# Piecewise Linear Approximation of Bézier-Curves*

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While geometric objects are often designed and represented as smooth curves or surfaces, many efficient algorithms (e.g. rendering tools) require their input in piecewise linear form. Therefore it becomes necessary to efficiently approximate objects of the first type by objects of the second type. We will investigate this problem for Bézier-curves in two or higher dimensions, i.e. parameterized curves of the form

$$C(t) = \sum_{i=0}^{n} B_{i,n}(t)\, p_i, \qquad t \in [0,1],$$

where $p_0, p_1, \ldots p_n \in I\!R^d$ are the *control points* and

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

are the *Bernstein polynomials.* We consider polygonal approximations $\hat{C}$ of $C$ with vertices on the curve $C$. That is, the vertices are $C(t_0), C(t_1), \ldots C(t_k)$ where $0 = t_0 < t_1 < \ldots < t_k = 1$ is a suitably chosen subdivision of the unit interval. We analyze the quality of this approximation, if the subdivision is pre-set (independently of the control points). In this way, the necessary coefficients $B_{i,n}(t_j)$ can be precomputed and stored in a table, and the vertices of the approximation can be easily retrieved as linear combinations of the control points.

Observe that $\hat{C}$ has a parameterization

$$\hat{C}(t) = \sum_{i=0}^{n} \hat{B}_{i,n}(t)\, p_i, \qquad t \in [0,1],$$

where the $\hat{B}_{i,n}$'s are piecewise linear approximations of the $B_{i,n}$ with $\hat{B}_{i,n}(t_j) = B_{i,n}(t_j)$ for $i = 0, 1, \ldots n$ and $j = 0, 1, \ldots k$ (see e.g. Figure 1). More specifically,

$$\hat{B}_{i,n}(t) = B_{i,n}(t_j) + (t - t_j)\frac{B_{i,n}(t_{j+1}) - B_{i,n}(t_j)}{t_{j+1} - t_j}$$

for $t \in [t_j, t_{j+1}]$. As a measure of its quality we use the distance function $d(C, \hat{C}) = \max_t \|C(t) - \hat{C}(t)\|$. This distance measure is related to the *Fréchet-distance* between curves (see [1]), where in addition the minimum is taken over all possible monotone parameterizations of the curves.

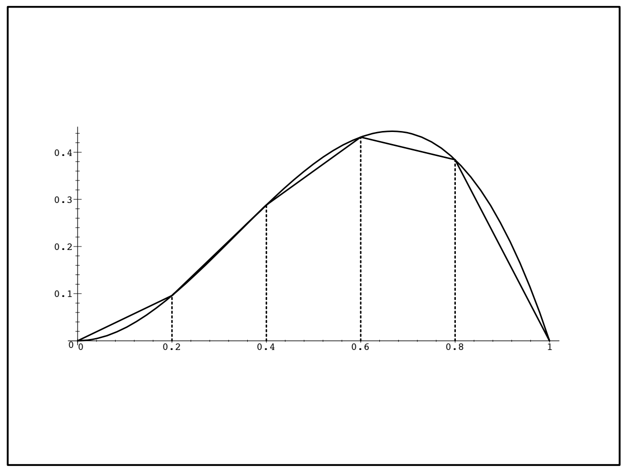


Figure 1: Piecewise linear approximation $\hat{B}_{2,3}$ of the Bernstein polynomial $B_{2,3}$

We have

$$d(C, \hat{C}) = \|\sum_{i=0}^{n} (B_{i,n}(\tau) - \hat{B}_{i,n}(\tau))\, p_i\| = \|\sum_{i=0}^{n} \alpha_i p_i\|$$

for some $\tau \in [0,1]$ and $\alpha_i = B_{i,n}(\tau) - \hat{B}_{i,n}(\tau)$ for $i = 0, \ldots, n$. $\sum_{i=0}^{n} \alpha_i = 0$, since $\sum_{i=0}^{n} B_{i,n}(t) = \sum_{i=0}^{n} \hat{B}_{i,n}(t) = 1$ for all $t$.

Let $\alpha = max_{i,t} |B_{i,n}(t) - \hat{B}_{i,n}(t)|$ be the maximum error in the approximations of the Bernstein polynomials, i.e. $|\alpha_i| \leq \alpha$ for $i = 0, \ldots, n$. Let $\beta_i = \alpha_i / \alpha$ for $i = 0, \ldots, n$. Then

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we want an upper bound on

$$\alpha \| \sum_{i=0}^{n} \beta_i p_i \|$$

where $|\beta_i| \leq 1$ for $i = 0, ..., n$ and $\sum_{i=0}^{n} \beta_i = 0$.

For a given point set $P = \{p_0, p_1, \ldots p_n\}$, the set

$$\text{bzh}\, P := \{\sum_{i=0}^{n} \lambda_i p_i \mid \sum_{i=0}^{n} \lambda_i = 0, \text{ and } |\lambda_i| \leq 1, \text{ for all } i\}$$

has interesting relations to the so-called $k$-set problem. In fact, it can be shown that this set is a convex polytope, whose vertices have a one-to-one correspondence to subsets of $P$ of cardinality*[1] $\lfloor (n+1)/2 \rfloor$ that can be separated from their complement in $P$ by a hyperplane. Moreover, it follows from the analysis of bzh $P$ that

$$\max\{\|p\| \mid p \in \text{bzh}\, P\} < \lfloor \frac{(n+1)}{2} \rfloor \,\text{diam}\, P,$$

where diam $P$ is the diameter of $P$.

For an analysis of the factor $\alpha$ (see e.g. [2]), fix some $B = B_{i,n}$ and some $j$, $0 \leq j \leq k-1$. Let $\mu \in [t_j, t_{j+1}]$ such that $|B(\mu) - \hat{B}(\mu)|$ is maximized in this interval, which implies that the derivative $B'(\mu) = (B(t_{j+1}) - B(t_j))/(t_{j+1} - t_j)$.

By Taylor expansion

$$B(t) = B(\mu) + (t - \mu)B'(\mu) + (t - \mu)^2 B''(\xi) \qquad (1)$$

for some $\xi$ between $t$ and $\mu$. The linear approximation $\hat{B}$ in this interval can be written as

$$\hat{B}(t) = \hat{B}(\mu) + (t - \mu)B'(\mu)\ . \qquad (2)$$

$\hat{B}(t_j) = B(t_j)$ and $\hat{B}(t_{j+1}) = B(t_{j+1})$. Now substitute $t_j$ for $t$ in (1) and (2), and subtract (1) from (2) (similar for $t_{j+1}$). This gives us

$$\hat{B}(\mu) - B(\mu) = (t_j - \mu)^2 B''(\xi_j) = (t_{j+1} - \mu)^2 B''(\xi_{j+1})$$

for some $\xi_j$ and $\xi_{j+1}$ with $t_j < \xi_j < \mu < \xi_{j+1} < t_{j+1}$. By exploiting the fact that one of $|t_j - \mu|$ and $|t_{j+1} - \mu|$ does not exceed $(t_{j+1} - t_j)/2$, the error in $\mu$ is bounded by

$$\left(\frac{t_{j+1} - t_j}{2}\right)^2 \max_{\xi \in [t_j, t_{j+1}]} |B''(\xi)|\ . \qquad (3)$$

The second derivative of the Bernstein polynomials can be written as ([3])

$$B''_{i,n} = n(n-1)(B_{i-2,n-2} - 2B_{i-1,n-2} + B_{i,n-2})$$

which shows that $\max_{t \in [0,1]} |B''(t)| < 2n(n-1)$.

Let us first assume that the $t_i$'s are evenly spaced with distance $\delta = 1/k$.

**Lemma 1** *In the equidistant (in parameter space) approximation with step size $\delta$ of a Bézier-curve of degree $n$, the error is bounded by*

$$\frac{1}{4}\delta^2 n(n-1) \lfloor \frac{n+1}{2} \rfloor \,\text{diam}\, P,$$

*where $P$ is the set of control points.* □

*[1] Note that the cardinality of $P$ is $n + 1$.

A closer look at (3) suggests a more adaptive step size which ideally makes the expression in (3) constant. More precisely, since we have to guarantee the bound for all Bernstein polynomials, we have to ensure that

$$\left(\frac{t_{j+1} - t_j}{2}\right)^2 \max_{\xi \in [t_j, t_{j+1}],\ 0 \leq i \leq n} |B''_{i,n}(\xi)|$$

stays below some given threshold. So for large values of the second derivative a small and for small values a large step size should be used. We analyzed this relationship in more detail and developed a method for choosing an adaptive sequence of breakpoints.

We have run experiments both for equidistant and adaptive step size. Here are the results for three representative curves with the following control point sequences (see Figures (2 - 4)).

curve 1: (0,0),(1,3),(0,1),(0,2),(4,0),(2,3).
curve 2: (1,1),(2.5,3),(5.5,1),(5.5,3.5),(8,2.5),(10,4.5).
curve 3: (1,1),(2,4.2),(6,4.2),(7,1).

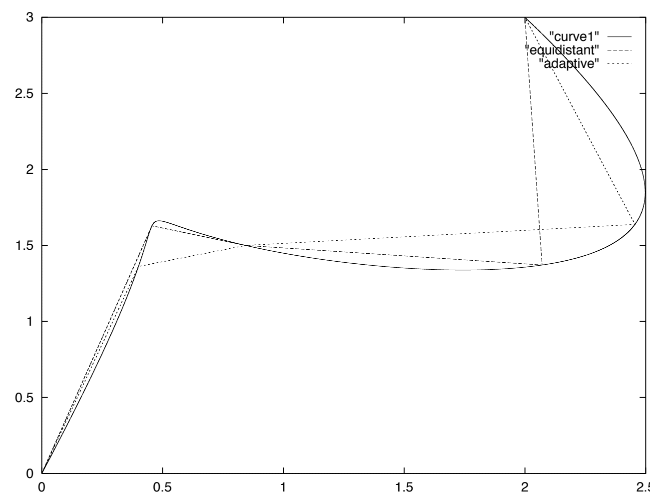


Figure 2: Curve 1.

Table 1 gives for all three curves in the case of 100 equidistant breakpoints a comparison between the bound on the error in the approximation derived in the analysis according to the formula in Lemma 1 and the actual value. We see that our formula is too pessimistic by a factor between approximately 3.5 and 15. One of the reasons is that Lemma 1 takes into account only the diameter of the set of control points, not the actual configuration (an improvement would be possible if we substitute $\max\{\|p\| \mid p \in \text{bzh}\, P\}$ for the factor $\lfloor (n+1)/2 \rfloor$ diam $P$).

| | Curve 1 | Curve 2 | Curve 3 |
|---|---|---|---|
| bound | 0.0067 | 0.0145 | 0.0018 |
| distance | 0.0019 | 0.0010 | 0.00032 |

Table 1: Derived bound and actual distance, $\delta = 0.01$ .

Table 2 gives the same information as Table 1 for the case of adaptive step size.

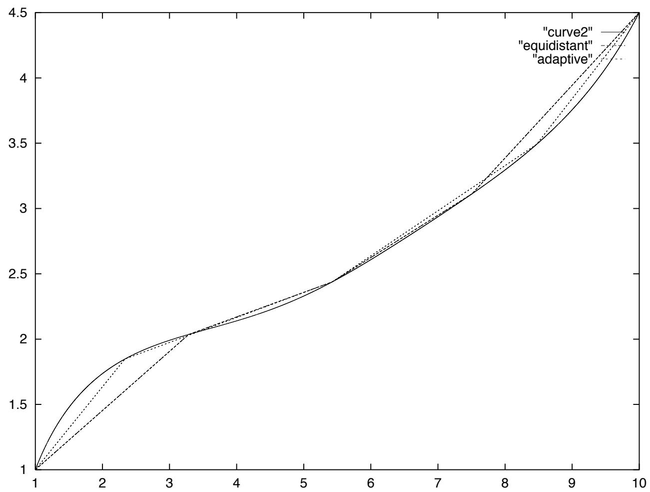


Figure 3: Curve 2.

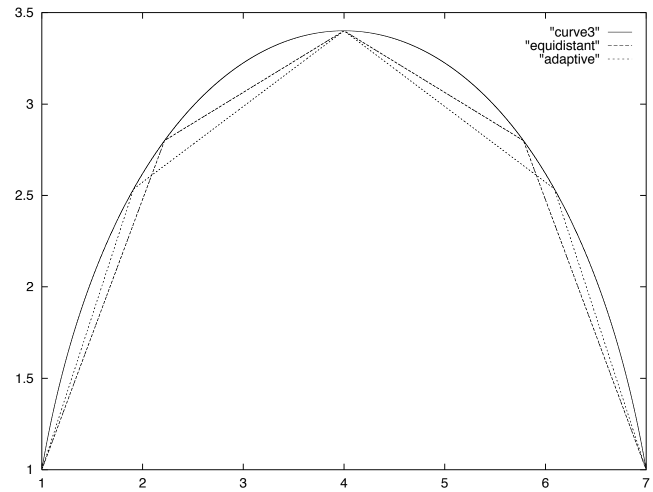


Figure 4: Curve 3.

| | Curve 1 | Curve 2 | Curve 3 |
|---|---|---|---|
| bound | 0.00219 | 0.00473 | 0.00109 |
| distance | 0.00070 | 0.00038 | 0.00053 |

Table 2: Derived bound and actual distance, adaptive subdivision for $k = 100$.

Finally, Table 3 gives the result of comparing the performance of equidistant vs. adaptive step size. For Curves 1 and 2 adaptation really pays yielding results nearly three times as good as the ones for equidistant breakpoints. For Curve 3 adaptation has a negative effect. The reason is that Curve 3 is bent more strongly in its middle part where the adaptive breakpoints are more sparse. However, more importantly, when comparing the first lines of Tables 1 and 2 we can see that the *guaranteed* upper bound is by a factor of 2-3 better in the adaptive case. This means that in our application for a given error tolerance we can choose a polygonal chain of correspondingly less segments.

| k | 4 | 6 | 8 | 10 | 20 | 40 | 100 |
|---|---|---|---|---|---|---|---|
| Curve1 | 0.47 | 0.43 | 0.44 | 0.46 | 0.37 | 0.37 | 0.36 |
| Curve2 | 0.52 | 0.46 | 0.43 | 0.41 | 0.38 | 0.37 | 0.36 |
| Curve3 | 1.22 | 1.40 | 1.44 | 1.59 | 1.72 | 1.29 | 1.60 |

Table 3: Error ratio: adaptive / equidistant.

## References


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