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Number of Crossing-Free Geometric Graphs vs. Triangulations

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Abstract

We show that there is a constant $\alpha > 0$ such that, for any set P of $n \geq 5$ points in general position in the plane, a crossing-free geometric graph on P that is chosen uniformly at random contains, in expectation, at least $(\frac{1}{2} + \alpha)M$ edges, where M denotes the number of edges in any triangulation of P. From this we derive (to our knowledge) the first non-trivial upper bound of the form $c^n \cdot \mathsf{tr}(P)$ on the number of crossing-free geometric graphs on P; that is, at most a fixed exponential in n times the number of triangulations of P. (The trivial upper bound of $2^M \cdot \mathsf{tr}(P)$, or $c = 2^{M/n}$, follows by taking subsets of edges of each triangulation.) If the convex hull of P is triangular, then $M = 3n - 6$, and we obtain $c < 7.98$.

Upper bounds for the number of crossing-free geometric graphs on planar point sets have so far applied the trivial 8^n factor to the bound for triangulations; we slightly decrease this bound to $O(343.11^n)$.

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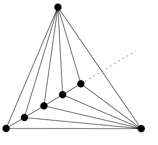
1 Introduction

Let P be a finite set of points in the plane. If no three points in P are collinear, then P is said to be in *general position*. A *geometric graph on* P is a graph with vertex set P whose edges are straight segments connecting corresponding endpoints. Such a straight-line embedded graph is *crossing-free* if no pair of its edges shares any point other than common endpoints. A maximal crossing-free graph on P is called a *triangulation*. We are interested in $\mathsf{pg}(P)$, the number of all crossing-free graphs on P. This quantity never exceeds a fixed exponential in $|P|$, a result first established by Ajtai et al. [3] with 10^{13} as base of the exponential. We will estimate $\mathsf{pg}(P)$ in terms of $\mathsf{tr}(P)$, the number of triangulations of P.

Upper bounds for the total number of crossing-free geometric graphs on a point set P also apply to specific classes of plane graphs: to triangulations, polygonizations, perfect matchings, and spanning trees, to name just a few. Yet better upper bounds for these classes are known. For a recent and detailed list of such results we refer to Aichholzer et al. [2].

In the following we assume that the point set P is fixed and we write $n := n(P) = |P|$ for its cardinality, and $k := k(P)$ for the number of points of P on the boundary of its convex hull; we assume $k \geq 3$. By Euler's polyhedral formula any triangulation of P contains exactly $M := 3n - k - 3$ edges. Since every crossing-free geometric graph is contained in some triangulation, and every triangulation has 2^M subgraphs, we have $\mathsf{pg}(P) \leq 2^M \cdot \mathsf{tr}(P) \leq 8^n \cdot \mathsf{tr}(P)$. The upper bound is tight in the following example: Consider a point set with triangular convex hull such that all interior points lie on a common line containing one of the three extreme points. Then there is a unique triangulation which has exactly $2^{3n-6} \approx 8^n$ crossing-free geometric subgraphs; these subgraphs constitute the set of all crossing-free graphs.



It is therefore surprising that even a small perturbation of the points to general position causes the ratio between the number of crossing-free graphs and that of triangulations to drop exponentially: We show that for any set P of points in general position, $\mathsf{pg}(P) \leq 2^{\gamma \cdot M} \cdot \mathsf{tr}(P)$ holds with $\gamma < 1$. In order to obtain this bound we will show in Section 2 that the expected number of edges in a crossing-free geometric graph on P chosen uniformly at random can be significantly bounded away from $\frac{M}{2}$. In Section 3 we prove via a mean vs. median argument that crossing-free geometric graphs with many edges account for a large fraction of all crossing-free graphs. This way we are able to improve on the pessimistic 2^M factor for points in general position.

2 Lower bound for the expected number of edges

For a geometric graph G on the point set P let $e(G) = |E(G)|$. We define a directed graph $\mathcal{D} = \mathcal{D}(P)$ on the set of all crossing-free geometric graphs on P with a directed arc from G to H iff $E(G) \subseteq E(H)$ and $e(G) = e(H) - 1$. Note that the empty graph has $\binom{n}{2}$ outgoing but 0 incoming arcs, while any triangulation of P has 0 outgoing but M incoming arcs. By $\mathbb{E}\big[e(G)\big]$ we denote the expected number of edges in a crossing-free geometric graph chosen uniformly at random.

Proposition 2.1 $\frac{M}{2} \leq \mathbb{E}\big[e(G)\big] \leq M$.

Proof. Clearly, $e(G) \leq M$ for any graph, implying the upper bound. Denote the in- and out-degree of G in $\mathcal{D}$ by $\deg^-(G)$ and $\deg^+(G)$. Notice that $\mathbb{E}\big[e(G)\big] = \mathbb{E}\big[\deg^-(G)\big] = \mathbb{E}\big[\deg^+(G)\big]$, where the first identity holds since $e(G) = \deg^-(G)$ for any graph G, and the second equality is true since both sides represent the number of arcs divided by the number of vertices of $\mathcal{D}$. Now, with $\deg(G) := \deg^-(G) + \deg^+(G)$ and by linearity of expectation

$$2 \cdot \mathbb{E}\big[e(G)\big] = \mathbb{E}\big[\deg^-(G)\big] + \mathbb{E}\big[\deg^+(G)\big] = \mathbb{E}\big[\deg(G)\big] \geq M, \qquad (1)$$

which gives the first inequality of the proposition. Why does the inequality in (1) hold? Consider some triangulation T with M edges containing the graph G. Then an edge $e \in E(T)$ either corresponds to an incoming arc of G if $e \in E(G)$, or to an outgoing arc of G if $e \notin E(G)$. Thus, $\deg(G) \geq M$. □

If P allows for more than one triangulation (which holds for all point sets in general position with $n \geq 5$) then $\binom{n}{2}$, the degree of the empty graph in $\mathcal{D}$, is strictly larger than M, hence $\mathbb{E}\big[\deg(G)\big] > M$. We will actually show that there is a constant $\alpha > 0$ such that $\mathbb{E}\big[e(G)\big] \geq (\frac{1}{2} + \alpha)M$, for n large enough.

Let G be a crossing-free graph on P and let $e \notin E(G)$ be an edge corresponding to an outgoing arc of G, hence adding e to G again yields a crossing-free graph. If every triangulation containing G also contains e, then we call e *forced* for G. Otherwise, we call edge e *optional* for G. For instance, if $n = 4$ and $k = 3$ then every edge not in G is forced. If $n = k = 4$ and G consists of the four edges of the convex hull then there is no forced edge, but two optional edges. Edges from the convex hull missing in G are always forced for G.

Arcs in $\mathcal{D}$ are labeled forced or optional to match their corresponding edge. Hence, for a graph G we may define its *forced degree*, $\mathrm{fdeg}(G)$, and its *optional degree*, $\mathrm{odeg}(G)$. We write $\mathrm{fdeg}^+(G)$ and $\mathrm{odeg}^+(G)$ for the corresponding forced and optional out-degree, respectively.

Lemma 2.2 *Adding to a crossing-free geometric graph G the set of all its forced edges results in a crossing-free geometric graph $\overline{G}$ without forced edges.*

The straightforward proof is omitted. Let $\overline{G}$ be as in Lemma 2.2 and let $u(G) := M - e(\overline{G})$, the number of edges we need to add to $\overline{G}$ in order to obtain a triangulation. Then by definition $M - u(G) = e(\overline{G}) = e(G) + \text{fdeg}^+(G) = \deg^-(G) + \text{fdeg}^+(G)$. With Equation (1) in mind we are interested in the expected value of $\text{excess}(G) := \deg(G) - M$ which we can rewrite as

$$\text{excess}(G) = \deg^+(G) + \deg^-(G) - M = \text{odeg}^+(G) - u(G), \tag{2}$$

using $\deg^+(G) = \text{fdeg}^+(G) + \text{odeg}^+(G)$. We want to show that $\text{odeg}^+(G)$ is large compared to $u(G) = u(\overline{G})$. Note that if e corresponds to an outgoing arc of $\overline{G}$ in $\mathcal{D}$ then any triangulation containing $\overline{G}$ but not e must necessarily contain an edge f that crosses e, otherwise we could add e to the triangulation, contradicting its maximality. Thus, also f corresponds to an outgoing arc of $\overline{G}$ in $\mathcal{D}$ and clearly both arcs, e and f, are optional for G. This suggests that one can find at least $2u(G)$ optional outgoing arcs of G.

Establishing this takes some care, since two edges e_1, e_2, corresponding to outgoing arcs of $\overline{G}$, could be crossed by a single edge f in a triangulation containing $\overline{G}$, requiring us to find a fourth optional outgoing arc of G. We repeatedly apply a result by Aichholzer et al. [1] stating that any two triangulations on the same point set have a perfect matching of their edges such that matched edges either cross or are identical.

Lemma 2.3 *For any crossing-free graph G we have* $\text{odeg}^+(G) \geq 2u(G)$.

Proof. We construct a matching C on the set of optional outgoing arcs of G in $\mathcal{D}$ such that $|C| = u(G)$. Let $\overline{G}$ be the graph from Lemma 2.2, T_1 a triangulation containing $\overline{G}$ and define $E_1 := E(T_1) \setminus E(\overline{G})$. We now match edges in E_1 (a set of $u(G)$ optional edges) with other edges optional for G. Start with $C := \emptyset$, and for $i \geq 1$ assume that $E_i := E(T_1) \cap \ldots \cap E(T_i) \setminus E(\overline{G})$ is not empty, where T_j, for $1 \leq j \leq i$, are triangulations constructed so far. Let $e \in E_i$. Since e is not forced for $\overline{G}$ there is an edge f crossing e which is not forced for $\overline{G}$. Let T_{i+1} be a triangulation containing $\overline{G}$ and f. The aforementioned result of [1] gives a perfect matching between $E(T_1)$ and $E(T_{i+1})$. Consider the matching partners of $E_i \setminus E(T_{i+1}) \subset E(T_1)$. These are edges of $E(T_{i+1}) \setminus \bigcup_{j=1}^{i} E(T_j)$, since they cross edges of $E_i \subset \bigcap_{j=1}^{i} E(T_j)$. These (crossing) pairs correspond to optional outgoing arcs of G. By construction these matching pairs are disjoint from all pairs in C, hence we may safely add them to C. In particular, all edges in E_i that cross f will be added to C in some matching pair. Now, we continue with E_{i+1} unless it is empty. Note that $e \in E_i \setminus E(T_{i+1})$ which implies that $|E_{i+1}| < |E_i|$. Furthermore, $|E_i| - |E_{i+1}|$ is exactly the number by which $|C|$ increases in the i-th round. Thus, the process terminates eventually with a matching of size $|E_1| = u(G)$. □

Lemma 2.4 *For a crossing-free graph G it holds that* $\operatorname{odeg}(G) \geq \frac{n-4}{2} + u(G)$.

Proof. Extend G to a triangulation T with $E(G) \subseteq E(T)$ and let $e \in E(T)$ be a flippable edge, i.e., an optional incoming arc of T in $\mathcal{D}$. If $e \in E(G)$ then e adds to $\operatorname{odeg}(G)$ as (optional) incoming arc of G in $\mathcal{D}$. If $e \notin E(G)$ then e adds to $\operatorname{odeg}(G)$ as (optional) outgoing arc of G. Hurtado et al. [5] proved that any triangulation on n points in general position contains at least $\frac{n-4}{2}$ flippable edges. Moreover, we know by Lemma 2.3 that G has at least $2u(G)$ optional outgoing edges. Since at most $u(G)$ of them are contained in T the remaining edges clearly add to $\operatorname{odeg}(G)$. □

With a similar argument as in the proof of Proposition 2.1 we find that $\mathbb{E}\big[\operatorname{excess}(G)\big] \geq \frac{1}{2}\mathbb{E}\big[\operatorname{odeg}^+(G)\big] = \frac{1}{4}\mathbb{E}\big[\operatorname{odeg}(G)\big] \geq \frac{n-4}{8}$, using (2) together with Lemma 2.3 and Lemma 2.4. Unfortunately, we were not able to give a lower bound for $\mathbb{E}\big[u(G)\big]$ other than the trivial one, $\mathbb{E}\big[u(G)\big] \geq 0$.
Note that $\mathbb{E}\big[e(G)\big] = \frac{1}{2} \cdot \big(M + \mathbb{E}\big[\operatorname{excess}(G)\big]\big)$ due to Equation (1).

Theorem 2.5 *For a set P of $n \geq 3$ points in general position in the plane with k points on the boundary of the convex hull, $M := 3n - k - 3$, it holds*

$$\mathbb{E}\big[e(G)\big] \geq \tfrac{M}{2} + \tfrac{n-4}{16} = (25n - 8k - 28)/16. \tag{3}$$

3 Upper bound for the number of crossing-free graphs

It remains to show how the lower bound from Equation (3) yields an upper bound on the total number of crossing-free graphs a set of n points can have. Let $\mathcal{H}(x) := -x\log_2 x - (1-x)\log_2(1-x)$ denote the binary entropy function.

Theorem 3.1 *Let P be a set of $n \geq 3$ points in general position in the plane with k points on the boundary of the convex hull. Define $M := 3n - k - 3$ and $\mu := \mathbb{E}\big[e(G)\big]$. Then* $\mathsf{pg}(P) \leq M \cdot 2^{\mathcal{H}(\frac{\mu}{M})M} \cdot \mathsf{tr}(P)$.

Proof. Markov's inequality for the random variable $M - e(G) \geq 0$ gives

$$\mathbb{P}\big[e(G) \geq \mu\big] = 1 - \mathbb{P}\big[M - e(G) \geq M - (\mu - 1)\big] \geq 1 - \frac{\mathbb{E}\big[M - e(G)\big]}{M - (\mu - 1)} \geq \frac{1}{M}.$$

Hence, crossing-free graphs with at least μ edges form a large fraction of all crossing-free graphs. On the other hand, since $\mu \geq \frac{M}{2}$ by Proposition 2.1, we can upper bound the number of crossing-free graphs with at least μ edges by

$$\sum_{m=\mu}^{M} \binom{M}{m} \cdot \mathsf{tr}(P) = \sum_{\ell=0}^{M(1-\frac{\mu}{M})} \binom{M}{\ell} \cdot \mathsf{tr}(P) \ \leq \ 2^{\mathcal{H}(1-\frac{\mu}{M})M} \cdot \mathsf{tr}(P) = 2^{\mathcal{H}(\frac{\mu}{M})M} \cdot \mathsf{tr}(P).$$

Now, lower and upper bound together yield $\frac{1}{M} \cdot \mathsf{pg}(P) \ \leq \ 2^{\mathcal{H}(\frac{\mu}{M})M} \cdot \mathsf{tr}(P)$. □

Note that the binary entropy function is strictly decreasing on the interval $[1/2, 1[$, thus the lower bound $\frac{\mu}{M} \geq \frac{1}{2} + \frac{n-4}{16M}$ from Equation (3) comes in quite handy. Theorem 3.1 gives a non-trivial upper bound for all point sets P with $n \geq 5$ since then $\mathcal{H}(\frac{\mu}{M}) < 1$.

A triangular convex hull of the point set P maximizes $\mathcal{H}(\frac{\mu}{M})$, hence we find that $\mathsf{pg}(P) = O\big(n\, 2^{\mathcal{H}(\frac{25}{48})3n}\big) \cdot \mathsf{tr}(P) = O\left(7.9792^n\right) \cdot \mathsf{tr}(P)$. Sharir and Welzl [6] showed that there are at most 43^n triangulations on a set of n points. Hence, n points in the plane allow for at most $O(343.11^n)$ crossing-free graphs.

Corollary 3.2 *For any set P of n points in general position in the plane,* $\mathsf{pg}(P) = O\left(7.98^n\right) \cdot \mathsf{tr}(P)$ *and* $\mathsf{pg}(P) = O(343.11^n)$*.*

For the other extreme case of a set P of n points in convex position, i.e., when $k = n$, we obtain that $\frac{\mathsf{pg}(P)}{\mathsf{tr}(P)}$ is at most $O\big(n\, 2^{\mathcal{H}(\frac{17}{32})2n}\big) = O\left(3.9844^n\right)$. This can be slightly improved to $O\big(n\, 2^{\mathcal{H}(\frac{9}{16})2n}\big) = O\left(3.9379^n\right)$, since any triangulation of the convex n-gon has $n-3$ flippable edges. However, the exact value is already known to be $\Theta\left((\frac{3}{2} + \sqrt{2})^n\right)$ where $\frac{3}{2} + \sqrt{2} \approx 2.9142$, cf. [4].

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