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On the Number of Crossing-Free Matchings, Cycles, and Partitions*

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Abstract

We show that a set of n points in the plane has at most $O(10.05^n)$ perfect matchings with crossing-free straight-line embedding. The expected number of perfect crossing-free matchings of a set of n points drawn i.i.d. from an arbitrary distribution in the plane is at most $O(9.24^n)$.

Several related bounds are derived: (a) The number of all (not necessarily perfect) crossing-free matchings is at most $O(10.43^n)$. (b) The number of *red-blue* perfect crossing-free matchings (where the points are colored red or blue, and each edge of the matching must connect a red point with a blue point) is at most $O(7.61^n)$. (c) The number of *left-right* perfect crossing-free matchings (where the points are designated as left or as right endpoints of the matching edges) is at most $O(5.38^n)$. (d) The number of perfect crossing-free matchings across a line (where all the matching edges must cross a fixed halving line of the set) is at most 4^n.

These bounds are employed to infer that a set of n points in the plane has at most $O(86.81^n)$ crossing-free spanning cycles (simple polygonizations), and at most $O(12.24^n)$ crossing-free partitions (these are partitions of the point set, so that the convex hulls of the individual parts are pairwise disjoint).

We also derive lower bounds for some of these quantities.

Keywords: Crossing-free geometric graphs, counting, simple polygonizations, crossing-free matchings, crossing-free partitions.

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1 Introduction

Let P be a set of n points in the plane. A *geometric graph* on P is a graph that has P as its vertex set and its edges are drawn as straight segments connecting the corresponding pairs of points. The graph is *crossing-free* if no pair of its edges cross each other, i.e., any two edges are not allowed to share any points other than common endpoints. Therefore, these are planar graphs with a plane embedding given by this specific drawing. We are interested in the number of crossing-free geometric graphs on P of several special types. Specifically, we consider the numbers $\mathsf{tr}(P)$, of *triangulations* (i.e., maximal crossing-free graphs), $\mathsf{pm}(P)$, of crossing-free *perfect matchings*, $\mathsf{sc}(P)$, of crossing-free *spanning cycles*, and, $\mathsf{cfp}(P)$, of *crossing-free partitions*[1] (these are partitions of P, so that the convex hulls of the individual parts are pairwise disjoint). We are primarily concerned with upper bounds for the numbers listed above in terms of n.

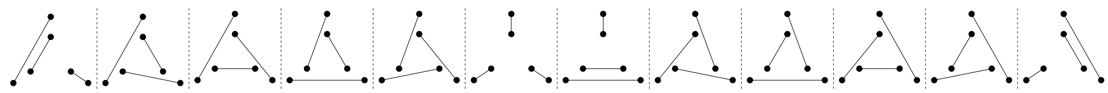


Figure 1: 6 points with 12 crossing-free perfect matchings, the maximum possible number; see [3] for the maximum numbers for up to ten points: 3 for 4 points, 12 for 6, 56 for 8, and 311 for 10.

History. This problem goes back to Newborn and Moser [32] in 1980 who ask for the maximal possible number of crossing-free spanning cycles in a set of n points[2]—they provide an upper bound of $2 \cdot 6^{n-2} \lfloor \frac{n}{2} \rfloor!$ but conjecture that the right bound should be of the form c^n for some constant c. This fact was established in 1982 by Ajtai, Chvátal, Newborn, and Szemerédi [4], who show[3] that there are at most 10^{13n} crossing-free graphs on n points. For motivation they mention—besides [32]—a question of David Avis about the maximum number of triangulations a set of n points can have.

Further developments were mainly concerned with deriving progressively better upper bounds for the number of triangulations[4] [38, 17, 35], so far culminating in a 59^n upper bound by Santos and Seidel [34] in 2003.[5] This compares to $\Omega(8.48^n)$, the largest known number of triangulations for a set of n points, recently derived by Aichholzer et al. [1]; this improves an earlier lower bound of $8^n/\text{poly}(n)$ given by García et al. [21]. (We let "poly(n)" denote a polynomial factor in n.)

[1]Our research was triggered by Marc van Kreveld asking about the number of crossing-free partitions, (see [10] for a motivation from geographic information systems) and, in the same week, by Michael Hoffmann and Yoshio Okamoto asking about the number of crossing-free spanning paths of a point set (motivated by their quest for good fixed parameter algorithms for the planar Euclidean Traveling Salesman Problem in the presence of a fixed number of inner points [14]); see also [23].

[2]In fact, Akl's work [6] appeared earlier, but it already refers to the manuscript by Newborn and Moser, and improves a lower bound (on the maximal number of crossing-free spanning cycles) of theirs.

[3]This paper is famous for its *Crossing Lemma*, proved in preparation of the singly exponential bound. The lemma gives an upper bound on the number of edges a geometric graph with a given number of crossings can have.

[4]Interest was also motivated by the obviously related question (from geometric modeling [38]) of how many bits it takes to encode a triangulation of a point set.

[5]Recently, this bound was improved to 43^n in [37]. However, the bounds on spanning cycles and crossing-free partitions we derive here via matchings are still better than the bounds obtained via this new triangulation bound.

Since every crossing-free graph is contained in some triangulation, and a triangulation has at most $3n-6$ edges, an upper bound of c^n for the number of triangulations immediately yields an upper bound of $2^{3n-6}c^n < (8c)^n$ for the number of all crossing-free graphs on a set of n points. Thus, with $c \leq 59$, this number is at most 472^n. To the best of our knowledge, *all* upper bounds derived so far on the number of crossing-free graphs of various types are derived via a bound on the number of triangulations, albeit in more refined ways.

One such approach is to exploit the fact that graphs of certain specific types have a fixed number of edges. For example, since a perfect matching has $\frac{n}{2}$ edges, we readily obtain $\mathsf{pm}(P) \leq \binom{3n-6}{n/2} \mathsf{tr}(P) < 227.98^n$ [18]. A short historical account of bounds on $\mathsf{sc}(P)$, with references including [6, 16, 21, 22, 24, 32, 33], can be found at the web site [15] (see also [12, Section 8.4, Problem 8]). The best bound published so far is $3.37^n \cdot \mathsf{tr}(P) \leq 198.83^n$, which relies on a bound of 3.37^n on the number of cycles in a planar graph [7].[6]

Crossing-free partitions fit into the picture, since every such partition can be uniquely identified with the graph of edges of the convex hulls of the individual parts—these edges form a crossing-free geometric graph of at most n edges; see Figure 2.

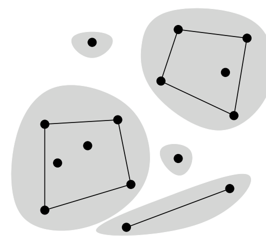


Figure 2: A crossing-free partition and its graph.

The situation is better understood for special configurations, for example for P a set of n points in convex position[7] (namely, the vertex set of a convex n-gon), where the *Catalan numbers* $C_m := \frac{1}{m+1}\binom{2m}{m} = \Theta(m^{-3/2}4^m)$, $m \in \mathbb{N}_0$, play a prominent role. In convex position $\mathsf{tr}(P) = C_{n-2}$ (the Euler-Segner problem, cf. [39, page 212] for a discussion of its history), $\mathsf{pm}(P) = C_{n/2}$ for n even (due to [20], cf. [39]), $\mathsf{sc}(P) = 1$, and $\mathsf{cfp}(P) = C_n$ ([9]).

Crossing-free partitions for point sets in convex position constitute a well-established notion because of its many connections to other problems, probably starting with "planar rhyme schemes" in Becker's note [9], cf. [39, Solution to 6.19pp]. The general case was considered by [13] (under the name of pairwise linearly separable partitions) for clustering algorithms. They show that the number of partitions into k parts is $O(n^{6k-12})$ for k constant.

Under the assumption of *general position* (no three points on a common line) it is known [21] that the number of crossing-free perfect matchings on a set of fixed size is minimized when the set is in convex position.[8] With little surprise, the same holds for spanning cycles, but it does not hold for triangulations [2, 25, 30]. For crossing-free partitions, this is an open question.

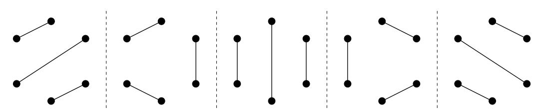


Figure 3: 6 points in convex position with $C_3 = 5$ crossing-free perfect matchings.

[6]In the course of our investigations, we showed that a graph with m edges and n vertices can have at most $\left(\frac{m}{n}\right)^n$ cycles; hence, a planar graph can have at most 3^n cycles. Then Raimund Seidel provided us with an argument, based on linear algebra, that a planar graph can have at most $\sqrt{6}^n < 2.45^n$ spanning cycles.

[7]For another example, it can be shown that the number of triangulations is at most $2^{3mn-m-n}$ for an $m \times n$ grid (with $(m+1)(n+1)$ points) [5] (cf. also [26]).

[8]Recently, Aichholzer et al. [1] showed that any family of acyclic graphs has the minimal number of crossing-free embeddings on a fixed point set when the set is in convex position.

New results. The main results of this paper are the following upper bounds, for a set P of n points in the plane: $\mathsf{pm}(P) = O(10.05^n)$, $\mathsf{sc}(P) = O(86.81^n)$, and $\mathsf{cfp}(P) = O(12.24^n)$. Also, the expected number of perfect crossing-free matchings of a set of n points drawn i.i.d. from *any* distribution in the plane (as long as two random points coincide with probability 0) is at most $O(9.24^n)$.

The new bound on the number of crossing-free perfect matchings is derived by an inductive technique that we have adapted from the method that Santos and Seidel [34] used for triangulations (the adaption however is far from obvious). We then go on to derive several improved bounds on the number of crossing-free matchings of various special types. Specifically, we show:

(a) The number of all (not necessarily perfect) crossing-free matchings is at most $O(10.43^n)$.

(b) The number of *red-blue* perfect crossing-free matchings (where half of the points are colored red and half blue, and each edge of the matching must connect a red point with a blue point) is at most $O(7.61^n)$.

(c) The number of *left-right* perfect crossing-free matchings (where the points are designated as left or as right endpoints of the matching edges) is at most $O(5.38^n)$.

(d) The number of perfect crossing-free matchings across a line (where all the matching edges must cross a fixed halving line of the set) is at most 4^n.

Finally, we derive upper bounds for the numbers of crossing-free spanning cycles and crossing-free partitions of P in terms of the number of certain types of matchings of certain point sets P' that are constructed from P. This yields the bounds $O(86.81^n)$ for the number of crossing-free cycles, and $O(12.24^n)$ for the number of crossing-free partitions.

We summarize the state of affairs in Table 1, including lower bounds which we will derive in Section 6, many of which use the *double-chain* configuration from [21].

	tr	pm	sc	cfp	ma	rbpm	lrpm	alpm	rdpm
$\forall P :\leq$	59 [34]	10.05	86.81	12.24	10.43	7.61	5.38	4	9.24
$\exists P :\geq$	8.48 [1]	3 [21]	4.64 [21]	5.23	4	2.23	2	2	3

Table 1: Entries c in the upper bound row should be read as $O(c^n)$, and entries c in the lower bound row should be read as $\Omega(c^n/\text{poly}(n))$, where $n := |P|$. "ma" stands for all (not necessarily perfect) crossing-free matchings, "rbpm" for perfect red-blue crossing-free matchings, "lrpm" for perfect left-right crossing-free matchings, "alpm" for perfect crossing-free matchings across a line, and "rdpm" for the expected number of perfect crossing-free matchings of a set of i.i.d. points.

This paper shows that significantly better bounds can be derived for matchings than those known earlier for other types of graphs, and, moreover, that matchings are a good basis for deriving bounds for crossing-free partitions and spanning cycles—as opposed to the situation before, where such bounds have always relied on triangulations.

2 Matchings: The Setup and a Recurrence

Let P be a set of n points in the plane in general position, no three on a line, no two on a vertical line. It is easy to see that this is no constraint when it comes to upper bounds on $\mathsf{pm}(P)$. A *crossing-free matching* is a collection of pairwise disjoint segments whose endpoints belong to P. Given such a matching M, each point of P is either *matched*, if it is an endpoint of a segment of M, or *isolated*, otherwise. The number of matched points is clearly always even. If $2m$ points are matched and s points are isolated, we call M a *crossing-free m-matching* or *(m, s)-matching*. We have $n = 2m + s$.

We denote by $\mathsf{ma}_m(P)$ the number of crossing-free matchings of P with m segments (for $m \in \mathbb{R}$—this number is clearly 0 unless $m \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$), and by $\mathsf{ma}(P)$ the number of all crossing-free matchings of P (i.e., $\mathsf{ma}(P) = \sum_m \mathsf{ma}_m(P)$). Recall that $\mathsf{pm}(P) = \mathsf{ma}_{n/2}(P)$.

Let M be a crossing-free (m, s)-matching on a set P of $n = 2m + s$ points, as above. The *degree* $d(p)$ of a point $p \in P$ in M is defined as follows. It is 0 if p is isolated in M. Otherwise, if p is a left (resp., right) endpoint of a segment of M, $d(p)$ is equal to the number of visible left (resp., right) endpoints of other segments of M, plus the number of visible isolated points; "*visible*" means *vertically* visible from the *relative interior* of the segment of M that has p as an endpoint. Thus p and the other endpoint of the segment are not counted in $d(p)$. See Figure 4 for an illustration.

Each left (resp., right) endpoint u in M can contribute at most 2 to the degrees of other points: 1 to each of the left (resp., right) endpoints of the segments lying vertically above and below u, if there exist such segments. Similarly, each isolated point u can contribute at most 4 to the degrees of other points: 1 to each of the endpoints of the segments lying vertically above and below u. It follows that

$$\sum_{p \in P} d(p) \le 4m + 4s.$$

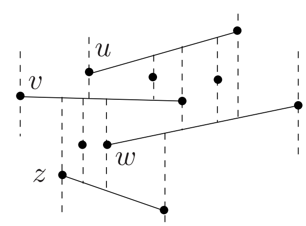


Figure 4: Degrees in a matching: $d(u) = 2$, $d(v) = 5$, $d(w) = 1$, $d(z) = 2$.

There are many segments ready for removal. The idea is to remove segments incident to points of low degree in an (m, s)-matching (points of degree at most 3 or at most 4, to be specific). We will show that there are many such points at our disposal. Then, in the next step, we show that segments with an endpoint of low degree can be reinserted in not too many ways. These two facts will be combined to derive a recurrence for the matching count.

For each integer $i \in \mathbb{N}_0$, let $v_i = v_i(M)$ denote the number of *matched* points of P with degree i in M. Hence, $\sum_{i \ge 0} v_i = 2m$.

Lemma 2.1 *Let $n, m, s \in \mathbb{N}_0$, with $n = 2m + s$. In every (m, s)-matching of any set of n points, we have*

$$2n \le 4v_0 + 3v_1 + 2v_2 + v_3 + 6s \ , \quad \text{and} \qquad (1)$$

$$3n \le 5v_0 + 4v_1 + 3v_2 + 2v_3 + v_4 + 7s \ . \qquad (2)$$

Proof. Let P be the underlying point set. We have

$$\sum_{i \geq 0} i\, v_i = \sum_{p \in P} d(p) \leq 4s + 4m = 4s + \sum_{i \geq 0} 2v_i \ .$$

Therefore, $0 \leq 4s + \sum_{i \geq 0}(2-i)v_i$. For $\kappa \in \mathbb{R}^+$, we add κ times $n = s + \sum_{i \geq 0} v_i$ to both sides to get

$$\kappa n \leq (4+\kappa)s + \sum_{i \geq 0}(2+\kappa-i)v_i \leq (4+\kappa)s + \sum_{0 \leq i < 2+\kappa}(2+\kappa-i)v_i \ . \tag{3}$$

We specialize[9] to $\kappa = 2$ for assertion (1) and $\kappa = 3$ for (2). ◻

There are not too many ways of inserting a segment. Fix some $p \in P$ and let M be a crossing-free matching which leaves p isolated. Now we match p with some other isolated point such that the overall matching continues to be crossing-free. For $i \in \mathbb{N}_0$, let $h_i = h_i(p, P, M)$ be the number of ways that can be done so that p has degree i after its insertion.

Lemma 2.2

$$\begin{aligned} 4h_0 + 3h_1 + 2h_2 + h_3 &\leq 24\ , \quad \textit{and} & (4)\\ 5h_0 + 4h_1 + 3h_2 + 2h_3 + h_4 &\leq 48\ . & (5) \end{aligned}$$

Proof. Let $\ell_i = \ell_i(p, P, M)$ be the number of ways we can match the point p as a left endpoint of degree i. First, we claim that $\ell_0 \in \{0, 1\}$.

To show this, form the *vertical decomposition* of M by drawing a vertical segment up and down from each (matched or isolated) point of $P \setminus \{p\}$, and extend these segments until they meet an edge of M, or else, all the way to infinity; see Figure 5 for an illustration of such a decomposition. We call these vertical segments *walls* in order to distinguish them from the segments in the matching.

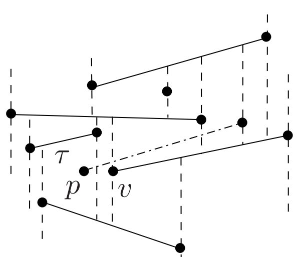


Figure 5: Inserting a segment at p; $d(p) = 1$ after insertion.

We obtain a decomposition of the plane into vertical trapezoids. Let τ be the trapezoid containing p (assuming general position, p lies in the interior of τ). See Figure 5.

We move from τ to the right through vertical walls to adjacent trapezoids until we reach a vertical wall that is determined by a point v that is either a left endpoint or an isolated point (if at all—we may make our way to infinity when p cannot be matched as a left endpoint to any point, in which case $\ell_i = 0$ for all i).

Note that up to that point there was always a unique choice for the next trapezoid to enter. Every crossing-free segment with p as its left endpoint will have to go through all of these trapezoids. It connects either to v (which *can* happen only if v is isolated), or crosses the vertical wall up or down from v. The former case yields a segment that gives p degree 0. In the latter case, v will contribute 1 to the degree of p. So pv, if an option, is the only possible segment that lets p have degree 0 as a left endpoint. (pv will not be an option when it crosses some segment, or when v is a left endpoint.)

[9]We list here explicitly the two values that lead to the best results in the further derivations, although at this point it clearly looks rather arbitrary.

We will return to this set-up when we consider degrees ≥ 1, in which case v acts as a *bifurcation point*. Before doing so, we first introduce a function f. It maps every nonnegative real vector $(\lambda_0, \lambda_1, \ldots, \lambda_k)$ of arbitrary length $k+1 \in \mathbb{N}$ to the maximum possible value[10] the expression

$$\lambda_0 \ell_0 + \lambda_1 \ell_1 + \cdots + \lambda_k \ell_k \tag{6}$$

can attain (for any isolated point in any matching of any finite point set *of any size*). We have already shown that $f(\lambda) \leq \lambda$ for $\lambda \in \mathbb{R}_0^+$. We claim that for all $(\lambda_0, \lambda_1, \ldots, \lambda_k) \in (\mathbb{R}_0^+)^{k+1}$, with $k \geq 1$, we have

$$f(\lambda_0, \lambda_1, \ldots, \lambda_k) \;\leq\; \max\{\lambda_0 + f(\lambda_1, \ldots, \lambda_k),\ 2f(\lambda_1, \ldots, \lambda_k)\}\ . \tag{7}$$

Assuming (7) has been established, we can conclude that $f(1) \leq 1$, $f(2,1) \leq 3$, $f(3,2,1) \leq 6$, and $f(4,3,2,1) \leq 12$; that is[11], $4\ell_0 + 3\ell_1 + 2\ell_2 + \ell_3 \leq 12$ and the first inequality of the lemma follows, since the same bound clearly holds for the case when p is a right endpoint. The second inequality follows similarly from $f(5,4,3,2,1) \leq 24$.

So it remains to prove (7). Consider a constellation with a point p that realizes the value of $f(\lambda_0, \lambda_1, \ldots, \lambda_k)$. We return to the set-up considered above, where we have traced a unique sequence of trapezoids from p to the right, till we encountered the first bifurcation point v (if v does not exist then all ℓ_i vanish).

Case 1: v is isolated. We know that $\lambda_0 \ell_0 \leq \lambda_0$. If we remove v from the point set, then every possible crossing-free segment emanating from p to its right has its degree decreased by 1. Therefore, $\lambda_1 \ell_1 + \cdots + \lambda_k \ell_k \;\leq\; f(\lambda_1, \ldots, \lambda_k)$, so the expression (6) cannot exceed $\lambda_0 + f(\lambda_1, \ldots, \lambda_k)$ in this case.

Case 2: v is a matched left endpoint. Then $\lambda_0 \ell_0 = 0$ (that is, we cannot connect p to v). Possible crossing-free segments with p as a left endpoint are discriminated according to whether they pass above or below v. We first concentrate on the segments that pass above v; we call them *relevant segments* (emanating from p). Let ℓ'_i be the number of relevant segments that give p degree i. We carefully remove isolated points from $P \setminus \{p\}$ and segments with their endpoints from the matching M (eventually also the segment of which v is a left endpoint), so that in the end all relevant segments are still available and each one, if inserted, makes the degree of p exactly 1 unit smaller than its original value (this deletion process may create new possibilities for segments from p). That will show $\lambda_1 \ell'_1 + \cdots + \lambda_k \ell'_k \;\leq\; f(\lambda_1, \ldots, \lambda_k)$. The same will apply to segments that pass below v, using a symmetric argument, which gives the bound of $2f(\lambda_1, \ldots, \lambda_k)$ for (6) in this second case.

The removal process is performed as follows. We define a relation $\prec$ on the set whose elements are the edges of M and the singleton sets formed by the isolated points of $P \setminus \{p\}$: $a \prec b$ if a point $a' \in a$ is vertically visible from a point $b' \in b$, with a' below b'. As is well known (cf. [19, Lemma 11.4]), $\prec$ is acyclic. Let $\prec^+$ denote the transitive closure of $\prec$, and let $\prec^*$ denote the transitive reflexive closure of $\prec$.

Let e be the segment with v as its left endpoint, and consider a minimal element a with $a \prec^+ e$. Such an element exists, unless e itself is a minimal element with respect to $\prec$.

[10] A priori, this value could be infinite.

[11] Note that $\ell_i \leq 2^i$ for each $i \geq 0$ (which can be shown to be tight); this only yields a bound of 26 for the linear combination in question. Moreover, $\sum_{i=0}^k \ell_i \leq 2^k$ (which again is tight), but this only improves the bound to 15, still short of what we need.

a is a singleton: So it consists of an isolated point; with abuse of notation we also denote by a the isolated point itself. a cannot be a point to which p can connect with a relevant edge. Indeed, if this were the case, we add that edge $e' = pa$ and modify $\prec$ to include e' too; more precisely, any pair in $\prec$ that involves a is replaced by a corresponding pair involving e', and new pairs involving e' are added (clearly, the relation remains acyclic and all pairs related under $\prec^+$ continue to be so related after e' is included and replaces a). See Figure 6(a). We have $e \prec e'$ (since, by assumption, the left endpoint v of e is vertically visible below e'), and $e' \prec^+ e$ (since the right endpoint a of e' satisfies $a \prec^+ e$)—a contradiction. With a similar reasoning we can rule out the possibility that a contributes to the degree of p when matched via a relevant edge pq. Indeed, if this were the case, let e'' be the segment directly above a, which is the first link in the chain that gives $a \prec^+ e$, i.e., $a \prec e'' \prec^* e$ (e'' must exist since $a \prec^+ e$). After adding pq with a contributing to its degree, we have either $a \prec pq$ and $pq \prec e''$ (see Figure 6(b)), or we have $pq \prec a$ (see Figure 6(c)). In the former case, we have $a \prec pq \prec e'' \prec^* e \prec pq$—contradicting the acyclicity of $\prec$. In the latter case, we have $pq \prec a \prec^+ e \prec pq$, again a contradiction. So if we remove a, then all relevant edges from p remain in the game and the degree of each of them (i.e., the degree of p that the edge induces when inserted) does not change.

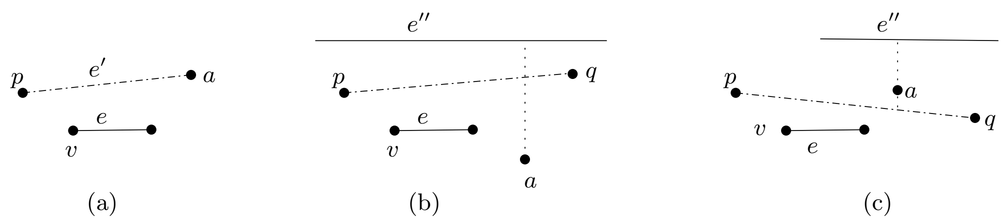


Figure 6: (a) The point a cannot be connected to p via a relevant edge. (b,c) a cannot contribute from below (in (b)) or from above (in (c)) to the degree of p when a relevant edge pq is inserted.

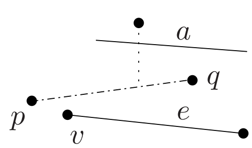


Figure 7: Edge a cannot obstruct a point from contributing from above to the degree of p when a relevant edge pq is inserted.

a is an edge: It cannot obstruct any isolated point or left endpoint below it from contributing to the degree of a relevant edge pq above v (because a is minimal with respect to $\prec$). If a obstructs a contribution to a relevant edge pq from above, then we add pq, thus $pq \prec a$ which, together with $a \prec^+ e$ and $e \prec pq$, contradicts the acyclicity of $\prec$. See Figure 7. Again, we can remove a without any changes to relevant possible edges from p.

We keep successively removing elements until e is minimal with respect to $\prec$. Note that so far all the relevant edges from p are still possible, and the degree of p that any of them induces when inserted has not changed.

Now we remove e with its endpoints. This cannot clear the way for any new contribution to the degree of a relevant edge. In fact, any such degree decreases by exactly 1 because v disappears. The claim is shown, and the proof of the lemma is completed. ⊔⊓

Deriving a recurrence.

Lemma 2.3 *Let $n, m \in \mathbb{N}_0$, such that $m \leq \frac{n}{2}$ and $s := n-2m$. For every set P of n points, we have*

$$\mathsf{ma}_m(P) \leq \begin{cases} \frac{12(s+2)}{n-3s}\,\mathsf{ma}_{m-1}(P) & \textit{if } s < \frac{n}{3}, \textit{ and} \\ \frac{16(s+2)}{n-7s/3}\,\mathsf{ma}_{m-1}(P) & \textit{if } s < \frac{3n}{7}. \end{cases}$$

Let us note right away that the first inequality supersedes the second for $s < \frac{n}{5}$ (i.e. $m > \frac{2n}{5}$), while the second one is superior for $s > \frac{n}{5}$.

Proof. Fix the set P, and let $\mathcal{X}$ and $\mathcal{Y}$ be the sets of all crossing-free m-matchings and $(m-1)$-matchings, respectively, in P.

Let us concentrate on the first inequality. We define an edge-labeled bipartite graph $\mathcal{G}$ on $\mathcal{X} \mathbin{\dot\cup} \mathcal{Y}$ as follows: Given an m-matching M, if p is an endpoint of a segment $e \in M$ and $d(p) \leq 3$, then we connect $M \in \mathcal{X}$ to the $(m-1)$-matching $M \setminus \{e\} \in \mathcal{Y}$ with an edge labeled $(p, d(p))$; $d(p)$ is the *degree label* of the edge. Note that M and $M \setminus \{e\}$ can be connected by two (differently labeled) edges, if both endpoints of e have degree at most 3.

For $0 \leq i \leq 3$, let x_i denote the number of edges in $\mathcal{G}$ with degree label i. We have

$$(2n-6s)\underbrace{|\mathcal{X}|}_{\mathsf{ma}_m(P)} \leq 4x_0 + 3x_1 + 2x_2 + x_3 \leq 24(s+2)\underbrace{|\mathcal{Y}|}_{\mathsf{ma}_{m-1}(P)}.$$

The first inequality is a consequence of inequality (1) of Lemma 2.1. The second inequality is implied by inequality (4) in Lemma 2.2, as follows. For a fixed $(m-1)$-matching M' in P, consider an edge of $\mathcal{G}$ that is incident to M' and is labeled by (p, i) (if there is such an edge). Then p must be one of the $s+2$ isolated points of P (with respect to M'), and there is a way to connect p to another isolated point in a crossing-free manner, so that p has degree i in the new matching. Hence, the contribution by p and M' to the sum $4x_0 + 3x_1 + 2x_2 + x_3$ is at most 24, by inequality (4) in Lemma 2.2, and the right inequality follows. The combination of both inequalities yields the second inequality in (8).

By considering endpoints up to degree 4 (instead of 3), we get the second inequality in an analogous fashion (with the help of inequality (2) in Lemma 2.1 and inequality (5) in Lemma 2.2). ⊡

For $m, n \in \mathbb{N}_0$, let $\mathsf{ma}_m(n)$ denote the maximum number of crossing-free m-matchings a set of n points can have.

Lemma 2.4 *Let $s, m, n \in \mathbb{N}_0$, with $n = 2m + s$. We have*

$$\begin{aligned} \mathsf{ma}_0(0) &= 1, \\ \mathsf{ma}_m(n) &\leq \begin{cases} \frac{n}{s}\,\mathsf{ma}_m(n-1), & \textit{for } s \geq 1, \\ \frac{12(s+2)}{n-3s}\,\mathsf{ma}_{m-1}(n), & \textit{for } s < \frac{n}{3}, \textit{ and} \\ \frac{16(s+2)}{n-7s/3}\,\mathsf{ma}_{m-1}(n), & \textit{for } s < \frac{3n}{7}. \end{cases} \end{aligned} \tag{8}$$

Proof. $\mathsf{ma}_0(0) = 1$ is trivial.

The first of the three inequalities in (8) is implied by

$$s \cdot \mathsf{ma}_m(P) = \sum_{p \in P} \mathsf{ma}_m(P \setminus \{p\}) \le n \cdot \mathsf{ma}_m(n-1) \ ,$$

for any set P of n points. The second and third inequality follow from Lemma 2.3. ◻

3 Solving a Recurrence

We derive an upper bound for a function

$$G \equiv G_{\lambda,\mu} : \ \mathbb{N}_0^2 \to \mathbb{R}^+ ,$$

for a pair of parameters $\lambda, \mu \in \mathbb{R}^+$, $\mu \ge 1$, which satisfies

$$\begin{aligned} G(0,0) &= 1, \\ G(m,n) &\le \begin{cases} \frac{n}{s}\, G(m,n-1) \ , & \text{for } s \ge 1, \text{ and} \\ \frac{\lambda(s+2)}{n-\mu s}\, G(m-1,n) \ , & \text{for } s < \frac{n}{\mu}, \end{cases} \end{aligned} \tag{9}$$

with the convention $s := n - 2m$.

The recurrence in (8) implies that an upper bound on $G_{12,3}(m,n)$ serves also as an upper bound for $\mathsf{ma}_m(n)$, and the same holds for $G_{16,7/3}(m,n)$. We will see how to best combine the two parameter pairs, to obtain even better bounds for $\mathsf{ma}_m(n)$. Later, we will encounter other instances of this recurrence, with other values of λ and μ.

We normalize by dividing by $\lambda^m \mu^{n-m}$. Then (9) becomes

$$\frac{G(m,n)}{\lambda^m \mu^{n-m}} \le \begin{cases} \frac{n}{\mu s} \frac{G(m,n-1)}{\lambda^m \mu^{n-1-m}} \ , & \text{for } s \ge 1, \text{ and} \\ \frac{\mu(s+2)}{n-\mu s} \frac{G(m-1,n)}{\lambda^{m-1} \mu^{n-m+1}} \ , & \text{for } s < \frac{n}{\mu}. \end{cases}$$

We set $H(m,n) = H_\mu(m,n) := \frac{G(m,n)}{\lambda^m \mu^{n-m}}$. Therefore, still with the convention $s := n - 2m$ and the assumption $\mu \ge 1$, we have

$$\begin{aligned} H(0,0) &= 1, \\ H(m,n) &\le \begin{cases} \frac{n}{\mu s}\, H(m,n-1) \ , & \text{for } s \ge 1, \text{ and} \\ \frac{\mu(s+2)}{n-\mu s}\, H(m-1,n) \ , & \text{for } s < \frac{n}{\mu}. \end{cases} \end{aligned} \tag{10}$$

We note that this recurrence depends only on μ.

Lemma 3.1 *Let $m, n \in \mathbb{N}_0$, with $m \le \frac{n}{2}$. Then $H(m,n) \le \binom{n}{m}$.*

Proof. $H(0,0) = 1 \le \binom{0}{0}$ forms the basis of a proof by induction on n and m. For all $n \in \mathbb{N}_0$, $H(0,n) \le \mu^{-n} \le 1 = \binom{n}{0}$ follows, since $\mu \ge 1$.

Let $1 \le m \le \frac{n}{2}$. If $m \le n - \mu s$ then $s \le \frac{n-m}{\mu} < \frac{n}{\mu}$. Hence, the second inequality in (10) can be applied, after which the first inequality can be applied. Hence,

$$\begin{aligned} H(m,n) &\le \frac{\mu(s+2)}{n-\mu s} H(m-1,n) \\ &\le \frac{\mu(s+2)}{n-\mu s} \frac{n}{\mu(s+2)} H(m-1,n-1) \\ &\le \frac{n}{m}\binom{n-1}{m-1} = \binom{n}{m} . \end{aligned}$$

Otherwise, $m > n - \mu s$ holds, which ensures $\mu s > n - m \ge 0$, i.e., $s \ge 1$. We can therefore employ the first inequality of (10), and obtain

$$H(m,n) \le \frac{n}{\mu s} H(m,n-1) < \frac{n}{n-m}\binom{n-1}{m} = \binom{n}{m} .$$

□

By expanding along the first inequality for a while before employing Lemma 3.1, we get

$$\begin{aligned} H(m,n) &\le \frac{n}{\mu s} \cdots \frac{n-k+1}{\mu(s-k+1)} H(m,n-k) \\ &\le \frac{1}{\mu^k}\left(\prod_{i=0}^{k-1} \frac{n-i}{s-i}\right)\binom{n-k}{m} \\ &= \frac{1}{\mu^k}\frac{\binom{n}{k}}{\binom{s}{k}}\binom{n-k}{m} && (11) \\ &= \frac{1}{\mu^k}\frac{\binom{2m}{m}}{\binom{n-m-k}{m}}\binom{n}{2m} , \quad \text{for } \mathbb{N}_0 \ni k \le s. && (12) \end{aligned}$$

When we stop this unwinding of the recurrence, we could have alternatively proceeded one more step, and upper bound $H(m, n-k)$ by $\frac{n-k}{\mu(s-k)}\binom{n-k-1}{m}$, provided $k < s$. As long as this expression is smaller than $\binom{n-k}{m}$, we should indeed have expanded further. That is, we expand as long as

$$\begin{aligned} & \frac{n-k}{\mu(s-k)}\binom{n-k-1}{m} < \binom{n-k}{m} \\ \Leftrightarrow\ & \frac{n-k}{\mu(s-k)}(n-k-m) < n-k \\ \Leftrightarrow\ & k < \frac{\mu s + m - n}{\mu - 1} = n - m\left(\frac{2\mu-1}{\mu-1}\right) = n - \frac{m}{\rho}, \end{aligned}$$

for $\rho := \frac{\mu-1}{2\mu-1}$. In other words, the best choice of k in (11) is

$$k = \left\lceil n - \frac{m}{\rho} \right\rceil = n - \left\lfloor \frac{m}{\rho} \right\rfloor . \qquad (13)$$

In fact, if this suggested value of k is negative (or if $\rho = 0$), we should not expand at all. Instead, we can try to expand along the second inequality of (10), to get (note that here reducing m by 1 increases s by 2)

$$
\begin{aligned}
H(m,n) &\le \frac{\mu(s+2)}{n-\mu s}\cdots\frac{\mu(s+2+2(k-1))}{n-\mu(s+2(k-1))}H(m-k,n) \\
&\le \left(\prod_{i=0}^{k-1}\frac{\frac{s}{2}+1+i}{\frac{n}{2\mu}-\frac{s}{2}-i}\right)\binom{n}{m-k} \\
&= \frac{\binom{\frac{s}{2}+k}{k}}{\binom{\frac{n}{2\mu}-\frac{s}{2}}{k}}\binom{n}{m-k},
\end{aligned}
\tag{14}
$$

for $\mathbb{N}_0 \ni k < \frac{n}{2\mu}-\frac{s}{2}+1 = m-\frac{\mu-1}{2\mu}\,n+1$; we employ here the usual generalization of binomial coefficients $\binom{a}{k}$ to $a\in\mathbb{R}$, namely, $\binom{a}{k} := \frac{a(a-1)\cdots(a-k+1)}{k!}$.

Rather than optimizing the value of k at which we stop the unwinding of the second recurrence inequality of (10), we approximate it by

$$
k = \left\lceil m - \frac{\mu-1}{2\mu-1}\,n\right\rceil = m - \lfloor \rho n\rfloor\ , \tag{15}
$$

and note that it lies in the allowed range, provided it is positive. (With some tedious calculations, one can show that the optimal stopping value is $k = m-\lfloor\rho(n+1)\rfloor$, which is either equal to the k in (15) or is smaller than it by 1.)

When $\frac{m}{n}=\rho$, both values suggested for k in (13) and (15) are 0, which indicates that we have to content ourselves with the bound $\binom{n}{m}$ from Lemma 3.1. Otherwise, it is clear which way to expand, since

$$
\begin{aligned}
\frac{m}{n} < \rho \quad &\Rightarrow \quad n-\left\lfloor \tfrac{m}{\rho}\right\rfloor \ge 0, \\
\frac{m}{n} > \rho \quad &\Rightarrow \quad m-\lfloor\rho n\rfloor \ge 0.
\end{aligned}
$$

We are now ready for an improved bound. For that we substitute k in (11) according to (13), and in (14) according to (15).

Lemma 3.2 *Let $m,n\in\mathbb{N}_0$, where $2m\le n$, and set $\rho := \frac{\mu-1}{2\mu-1}$. If $\frac{m}{n}\le\rho$, then*

$$
H_\mu(m,n) \le \frac{1}{\mu^{n-\lfloor m/\rho\rfloor}}\frac{\binom{n}{n-\lfloor m/\rho\rfloor}}{\binom{n-2m}{n-\lfloor m/\rho\rfloor}}\binom{\lfloor m/\rho\rfloor}{m}\ ,
$$

and for $\frac{m}{n}>\rho$, we have

$$
H_\mu(m,n) \le \frac{\binom{\frac{n}{2}-\lfloor\rho n\rfloor}{m-\lfloor\rho n\rfloor}}{\binom{m-\frac{n}{2}(1-\frac{1}{\mu})}{m-\lfloor\rho n\rfloor}}\binom{n}{\lfloor\rho n\rfloor}\ .
$$

Thus, $G_{\lambda,\mu}(m,n)\le\overline{G}_{\lambda,\mu}(m,n)$ with

$$
\overline{G}_{\lambda,\mu}(m,n) := \begin{cases}
\lambda^m\mu^{\lfloor m/\rho\rfloor-m}\,\dfrac{\binom{n}{n-\lfloor m/\rho\rfloor}}{\binom{n-2m}{n-\lfloor m/\rho\rfloor}}\dbinom{\lfloor m/\rho\rfloor}{m}\ , & \text{for } \frac{m}{n}\le\rho, \text{ and} \\[2ex]
\lambda^m\mu^{n-m}\,\dfrac{\binom{\frac{n}{2}-\lfloor\rho n\rfloor}{m-\lfloor\rho n\rfloor}}{\binom{m-\frac{n}{2}(1-\frac{1}{\mu})}{m-\lfloor\rho n\rfloor}}\dbinom{n}{\lfloor\rho n\rfloor}\ , & \text{for } \frac{m}{n}>\rho.
\end{cases}
$$

Next we work out a number of properties of the upper bound $\overline{G}_{\lambda,\mu}$.

Estimates up to a polynomial factor. In the following derivations, we sometimes use "$\approx_n$" to denote equality up to a polynomial factor in n.

We will frequently use the following estimate (implied by Stirling's formula, cf. [29, Chapter 10, Corollary 9])

$$\binom{\alpha n}{\lceil \beta n \rceil} \approx_n \binom{\alpha n}{\lfloor \beta n \rfloor} \approx_n \left(\frac{\alpha^\alpha}{\beta^\beta (\alpha - \beta)^{\alpha - \beta}} \right)^n, \qquad \text{for } \alpha, \beta \in \mathbb{R},\ \alpha \geq \beta \geq 0.$$

Big m. We note that for $\frac{m-1}{n} \geq \rho$

$$\overline{G}_{\lambda,\mu}(m,n) = \frac{\lambda(s+2)}{n - \mu s}\, \overline{G}_{\lambda,\mu}(m-1,n) \qquad \text{with } s := n - 2m.$$

Since $\frac{\lambda(s+2)}{n-\mu s} < 1 \Leftrightarrow s < \frac{n-2\lambda}{\lambda+\mu} \Leftrightarrow m > \frac{(\lambda+\mu-1)n+2\lambda}{2(\lambda+\mu)}$, the function $\overline{G}_{\lambda,\mu}(m,n)$ maximizes for integers m in the range $\rho n \leq m \leq \frac{n}{2}$ at

$$m^* := \left\lfloor \frac{(\lambda+\mu-1)n + 2\lambda}{2(\lambda+\mu)} \right\rfloor = \left\lfloor \frac{n}{2} - \frac{n - 2\lambda}{2(\lambda+\mu)} \right\rfloor, \tag{16}$$

unless this value is not in the provided range. However, $m^* \leq \frac{n}{2}$ unless n is very small ($n < 2\lambda$). And $m^* \geq \rho n$ unless $\lambda < \mu - 1$.

Small m. With the identity indicated in (12) we have, for $\frac{m}{n} \leq \rho$, that $\overline{G}$ can also be written as

$$\overline{G}_{\lambda,\mu}(m,n) \;=\; \lambda^m \mu^{\lfloor m/\rho \rfloor - m} \frac{\binom{2m}{m}}{\binom{\lfloor m/\rho \rfloor - m}{m}} \binom{n}{2m} \;\approx_m\; (4\lambda(\mu-1))^m \binom{n}{2m}. \tag{17}$$

This bound peaks (up to an additive constant) at

$$m^{**} := \left\lfloor \frac{\sqrt{\lambda(\mu-1)}}{1 + 2\sqrt{\lambda(\mu-1)}}\, n \right\rfloor .$$

We observe that $m^{**} \leq \rho n$ for $\lambda \leq \mu - 1$.

We can summarize, that the function $\overline{G}_{\lambda,\mu}(m,n)$ attains its maximum—up to a poly(n)-factor—over m at

$$m = \begin{cases} m^{**} & \text{if } \lambda \leq \mu - 1, \text{ and} \\ m^* & \text{otherwise.} \end{cases} \tag{18}$$

In all applications in this paper we have $\lambda > \mu - 1$, so the peak occurs at m^*.

4 Bounds for Matchings

4.1 Perfect Matchings

For perfect matchings we consider the case where n is even, $m = \frac{n}{2}$, and $s = 0$. We note that in this case $m/n = 1/2 > \rho$, for any value of μ. Hence, the second bound of Lemma 3.2 applies. We first calculate $\frac{n}{2} - \frac{n}{2}(1-\frac{1}{\mu}) = \frac{1}{2\mu}\, n$, and $\frac{n}{2} - \lfloor \rho n \rfloor = \left\lceil \frac{n}{2} - \frac{\mu-1}{2\mu-1}\, n \right\rceil = \left\lceil \frac{1}{2(\mu-1)}\, n \right\rceil$. Hence,

$$\begin{aligned}
\overline{G}_{\lambda,\mu}\left(\frac{n}{2}, n\right) &= (\lambda\mu)^{n/2} \binom{\frac{1}{2\mu}\, n}{\left\lceil \frac{1}{2(2\mu-1)}\, n \right\rceil}^{-1} \binom{n}{\left\lfloor \frac{\mu-1}{2\mu-1}\, n \right\rfloor} \\
&\approx_n (\lambda\mu)^{n/2} \left(\frac{\left(\frac{1}{2(2\mu-1)}\right)^{\frac{1}{2(2\mu-1)}} \left(\frac{\mu-1}{2\mu(2\mu-1)}\right)^{\frac{\mu-1}{2\mu(2\mu-1)}}}{\left(\frac{1}{2\mu}\right)^{\frac{1}{2\mu}} \left(\frac{\mu-1}{2\mu-1}\right)^{\frac{\mu-1}{2\mu-1}} \left(\frac{\mu}{2\mu-1}\right)^{\frac{\mu}{2\mu-1}}} \right)^n \\
&= (\lambda\mu)^{n/2} \left(\mu^{\frac{1}{2(2\mu-1)} - \frac{\mu}{2\mu-1}} (\mu-1)^{\frac{\mu-1}{2\mu(2\mu-1)} - \frac{\mu-1}{2\mu-1}} (2\mu-1)^{-\frac{1}{2\mu}+1} \right)^n \\
&= (\lambda\mu)^{n/2} \left((\mu-1)^{-\frac{\mu-1}{2\mu}} \mu^{-\frac{1}{2}} (2\mu-1)^{\frac{2\mu-1}{2\mu}} \right)^n \\
&= \left(\lambda^{\frac{1}{2}} (\mu-1)^{-\frac{\mu-1}{2\mu}} (2\mu-1)^{\frac{2\mu-1}{2\mu}} \right)^n .
\end{aligned}$$

Substituting $(\lambda, \mu) = (12, 3)$ and $(16, \frac{7}{3})$, as suggested by Lemma 2.4, we obtain the following upper bounds for the number of crossing-free perfect matchings:

$$\begin{aligned}
\overline{G}_{12,3}\left(\frac{n}{2}, n\right) &\approx_n \left(2^{\frac{2}{3}} \cdot 3^{\frac{1}{2}} \cdot 5^{\frac{5}{6}}\right)^n = O(10.5129^n) , \quad \text{and} \\
\overline{G}_{16,\frac{7}{3}}\left(\frac{n}{2}, n\right) &\approx_n \left(2^{\frac{10}{7}} \cdot 3^{-\frac{1}{2}} \cdot 11^{\frac{11}{14}}\right)^n = O(10.2264^n) .
\end{aligned}$$

While the second bound is obviously superior, we remember that the recurrence with $(\lambda, \mu) = (12, 3)$ is better for $m > \frac{2n}{5}$ (or $s < \frac{n}{5}$). This observation leads to the following better bound for P a set of n points and for $k = \lfloor \frac{n}{2} - \frac{2n}{5} \rfloor = \lfloor \frac{n}{10} \rfloor$, where we expand as in the first inequality of Lemma 2.3.

$$\begin{aligned}
\mathsf{pm}(P) &\leq \left(\prod_{i=0}^{k-1} \frac{12(2i+2)}{n-6i} \right) \mathsf{ma}_{n/2-k}(P) \leq 4^k \binom{\frac{n}{6}}{k}^{-1} \overline{G}_{16,7/3}(n/2-k, n) \\
&\approx_n \left(2^{20/21}\, 3^{-2/7}\, 5^{1/21}\, 11^{11/14}\right)^n = O(10.0438^n).
\end{aligned}$$

Perfect versus all matchings. Recall from Lemma 2.3 that $\mathsf{ma}_m(P) \leq \frac{12(s+2)}{n-3s} \mathsf{ma}_{m-1}(P)$. Note that $\frac{12(s+2)}{n-3s} < 1$ for $m > \frac{7n}{15} + \frac{4}{5}$ (and in this range the factor $\frac{12(s+2)}{n-3s}$ is smaller than the alternative offered in Lemma 2.3). That is, there are always fewer perfect matchings than there are $\lfloor \frac{7n}{15} + \frac{4}{5} \rfloor$-matchings. More specifically, for sets P with $n := |P|$ even, and for $k = \frac{n}{2} - \lfloor \frac{7n}{15} + \frac{4}{5} \rfloor = \lceil \frac{n}{30} - \frac{4}{5} \rceil$, we have

$$\begin{aligned}
\mathsf{pm}(P) = \mathsf{ma}_{n/2}(P) &\leq \prod_{i=0}^{k-1} \frac{12(2i+2)}{n-6i}\, \mathsf{ma}_{n/2-k}(P) \\
&= \left(\frac{12 \cdot 2}{6}\right)^k \binom{\frac{n}{6}}{k}^{-1} \mathsf{ma}_{n/2-k}(P) \\
&\approx_n 4^{n/30} \left(\left(\frac{1}{5}\right)^{1/5} \left(\frac{4}{5}\right)^{4/5}\right)^{n/6} \mathsf{ma}_{\lfloor 7n/15+4/5 \rfloor}(P) \\
&= \left(2^{1/3}\, 5^{-1/6}\right)^n \mathsf{ma}_{\lfloor 7n/15+4/5 \rfloor}(P)\,.
\end{aligned}$$

This implies that $\mathsf{pm}(P) \leq \left(2^{1/3}\, 5^{-1/6}\right)^n \mathsf{ma}(P)\, \mathrm{poly}(n) = O(0.9635^n)\, \mathsf{ma}(P)$. In every point set there are exponentially (in the size of the set) more crossing-free matchings than there are crossing-free perfect matchings.

4.2 All Matchings

Our considerations in the derivation of the bound for perfect matchings imply the following upper bound for matchings with m segments.

$$\mathsf{ma}_m(P) \leq \begin{cases} \overline{G}_{16,7/3}(m,n)\,, & m \leq \frac{2n}{5}, \text{ and} \\ \overline{G}_{12,3}(m,n)\, \frac{\overline{G}_{16,7/3}(\frac{2n}{5},n)}{\overline{G}_{12,3}(\frac{2n}{5},n)}\,, & \text{otherwise.} \end{cases} \tag{19}$$

To determine where the expression (19) maximizes, we note that $\overline{G}_{16,7/3}$ does not peak in its "small m"-range ($m \leq \frac{4}{11}$) since $16 > \frac{7}{3} - 1$ (recall (18)). In the "big m"-range, it peaks at roughly $\frac{26n}{55}$ (see (16)), which exceeds $\frac{2}{5}$. Therefore, the maximum occurs when $\overline{G}_{12,3}$ comes into play, which peaks at roughly $\frac{7n}{15}$. For that value the upper bound evaluates to $\approx_n (2^{13/21}\, 3^{-2/7}\, 5^{3/14}\, 11^{11/14})^n = O(10.4244^n)$.

We summarize in the following main theorem.

Theorem 4.1 *Let P be a set of n points in the plane. Then*

(1) $\mathsf{pm}(P) \leq \left(2^{20/21}\, 3^{-2/7}\, 5^{1/21}\, 11^{11/14}\right)^n \mathrm{poly}(n) = O(10.0438^n)$.

(2) $\mathsf{pm}(P) \leq \left(2^{1/3}\, 5^{-1/6}\right)^n \mathsf{ma}(P)\, \mathrm{poly}(n) = O(0.9635^n)\, \mathsf{ma}(P)$.

(3) $\mathsf{ma}(P) \leq \left(2^{13/21}\, 3^{-2/7}\, 5^{3/14}\, 11^{11/14}\right)^n \mathrm{poly}(n) = O(10.4244^n)$.

We note, by the way, that the first inequality in the theorem is a direct consequence of the other two inequalities.

4.3 Random Point Sets

Let P be any set of $N \in \mathbb{N}$ points in the plane, no three on a line, and let $r \in \mathbb{N}$ with $r \le N$. If R is a subset of P chosen uniformly at random from $\binom{P}{r}$, then, for $\lambda = 16$, $\mu = \frac{7}{3}$, and provided $m \le \frac{\mu-1}{2\mu-1}N = \frac{4}{11}N$, and that $r \ge 2m$, we have, using (17),[12]

$$\begin{aligned}
\mathbf{E}[\mathsf{ma}_m(R)] &= \left(\sum_{R\in\binom{P}{r}} \mathsf{ma}_m(R)\right) / \binom{N}{r} = \mathsf{ma}_m(P) \binom{N-2m}{r-2m} / \binom{N}{r} \\
&\le (4\lambda(\mu-1))^m \binom{N}{2m} \left(\binom{N-2m}{r-2m} / \binom{N}{r}\right) \mathrm{poly}(m) \\
&\approx_m (4\lambda(\mu-1))^m \binom{r}{2m} = \left(2^8\, 3^{-1}\right)^m \binom{r}{2m} .
\end{aligned}$$

We see that if we sample r points from a large enough set, then the expected number of crossing-free matchings observes for all m the upper bound derived for the range of small m.

Suppose now that, for n even, we sample n i.i.d. points from an arbitrary distribution, for which we only require that two sampled points coincide with probability 0. Then we can first sample a set P of $N > \frac{11}{8}n$ points, and then choose a subset of size n uniformly at random from the family of all subsets of this size. We obtain a set R of n i.i.d. points from the given distribution. If P is in general position, by the argument above the expected number of perfect crossing-free matchings is at most $\approx_n (2^8\, 3^{-1})^{n/2}$. If P exhibits collinearities, we perform a small perturbation yielding a set $\tilde{P}$ and the subset $\tilde{R}$. Now the bound applies to $\tilde{R}$, and also to R since a sufficiently small perturbation cannot decrease the number of crossing-free perfect matchings.

Theorem 4.2 *For any distribution in the plane for which two sampled points coincide with probability* 0*, the expected number of crossing-free perfect matchings of n i.i.d. points is at most*

$$\left(2^4\, 3^{-1/2}\right)^n \mathrm{poly}(n) = O(9.2377^n) .$$

4.4 Red-Blue Perfect Matchings

We next consider several variants of crossing-free *bipartite* matchings, for which better upper bounds can be derived.

Here we assume that the given set P of n points is the disjoint union $R \mathbin{\dot\cup} B$ of two subsets, and each edge in the matching has to connect a point of R with a point of B. We refer to the points of R as red points, and to those of B as blue.

We repeat the preceding analysis, but we modify the definition of the degree $d(p)$ of a point: If p is a matched point in R, say the left endpoint of its edge e, then $d(p)$ is equal to the number of left endpoints plus the number of *blue* isolated points that are vertically visible from (the relative interior of) e. A symmetric definition holds for right endpoints

[12] There is a small subtlety in that the second identity in the derivation relies on the fact that P is in general position. For that consider three points on a line.

and for points $p \in B$. (Intuitively, a blue isolated point q has to contribute only to the degrees of red points, because, when we insert an edge emanating from a blue point p, it cannot connect to q, and it does not matter whether it passes above or below q; that is, q does not cause any bifurcation in the ways in which p can be connected.)

In this case we have

$$\sum_{p \in P} d(p) \leq 4m + 2s,$$

because each isolated point contributes to the degree of only two matched points. This changes the bounds in Lemma 2.1 to

$$\begin{aligned} 2n &\leq 4v_0 + 3v_1 + 2v_2 + v_3 + 4s \ , \quad \text{and} \\ 3n &\leq 5v_0 + 4v_1 + 3v_2 + 2v_3 + v_4 + 5s \ . \end{aligned}$$

The rest of the analysis continues verbatim, except that now the recurrence (8) involves the factors $\frac{12(s+2)}{n-2s}$ and $\frac{16(s+2)}{n-5s/3}$, or, in other words, $(\lambda, \mu) = (12, 2)$ (with $\rho = 1/3$) and $(16, \frac{5}{3})$ (with $\rho = 2/7$), respectively. The first factor is superior for $s < \frac{n}{3}$, i.e., $m > \frac{n}{3}$.

We thus obtain, with $k = \lfloor \frac{n}{6} \rfloor$, a bound of

$$\left(\prod_{i=0}^{k-1} \frac{12(2i+2)}{n-4i} \right) \overline{G}_{16,5/3}(n/2 - k, n)$$

for the number of perfect red-blue matchings. Manipulating it, as above, yields:

Theorem 4.3 *Let P be a set of n points in the plane each one colored red or blue. Then the number of red-blue perfect crossing-free matchings in P is at most*

$$\left(2^{6/5}\, 3^{-3/20}\, 7^{7/10} \right)^n \mathrm{poly}(n) = O(7.6075^n) \,.$$

4.5 Left-Right Perfect Matchings

Here we assume that P is partitioned into two disjoint subsets L, R and consider bipartite matchings in $L \times R$ such that, for each edge of the matching, its left endpoint belongs to L and its right endpoint to R.

We modify the definition of the degrees of the points, as in the red-blue case, and have, as above,

$$\sum_{p \in P} d(p) \leq 4m + 2s.$$

The analysis further improves, because when we insert an edge emanating from a point $p \in L$, say, the corresponding numbers h_i must be equal to ℓ_i, since p can only be the left endpoint of the edge. A similar improvement holds for points $q \in R$. Hence, we can bound the sum $4h_0 + 3h_1 + 2h_2 + h_3$ by 12, rather than 24; similarly, we have $5h_0 + 4h_1 + 3h_2 + 2h_3 + h_4 \leq 24$. That is, we have the two options $(\lambda, \mu) = (6, 2)$ and $(8, \frac{5}{3})$. We thus obtain the bound

$$\left(\prod_{i=0}^{k-1} \frac{6(2i+2)}{n-4i} \right) \overline{G}_{8,5/3}(n/2 - k, n) \ , \quad \text{for } k = \lfloor \tfrac{n}{6} \rfloor,$$

which leads to the following result.

Theorem 4.4 *Let P be a set of n points in the plane and assume that the points are classified as left endpoints or right endpoints. Then the number of left-right perfect crossing-free matchings in P that obey this classification is at most*

$$\left(2^{7/10}\, 3^{-3/20}\, 7^{7/10}\right)^n \mathrm{poly}(n) = O(5.3793^n)\ .$$

4.6 Matchings Across a Line

Consider next the special case of crossing-free bipartite perfect matchings between two sets of $\frac{n}{2}$ points each that are separated by a line. Here we can obtain an upper bound that is smaller than the one in Theorem 4.4.

Theorem 4.5 *Let n be an even integer. The number of crossing-free perfect bipartite matchings between two separated sets of $\frac{n}{2}$ points each in the plane is at most ${C_{n/2}}^2 < 4^n$; (recall that C_m is the mth Catalan number).*

Proof. Let L and R be the given separated sets. Without loss of generality, take the separating line λ to be the y-axis, and assume that the points of L lie to the left of λ and the points of R lie to its right. Let M be a crossing-free perfect bipartite matching in $L \times R$. For each edge e of M, let e_L (resp., e_R) denote the portion of e to the left (resp., right) of λ, and refer to them as the *left half-edge* and the *right half-edge* of e, respectively. We will obtain an upper bound for the number of combinatorially different ways to draw the left half-edges of a crossing-free perfect matching in $L \times R$. The same bound will apply symmetrically to the right half-edges, and the final bound will be the square of this bound.

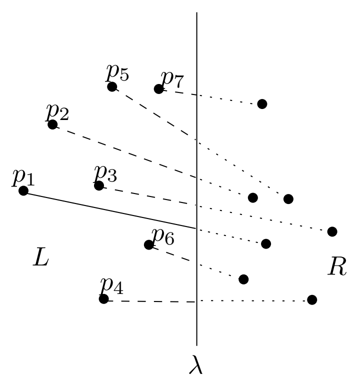


Figure 8: Recursively counting permutations induced on λ by left half-edges.

In more detail, we ignore R, and consider collections S of $\frac{n}{2}$ pairwise disjoint segments, each connecting a point of L to some point on λ, so that each point of L is incident to exactly one segment. For each segment in S, we label its λ-endpoint by the point of L to which it is connected. The increasing y-order of the λ-endpoints of the segments thus defines a permutation of L, and our goal is to bound the number of different permutations that can be generated in this way. (In general, this is a *strict* upper bound on the quantity we seek—see below.)

We obtain this bound in the following recursive manner. Write $m := |L| = \frac{n}{2}$. Sort the points of L from left to right (we may assume that there are no ties—they can be eliminated by a slight rotation of λ), and let $p_1, p_2, \ldots, p_m$ denote the points in this order. Consider the half-edge e_1 emanating from the leftmost point p_1. Any other point p_j lies either above or below e_1. By rotating e_1 about p_1, we see that there are at most m (exactly m, if we assume general position) ways to split $\{p_2, \ldots, p_m\}$ into a subset L_1^+ of points that lie above e_1 and a complementary subset L_1^- of points that lie below e_1, where in the i-th split, $|L_1^+| = i - 1$ and $|L_1^-| = m - i$. Note that, in any crossing-free perfect bipartite matching that has e_1 as a left half-edge incident to p_1, all the points of L_1^+ (resp., of L_1^-) must be incident to half-edges that terminate on λ *above* (resp., *below*) the λ-endpoint of e_1; see Figure 8.

Hence, after having fixed i, we can proceed to bound recursively and separately the number of permutations induced by L_1^+, and the number of those induced by L_1^-. In other words, denoting by $\Pi(m)$ the maximum possible number of different permutations induced in this way by a set L of m points (in general position), we get the following recurrence

$$\Pi(m) \leq \sum_{i=1}^{m} \Pi(i-1)\Pi(m-i),$$

for $m \geq 1$, where $\Pi(0) = 1$. However, this is the recurrence that (with equality) defines the Catalan numbers, so we conclude that $\Pi(m) \leq C_m$.

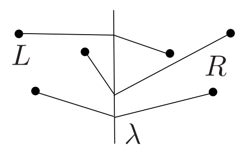


Figure 9: A left and a right permutation which are not compatible.

A (probably weak) upper bound for the number of crossing-free perfect bipartite matchings in $L \times R$ is thus ${C_m}^2$. Indeed, for any permutation π_L of L and any permutation π_R of R, there is at most one crossing-free perfect bipartite matching in $L \times R$ that induces both permutations. Namely, it is the matching that connects the j-th point in π_L to the j-th point in π_R, for each $j = 1, \ldots, m$. See Figure 9 for an example of two such permutations that do not yield a (straight-edge) crossing-free matching.

We thus obtain the asserted upper bound ${C_m}^2 = {C_{n/2}}^2 < 4^n$. ◻

5 Two Implications

5.1 Spanning Cycles

Theorem 5.1 *Let P be a set of n points in the plane. Then the number of crossing-free spanning cycles satisfies*

$$\mathsf{sc}(P) \leq (2^{7/5}\, 3^{7/10}\, 7^{7/5})^n \mathrm{poly}(n) = O(86.8089^n)\ .$$

Proof. Let P be a given set of n points. We construct a new set P' of $2n$ points by creating two copies p^+, p^- of each point $p \in P$, and by placing these copies co-vertically very close to the original location of p, with p^+ lying above p^-.

Let π be a cycle in P. We map π to a perfect matching in P' as follows. For each $p \in P$, let q, r be its neighbors in π. (i) If both q, r lie to the left of p, with the edge qp lying above rp, we connect p^+ to either q^+ or q^-, and connect p^- to either r^+ or r^- (the actual choices will be determined at q and r by similar rules). (ii) The same rule applies in the case where both q, r lie to the right of p. (iii) If q lies to the left of p and r lie to the right of p, then we connect p^+ to either q^+ or q^-, and connect p^- to either r^+ or r^-. It is clear that the resulting graph π^* is a crossing-free perfect matching in P', assuming general position of the points of P, if we draw each pair of points p^+, p^- sufficiently close to each other. See Figure 10 for an illustration.

We assign to each point $p \in P$ a label that depends on π. A point whose two neighbors in π lie to its left is labeled as a *right point*, a point whose two neighbors in π lie to its right is labeled as a *left point*, and a point having one neighbor in π to its right and one to its left is labeled as a *middle point*.

We assign the cycle π to the pair (π^*, λ), where π^* is the resulting perfect matching on P' and λ is the labeling of P, as just defined.

Each pair (π^*, λ) can be realized by at most one cycle π in P, by simply merging each pair p^+, p^- back into the original point p. (The resulting graph need not be a cycle; in general it is a collection of pairwise disjoint cycles.) It therefore suffices to bound the number of such pairs (π^*, λ).

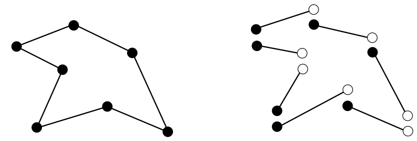


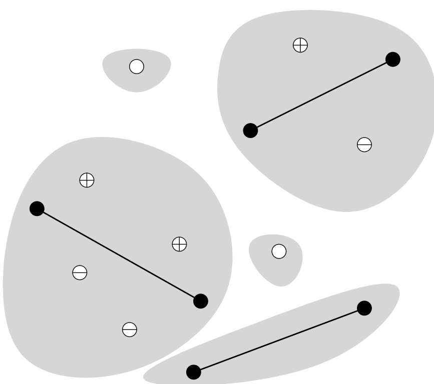
Figure 10: A cycle in P induces a left-right perfect matching in P'.

A given labeling λ of P uniquely classifies each point of P' as being either a left point of an edge of the matching or a right endpoint of such an edge. Hence, the number of crossing-free perfect matchings π' on P' that respect this left-right assignment is at most $(2^{7/10}\, 3^{-3/20}\, 7^{7/10})^{2n}\text{poly}(n)$. The number of labellings of P is 3^n. Hence, the number of crossing-free cycles in P is at most $(2^{7/5}\, 3^{7/10}\, 7^{7/5})^n\text{poly}(n)$, as asserted. ◻

Clearly, it follows from the proof that the bound holds for the number of crossing-free spanning paths as well, and also for the number of cycle covers (or path covers) of P.[13]

5.2 Crossing-free Partitions

We now relate crossing-free partitions of a point set P to matchings, thereby establishing an upper bound on $\mathsf{cfp}(P)$.



Spines, isolated (○), top (⊕), and bottom (⊖) points.

Figure 11: Encoding a crossing-free partition.

To this end, every crossing-free partition of P is mapped to a tuple (M, S, I^+, I^-) where (see Figure 11)

(i) M is the matching in P, whose edges connect the leftmost point to the rightmost point of each set in the partition with at least two elements (we refer to each such segment as the *spine* of its set),
(ii) S is the set of all points that form singleton sets in the partition, and
(iii) I^+ (resp., I^-) is the set of points in $P \setminus S$ that are neither the leftmost nor the rightmost in their set, and which lie *above* (resp., *below*) the spine of their set.

We observe that M is crossing-free, and that the partition is uniquely determined by (M, S, I^+, I^-). Therefore, any upper bound on the number of such tuples will establish an upper bound on the number of crossing-free partitions. For every crossing-free matching M on P there are $3^{n-2|M|}$ triples (S, I^+, I^-) which form a 4-tuple with M (clearly, not all of them have to come from a crossing-free partition, so we overcount). Therefore $\sum_m 3^{n-2m}\mathsf{ma}_m(P)$ is an upper bound on the number of crossing-free partitions.

Ignoring the 3^n-factor for the time being, we have to determine an upper bound on $3^{-2m}\mathsf{ma}_m(P)$, for which we employ the bound from (19). We observe that $3^{-2m}\overline{G}_{\lambda,\mu}(m, n) =$

[13] A slight improvement can be obtained by noting that when a cycle has j middle points, we can derive from it 2^j distinct matchings in P', by flipping the connections to some of the pairs of P' that represent middle points.

$\overline{G}_{\lambda/9,\mu}(m,n)$, and therefore

$$3^{-2m}\mathsf{ma}_m(P) \leq \begin{cases} \overline{G}_{16/9,7/3}(m,n)\ , & m \leq \frac{2n}{5}, \text{ and} \\ \overline{G}_{4/3,3}(m,n)\ \frac{\overline{G}_{16,7/3}(\frac{2n}{5},n)}{\overline{G}_{12,3}(\frac{2n}{5},n)}\ , & \text{otherwise.} \end{cases} \tag{20}$$

Since $\frac{16}{9} \geq \frac{7}{3} - 1$ (see (18)) the peak will not occur in the "small m"-range of $\overline{G}_{16/9,7/3}$. In its "big m"-range, the maximum occurs at m roughly $\frac{14n}{37}$ (see (16)) which lies in the interval $[\frac{4}{11}, \frac{2}{5}]$. Also, $G_{4/3,3}$ peaks for $m \leq \frac{2n}{5}$ since $\frac{4}{3} \leq 3 - 1$ (consult (18)). Therefore, the bound peaks at m roughly $\frac{14n}{37}$ with the value

$$3^n\,\overline{G}_{16/9,7/3}(\lfloor\tfrac{14n}{37}\rfloor, n) \approx_n (2^{4/7}\,3^{-1/2}\,11^{11/14}\,37^{3/14})^n\ .$$

Note that we could have estimated the number of 4-tuples by first choosing a subset Q, which is the union of S and the endpoints of M, then choose a matching in Q, and then partition $P \setminus Q$ into $I^+ \cup I^-$. This leads to a bound of $\approx_n \sum_k \binom{n}{k} c^k 2^{n-k} = (c+2)^n$, where c is the constant in the bound for all matchings. This yields a bound of $O(12.43^n)$ which falls short of our bound obtained above.

Theorem 5.2 *Let P be a set of n points in the plane. Then the number of crossing-free partitions satisfies*

$$\mathsf{cfp}(P) \leq \left(2^{4/7}\,3^{-1/2}\,11^{11/14}\,37^{3/14}\right)^n \mathrm{poly}(n) = O(12.2388^n)\ .$$

6 Lower Bounds

In this section we briefly derive the lower bounds mentioned in Table 1. Most of them rely on an analysis of the so-called *double chain*, as it was first considered by García, Noy, and Tejel [21] in the context of crossing-free graphs. For matchings across a line (and left-right matchings) we use a different configuration.

6.1 The Double Chain

Given $m \in \mathbb{N}$, the double chain D_{2m} consists of $n := 2m$ points. There is an upper half U_m of m points on the parabola $y = \frac{x^2+1}{2}$ with their x-coordinates in $[-1, +1]$, and there is a lower half L_m of m points on the parabola $y = -\frac{x^2+1}{2}$ in the same x-range. The important property is that U_m and L_m are in convex position, and the relative interior of each segment connecting a point from U_m with a point from L_m is disjoint from the convex hulls of U_m and of L_m, and thus cannot cross any segment connecting points within these sets.

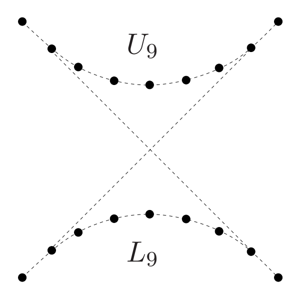


Figure 12: The double chain D_{18}.

García et al. [21] show, among others, that $\mathsf{sc}(D_{2m}) = \Omega(4.64^n)$ and that

$$\mathsf{pm}(D_{2m}) = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m}{2k}^2 {C_k}^2 \approx_n 3^n\ . \tag{21}$$

We wish to recapitulate the argument for the latter bound. A crossing-free perfect matching with k inner edges within U_m leaves $m-2k$ points in U_m to be matched to the same number of points in L_m. So within L_m, we also have k inner edges. If we choose the $2k$ endpoints in U_m for the inner edges ($\binom{m}{2k}$ choices) then we have C_k possibilities to connect them in a perfect crossing-free matching; the same bound applies to L_m. The remaining points from U_m and L_m allow exactly one crossing-free perfect matching from the upper set to the lower set. This gives the bound in (21). (The estimate for the sum builds on the observation that $\sum_{i=0}^{N} {a_i}^2 \approx_N \left(\sum_{i=0}^{N} a_i\right)^2$ for nonnegative real numbers a_i.)

In a similar fashion we can argue now for

$$\mathsf{ma}(D_{2m}) = \sum_{k=0}^{m} \binom{m}{k}^2 {M_k}^2 \approx_n 4^n \ ,$$

where $M_k = \sum_i \binom{k}{2i} C_i = \Theta(k^{-3/2} 3^k)$ is the kth Motzkin number that counts the number of all matchings of k points in convex position [31].

Crossing-free partitions. Along similar lines we easily get a lower bound of

$$\mathsf{cfp}(D_{2m}) \geq \sum_{k=0}^{m} \binom{m}{k}^2 {C_k}^2 \approx_n 5^n \ .$$

This bound for crossing-free partitions counts only a restricted class of such partitions, namely those composed of a matching between $m-k$ points in U_m with $m-k$ points in L_m, together with crossing-free partitions among the remaining k points in U_m and among the remaining k points in L_m.

Let us perform an exhaustive counting of crossing-free partitions of the double chain. Here are the ingredients.

Recall first that for $N \in \mathbb{N}_0$, $i \in \mathbb{N}$, the number N can be written as an ordered sum of i nonnegative integers in $\binom{N+i-1}{i-1}$ ways, and as an ordered sum of i positive integers in $\binom{N-1}{i-1}$ ways.

Now we "prepare" the upper half U_m for a crossing-free partition as follows. We specify the number k of parts that extend to the lower half, and we also specify which k contiguous nonempty subsequences of points of U_m form the upper portions of these extended parts; we refer to these sequences as *docking places*. If the overall size of these docking places is $k+\ell$, we have to specify numbers $a_i \in \mathbb{N}_0$, $0 \leq i \leq k$, which are the sizes of intermediate non-docking parts, and numbers $b_i \in \mathbb{N}$, $1 \leq i \leq k$, which are the sizes of docking parts, so that $m = a_0 + b_1 + a_1 + \cdots b_k + a_k$, with $\sum a_i = m - k - \ell$ (and so $\sum b_i = k + \ell$).

There are $\binom{m-k-\ell+(k+1)-1}{(k+1)-1} = \binom{m-\ell}{k}$ ways to choose the a_i's, and $\binom{k+\ell-1}{k-1}$ ways to choose the b_i's. That is, the number of configurations with k docking places (with the non-docking points already forming a crossing-free partition within U_m) is exactly

$$\sum_{\ell=0}^{m-k} \binom{m-\ell}{k} \binom{k+\ell-1}{k-1} C_{m-k-\ell} \ .$$

Hence, repeating the same analysis to the lower half L_m, and observing that, as in the case of matchings, there is a unique way to connect the upper and lower docking places in a non-crossing manner, we obtain

$$\mathsf{cfp}(D_{2m}) = {C_m}^2 + \sum_{k=1}^{m} \left(\sum_{\ell=0}^{m-k} \binom{m-\ell}{k} \binom{k+\ell-1}{k-1} C_{m-k-\ell} \right)^2 .$$

So for an estimate up to a polynomial factor in m, it remains to find k and ℓ so that $f(m, \ell, k) := \binom{m-\ell}{k}\binom{k+\ell-1}{k-1} C_{m-k-\ell}$ is large. We have

$$f(m, \lfloor 0.05m \rfloor, \lfloor 0.22m \rfloor) > 5.23^m \mathrm{poly}(m) \ ,$$

which gives $\mathsf{cfp}(D_{2m}) > (5.23^m \mathrm{poly}(m))^2 = 5.23^n \mathrm{poly}(n)$. (The coefficients 0.05 and 0.22 were chosen via a numerical experimentation.)

Red-blue matchings. It is worthwhile to notice that if we color n points in convex position, n even, alternately red and blue along the boundary of their convex hull, then all perfect matchings on this set are compatible with this coloring. That is, we have a colored set of n points with $C_{n/2} \approx 2^n$ crossing-free perfect red-blue matchings. Again, we will employ the double chain for a better lower bound.

Assume m to be even, consider D_{2m}, and color the points in U_m alternately red and blue, starting with red at the leftmost point. Then color L_m alternately blue and red, starting with blue at the leftmost point. Given that coloring we generate perfect red-blue matchings as follows.

- Choose some k, $0 \le k \le \frac{m}{2}$.
- Select k red points in U_m ($\binom{m/2}{k}$ possibilities).
- Select k blue points in L_m ($\binom{m/2}{k}$ possibilities).
- Match the selected red points and their next (to the right) blue neighbors in U_m with the selected blue points and their next (to the right) red neighbors in L_m. This can be done in a *unique* crossing-free manner, which is also red-blue compatible.
- Match the remaining $m - 2k$ points in U_m. By the way points were selected, the remaining points are still alternately red and blue and thus allow $C_{m/2-k}$ red-blue matchings, and the same holds for the lower chain L_m.

This gives

$$\sum_{k=0}^{m/2} \binom{m/2}{k}^2 {C_{m/2-k}}^2 \approx_m \sum_{k=0}^{m/2} \binom{m/2}{k}^2 \left(4^{m/2-k}\right)^2 \approx_m 5^m = \sqrt{5}^{\,n} = \Omega(2.23^n)$$

perfect crossing-free red-blue matchings as claimed in Table 1. The above procedure does not catch all possible perfect crossing-free red-blue matchings—a more accurate analysis might lead to a better bound.

Perfect matchings in random sets. Finally, let us describe a distribution in the plane such that the expected number of crossing-free perfect matchings of n i.i.d. points, for n even, is at least $3^n/\mathrm{poly}(n)$. We draw a random point p by first choosing an x uniformly at random in $[-1,+1]$, and then by letting $p = (x, \frac{x^2+1}{2})$ or $p = (x, -\frac{x^2+1}{2})$, each of the two possibilities with probability $\frac{1}{2}$. A set P of n i.i.d. points from this distribution is of the form $U_k \cup L_{n-k}$ with probability $\frac{1}{2^n}\binom{n}{k}$. Therefore,

$$\mathbf{E}[\mathsf{pm}(P)] = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\mathsf{pm}(U_k \cup L_{n-k}) \geq \frac{1}{2^n}\binom{n}{n/2}\mathsf{pm}(\underbrace{U_{n/2} \cup L_{n/2}}_{D_n}) \approx_n 3^n .$$

6.2 Matchings Across a Line

We present a simple construction with about 2^n different crossing-free perfect bipartite matchings across a line.

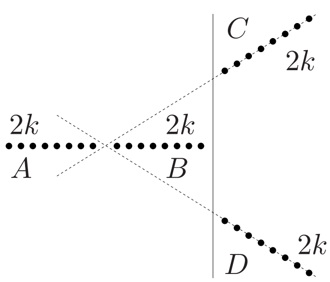


Figure 13: The lower bound construction for crossing-free perfect matchings across a line.

Assume that $n = 8k$, and refer to Figure 13. Take two disjoint horizontal segments that lie on the x-axis to the left of the y-axis, and place on each of them $2k$ points. Denote by A (resp., B) the set of points on the left (resp., right) segment. The set L is $A \cup B$. To form R, draw two lines that separate A and B, one with positive slope and one with negative slope. Place $2k$ points on each of these lines to the right of the y-axis, and denote the set on the line with positive (resp., negative) slope by C (resp., D). The set R is $C \cup D$.

In order to specify a crossing-free perfect bipartite matching, we proceed as follows: Split A into two sets A_C and A_D of size k each, split B into two sets B_C and B_D of size k each, split C into two sets C_A and C_B of size k each, and split D into two sets D_A and D_B of size k each. The total number of choices is $\binom{2k}{k}^4 \approx_k 2^{8k} = 2^n$. Now we match A_C with C_A, A_D with D_A, B_C with C_B, and B_D with D_B, which can always be done in a unique way that is non-crossing; see Figure 13.

We have thus shown:

Theorem 6.1 *The maximum number of crossing-free perfect bipartite matchings between two separated sets, each of $\frac{n}{2}$ points, is at least $\binom{2\lfloor n/8\rfloor}{\lfloor n/8\rfloor}^4 \approx_n 2^n$.*

Clearly, this serves also as a lower bound for the more general case of perfect left-right matchings, for which we were not able to improve over the 2^n bound.

7 Discussion, Open Problems

Relating the basis-constants. For $n \in \mathbb{N}$, let $\mathsf{pm}(n) := \max_{|P|=n} \mathsf{pm}(P)$ and[14] $c_{\mathsf{pm}} := \limsup_{n\to\infty} \sqrt[n]{\mathsf{pm}(n)}$. In an analogous fashion, define the constants c_{ma}, c_{sc}, c_{cfp}, and c_{lrpm}

[14]In fact, there is a unique limit for n over the even integers.

for the corresponding matching bounds. Also, define

$$\mathsf{rdpm}(n) := \sup_{\mu} \mathbf{E}\left[\mathsf{pm}(P) \,|\, P \text{ a set of } n \text{ i.i.d. points from distribution } \mu\right],$$

and put $c_{\mathsf{rdpm}} := \limsup_{n\to\infty} \sqrt[n]{\mathsf{rdpm}(n)}$.

Apart from the absolute bounds that we derived for these constants, we have shown a number of relations among them, e.g.

$$\begin{array}{rcll} c_{\mathsf{pm}} & \le & 2^{1/3}\,5^{-1/6}\,c_{\mathsf{ma}} & \text{(note also that } c_{\mathsf{ma}} \le c_{\mathsf{pm}}+1\text{),} \\ c_{\mathsf{sc}} & \le & 3\,{c_{\mathsf{lrpm}}}^2 & \text{(also } c_{\mathsf{sc}} \le {c_{\mathsf{pm}}}^2\text{), and} \\ c_{\mathsf{cfp}} & \le & c_{\mathsf{ma}}+2 & \text{(see the remark preceding Theorem 5.2).} \end{array}$$

We also derived a better upper bound on c_{rdpm} than on c_{pm} (while these constants still share the same lower bound of 3). It would be interesting to know whether that is an artifact of our proof. We believe not, supported by the following observation: If we consider four points, then in non-convex position they have three crossing-free perfect matchings. If, however, we choose four i.i.d. points from any distribution, then they are in non-convex position with probability less than $\frac{5}{8}$ [28], and thus the expected number of crossing-free perfect matchings is less than $\frac{5}{8}\cdot 3 + \frac{3}{8}\cdot 2 = 2.625$.

Conjecture 1 $c_{\mathsf{rdpm}} < c_{\mathsf{pm}}$.

Also, can the bound for i.i.d. points be improved for specific distributions, uniform distribution in a disk, say?

Counting and enumeration. As far as we know, the algorithmic complexity of computing the number $\mathsf{pm}(P)$ of crossing-free perfect matchings for a set P of points is open—neither a polynomial algorithm is known, nor any lower bounds, $\#\mathcal{P}$-complete, say. The same is true for the numbers $\mathsf{tr}(P)$, $\mathsf{sc}(P)$, etc.

The situation looks somewhat more promising for enumeration. For triangulations and crossing-free spanning trees of a point set, Avis and Fukuda [8] show how to enumerate these objects in time poly(n) times the size of the output (see [27] for an application for enumeration of crossing-free graphs on a point set).

Nothing of the kind is known for perfect crossing-free matchings and spanning cycles. We mention on the side that *maximal* crossing-free matchings can be enumerated efficiently, due to a general result of that kind for maximal cliques in graphs [11]. To see this, define a graph for an n point set as follows. Let the vertices be the $\binom{n}{2}$ segments connecting pairs of points. Two such segments are connected by an edge if they are disjoint, i.e. they neither cross nor share an endpoint. Now cliques in this graphs correspond to crossing-free matchings of the point set.

For perfect crossing-free matchings, we would need maximum cliques in the constructed graph. For these, no efficient enumeration algorithms exist (and are unlikely to exist at all), but it is still feasible that the special geometric structure allows such an algorithm for our problem.

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