively, *enters*) *through the ith A-gate* if it moves from position $k-i+1$ to position $k-i+2$ (respectively, from position $k-i+2$ to position $k-i+1$) during a transposition with another element. Similarly, x *exits* (respectively, *enters*) *through the ith C-gate* if it moves from position $m+k+i$ to position $m+k+i-1$ (respectively, from $m+k+i-1$ to $m+k+i$) during a transposition. Observe that for $1 \leq i \leq j \leq k$, a_j has to exit through the ith A-gate and to enter through the i-th C-gate at least once, and analogously for c_j.

Further, we say that $a \in \{a_1, \ldots, a_k\}$ (respectively, $c \in \{c_1, \ldots, c_k\}$) is *confined* until the first time it exits through the 1st A-gate (respectively, C-gate); then it becomes *free*. Elements $b \in \{b_1, \ldots, b_m\}$ are always free.

Simplifying Observation. For every circular sequence $\mathbf{\Pi'}$, there is another sequence $\mathbf{\Pi}$ with the same number of $(\leq k)$-critical transpositions and without transpositions between confined elements. Thus, we may restrict our attention to sequences without such *confined transpositions*.

Proof of the observation. To see why this is so, consider the first confined transposition in $\mathbf{\Pi'}$ (if there isn't any, we are done). Clearly, this first transposition must be either between two a's or between two c's. But before a_i and a_j, say, can be transposed, every a_s with $i < s < j$ has to be transposed with either a_i or a_j. And as long as a_j is confined, every element a_s, $s < j$ which has not yet been transposed with a_j is also confined.

Therefore, the first confined transposition has to happen between two a's (or between two c's) that are adjacent in the starting permutation Π'_0, say between a_i and a_{i+1}. Now we can modify $\mathbf{\Pi'}$ as follows: Instead of transposing a_i and a_{i+1} when it happens in $\mathbf{\Pi'}$, let a_{i+1} follow the "path" of a_i in $\mathbf{\Pi'}$ and vice versa, and only transpose a_i and a_{i+1} in the end. (Observe that for this to be feasible, it is crucial that a_i and a_{i+1} are adjacent in Π'_0.) This does not affect the number of $(\leq k)$-critical transpositions and deletes one confined transposition without generating any new ones, which (by induction, say) proves the observation.

So we may assume that the circular sequence $\mathbf{\Pi}$ does not contain any confined transpositions. Now, let us write down the *liberation sequence* σ of all a's and c's in the the order in which they become free. Since $\mathbf{\Pi}$ does not contain any confined transpositions, the a's appear in σ in increasing order (i.e., a_i precedes a_j in σ if $i < j$) and the same holds for the c's.

We are now ready to estimate the number of $(\leq k)$-critical transpositions. As observed above, for $1 \leq i \leq j \leq k$, a_j has to exit through the ith A-gate and to enter through the i-th C-gate at least once, and c_j has to exit through the ith C-gate and to enter through the ith A-gate at least once. For each of these events, we count the first time it happens. This gives a total count of $4\binom{k+1}{2}$ transpositions, all of which are $(\leq k)$-critical.

The transpositions that are counted twice are precisely the transpositions between some a_j and some c_l during which, for some $i \leq \min\{j, l\}$,

1. either a_j enters and c_l exits through the ith C-gate (both for the first time),

2. or a_j exits and c_l enters through the ith A-gate (both for the first time).

In order to estimate the number of such transpositions, we "credit" each transposition to the entering element. More precisely, we define a *savings digraph* D with vertex set $\{a_1, \ldots, a_k\} \cup \{c_1, \ldots, c_k\}$ and the following edges: In Case 1, we put in a directed edge from c_l to a_j, and in Case 2 a directed edge from a_j to c_l.

Thus, the number of $(\leq k)$-critical transpositions is at least $4\binom{k+1}{2}$ minus the number of edges in D, and it suffices to show that the latter is at most $\binom{k+1}{2}$.

For this, we estimate the in-degree of each vertex. On the one hand, observe that the in-degree of a_j is at most j (there is at most one incoming edge for each ith C-gate, $1 \leq i \leq j$, since we only count the first time that a_j enters through a gate). On the other hand, we observe that if there is a directed edge from c_l to a_j, then a_j precedes c_l in the liberation sequence σ (observe that a_j must have become free before entering through any C-gates, while c_l is still confined when it exits through a C-gate for the first time. Note that the first A-gate and the first C-gate do not coincide since we assume $k < n/2$. Thus, since any two elements are transposed at most (in fact, exactly) once, the in-degree of a_j is also at most the number of c's that come after it in the sequence σ. Hence, the in-degree of a_j is at most the minimum $\mu_\sigma(a_j)$ of j and the number of c's that come after a_j in the sequence σ. Similarly, the in-degree of c_l is at most the minimum $\mu_\sigma(c_l)$ of l and the number of a's which come after c_l in the sequence σ.

The proof is concluded by the following observation: For all σ (subject to the constraint that the a's and the c's appear in increasing order),

$$\sum_{j=1}^{k} \big(\mu_\sigma(a_j) + \mu_\sigma(c_j)\big) = \binom{k+1}{2}. \tag{6.1}$$

To prove this, first note that it obviously holds true for the sequence $\langle a_1, a_2, \ldots, a_k, c_1, c_2, \ldots, c_k \rangle$. So it suffices to show that the sum is invariant under swaps of adjacent a's and c's. Suppose that $\sigma = \rho * \langle a_j, c_l \rangle * \tau$ and $\sigma' = \rho * \langle c_l, a_j \rangle * \tau$ (where "$*$" denotes concatenation of sequences). First observe that $\mu_\sigma(x) =$

$\mu_{\sigma'}(x)$ for all $x \neq a_j, c_l$. Moreover,

$$\begin{aligned} \mu_\sigma(a_j) &= \min\{j, k-l+1\}, & \mu_\sigma(c_l) &= \min\{l, k-j\}, \\ \mu_{\sigma'}(a_j) &= \min\{j, k-l\}, & \mu_{\sigma'}(c_l) &= \min\{l, k-j+1\}. \end{aligned}$$

We distinguish two cases: On the one hand, if $j + l \leq k$, then $\mu_\sigma(a_j) = j = \mu_{\sigma'}(a_j)$ and $\mu_\sigma(c_l) = l = \mu_{\sigma'}(c_l)$, i.e. nothing changes. On the other hand, if $j + l > k$, then $\mu_\sigma(a_j) = k - l + 1 = \mu_{\sigma'}(a_j) + 1$ and $\mu_\sigma(c_l) = k - j = \mu_{\sigma'}(c_l) - 1$, so the sum remains unaffected. This proves (6.1) and hence the theorem. □

6.2 Higher Dimensions

The number of j-facets of a point set $S \subseteq \mathbf{R}^d$ is certainly not less than the number of j-facets that are intersected by a given line ℓ which is in general position w.r.t. S. Thus, for either orientation of ℓ,

$$e_j(S) \geq h_j(S, \ell) + h_{n-d-j}(S, \ell).$$

If we combine this with the existence of centerpoints, we can prove the following:

Theorem 6.3. *For a set S of n points in general position in $\mathbf{R}^d$,*

$$e_j(S) \geq 2 \min \left\{ \binom{j+d-1}{d-1}, \binom{n-1-j}{d-1}, \binom{\lceil \frac{n}{d} \rceil - 1}{d-1} \right\}.$$

Proof. W.l.o.g., the orthogonal projection $\overline{S}$ of S onto the hyperplane $\{\boldsymbol{x} \in \mathbf{R}^d : x_d = 0\} \cong \mathbf{R}^{d-1}$ is a set of n points in general position. By Observation 4.34, there exists a point $\overline{\boldsymbol{o}} \in \mathbf{R}^{d-1}$ which is almost a centerpoint for $\overline{S}$, i.e., which has depth at least $\lceil n/d \rceil - d + 1$ in $\overline{S}$. Thus, by Lemma 4.33, we get for the line $\ell := \overline{\boldsymbol{o}} \times \mathbf{R} \subseteq \mathbf{R}^d$ that

$$h_j(S, \ell) = h_j(\overline{S}, \overline{\boldsymbol{o}}) \geq \begin{cases} \binom{j+d-1}{d-1} & \text{if } 0 \leq j \leq \lceil n/d \rceil - d, \\ \binom{\lceil n/d \rceil - 1}{d-1} & \text{if } \lceil n/d \rceil - d < j \leq (n-d)/2, \end{cases}$$

from which the theorem follows. □

In particularly, we get as a corollary that for $0 \leq j \leq \lceil n/d \rceil - d$,

$$E_j(S) \geq 2\binom{j+d}{d}.$$

On the other hand, consider the obvious generalization of Example 5.1 to higher dimensions: Let $r_1, \ldots, r_{d+1}$ be rays emanating from the origin $\mathbf{0} \in \mathbf{R}^d$ through te vertices of a regular simplex centered at $\mathbf{0}$. If we assume that n is divisible by $d+1$ and if for each of these rays r_i, we place $\frac{n}{d+1}$ points very close to r_i and at distance at least 1 from the origin and each other, then we obtain an n-point set S such that $E_j(S) = (d+1)\binom{j+d}{d}$ for $0 \leq j < \frac{n}{d+1}$.

It is tempting to conjecture that this is the lower-bound example, but I have so far been unable to prove this.

Conjecture 6.4. *For every set S of n points in general position and every $0 \leq j < \lfloor \frac{n}{d+1} \rfloor$, we have*

$$E_j(S) \geq (d+1)\binom{j+d}{d}.$$

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