On the Number of Crossing-Free Matchings, (Cycles, and Partitions)*

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Abstract

We show that a set of n points in the plane has at most $O(10.05^n)$ perfect matchings with crossing-free straight-line embedding. The expected number of perfect crossing-free matchings of a set of n points drawn i.i.d. from an arbitrary distribution in the plane is at most $O(9.24^n)$.

Several related bounds are derived: (a) The number of all (not necessarily perfect) crossing-free matchings is at most $O(10.43^n)$. (b) The number of *left-right* perfect crossing-free matchings (where the points are designated as left or as right endpoints of the matching edges) is at most $O(5.38^n)$. (c) The number of perfect crossing-free matchings across a line (where all the matching edges must cross a fixed halving line of the set) is at most 4^n.

These bounds are employed to infer that a set of n points in the plane has at most $O(86.81^n)$ crossing-free spanning cycles (simple polygonizations), and at most $O(12.24^n)$ crossing-free partitions (partitions of the point set, so that the convex hulls of the individual parts are pairwise disjoint).

1 Introduction

Let P be a set of n points in the plane. A *geometric graph* on P is a graph that has P as its vertex set and its edges are drawn as straight segments connecting the corresponding pairs of points. The graph is *crossing-free* if no pair of its edges cross each other, i.e., any two edges are not allowed to share any points other than common endpoints. Therefore, these are planar graphs with a plane embedding given by this specific drawing. We are interested in the number of crossing-free geometric graphs on P of several special types. Specifically, we consider the numbers $\mathsf{tr}(P)$, of *triangulations* (i.e., maximal crossing-free graphs), $\mathsf{pm}(P)$, of crossing-free *perfect matchings*, $\mathsf{sc}(P)$, of crossing-free *spanning cycles*, and, $\mathsf{cfp}(P)$, of *crossing-free partitions*[1] (partitions of P, so that the convex hulls of the parts are pairwise disjoint). We are concerned with upper bounds for the numbers listed above in terms of n.

History. This problem goes back to Newborn and Moser [25] in 1980 who ask for the maximal possible number of crossing-free spanning cycles in a set of n points[2]; they give an upper bound of $2 \cdot 6^{n-2} \left\lfloor \frac{n}{2} \right\rfloor!$ but conjecture that the bound should be of the form c^n, c a constant. This was established in 1982 by Ajtai, Chvátal, Newborn, and Szemerédi [4], who show[3] that there are at most 10^{13n} crossing-free graphs.[4]

Further developments were mainly concerned with deriving progressively better upper bounds for the number of triangulations[5] [29, 13, 28], so far culminating in a 59^n upper bound by Santos and Seidel [27] in 2003. It compares to $\Omega(8.48^n)$, the largest known number of triangulations for a set of n points, recently derived by Aichholzer et al. [1]; this improves an earlier lower bound of $8^n/\mathrm{poly}(n)$ given by García et al. [17]. (We let "poly(n)" denote a polynomial factor in n.)

Every crossing-free graph is contained in some triangulation (with at most $3n-6$ edges). Hence, a c^n bound for the number of triangulations yields a bound of $2^{3n-6}c^n < (8c)^n$ for the number of crossing-free graphs on a set of n points; with $c \leq 59$, this is at most 472^n. To the best of our knowledge, *all* upper bounds so far on the number of crossing-free graphs of various types are derived via a bound on the number of triangulations,

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[1] Our research was triggered by M. van Kreveld asking about the number of crossing-free partitions, and, in the same week, by M. Hoffmann and Y. Okamoto asking about the number of crossing-free spanning paths of a point set (motivated by their quest for good fixed parameter algorithms for the planar Euclidean Traveling Salesman Problem in the presence of a fixed number of inner points [10]); see also [19].

[2] Akl's work [5] appeared earlier, but it refers to the manuscript by Newborn and Moser, and improves a lower bound (on the maximal number of crossing-free spanning cycles) of theirs.

[3] This paper is famous for its *Crossing Lemma*, proved in preparation of the singly exponential bound. The lemma gives an upper bound on the number of edges a geometric graph with a given number of crossings can have.

[4] For motivation they mention also a question of D. Avis about the maximum number of triangulations a set of n points can have.

[5] Interest was also motivated by the related question (from geometric modeling [29]) of how many bits it takes to encode a triangulation.

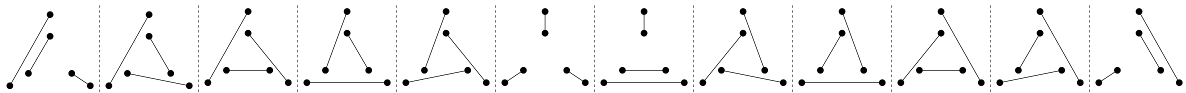


Figure 1: 6 points with 12 crossing-free perfect matchings, the maximum possible number; see [3] for the maximum numbers for up to ten points: 3 for 4 points, 12 for 6, 56 for 8, and 311 for 10.

albeit in more refined ways. One idea is to exploit the fact that graphs of certain types have a fixed number of edges; e.g., since a perfect matching has $\frac{n}{2}$ edges, we readily obtain $\mathsf{pm}(P) \le \binom{3n-6}{n/2} \mathsf{tr}(P) < 227.98^n$ [14]. A short historical account of bounds on $\mathsf{sc}(P)$, with references including [5, 12, 17, 18, 20, 25, 26], can be found at the web site [11] (see also [8]). The best bound published is $3.37^n \, \mathsf{tr}(P) \le 198.83^n$. It relies on a 3.37^n bound on the number of cycles in a planar graph [6]. In the course of our investigations, we showed that a graph with m edges and n vertices has at most $\left(\frac{m}{n}\right)^n$ cycles; hence, a planar graph has at most 3^n cycles. Then R. Seidel provided us with an argument, based on linear algebra, that a planar graph has at most $\sqrt{6}^n < 2.45^n$ spanning cycles.

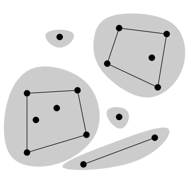


Figure 2: Graph of a crossing-free partition.

Crossing-free partitions fit into the picture, since every such partition can be uniquely identified with the graph of edges of the convex hulls of the individual parts—these edges form a crossing-free geometric graph of at most n edges; see Fig. 2.

The situation is better understood for special configurations, for example for P a set of n points in convex position (the vertex set of a convex n-gon), where the *Catalan numbers* $C_m := \frac{1}{m+1}\binom{2m}{m} = \Theta(m^{-3/2}4^m)$, $m \in \mathbb{N}_0$, play a prominent role. In convex position $\mathsf{tr}(P) = C_{n-2}$ (the Euler-Segner problem, cf. [30, pg. 212] for its history), $\mathsf{pm}(P) = C_{n/2}$ for n even ([16], cf. [30]), $\mathsf{sc}(P) = 1$, and $\mathsf{cfp}(P) = C_n$ ([7]).

Crossing-free partitions for point sets in convex position constitute a well-established notion because of its many connections to other problems, probably starting with "planar rhyme schemes" in Becker's note [7], cf. [30, Solution to 6.19pp]. The general case was considered by [9] (under the name of pairwise linearly separable partitions) for clustering algorithms. They show that that the number of partitions into k parts is $O(n^{6k-12})$ for k constant.

Under the assumption of *general position* (no three points on a common line) it is known [17] that the number of crossing-free perfect matchings on a set of fixed size is minimized when the set is in convex position. (Recently, Aichholzer et al. [1] showed that any family of acyclic graphs has the minimal number of crossing-free embeddings on a point set in convex position.) With little surprise, the same holds for spanning cycles, but it does not hold for triangulations [21, 2, 23]. For crossing-free partitions, this is open.

Results. We show the following bounds, for a set P of n points in the plane: $\mathsf{pm}(P) = O(10.05^n)$, $\mathsf{sc}(P) = O(86.81^n)$, and $\mathsf{cfp}(P) = O(12.24^n)$. Also, the expected number of perfect crossing-free matchings of a set of n points drawn i.i.d. from *any* distribution in the plane (where two random points coincide with probability 0) is $O(9.24^n)$.

The bound on the number of crossing-free perfect matchings is derived by an inductive technique that we have adapted from the method that Santos and Seidel [27] used for triangulations (the adaption however is far from obvious). We then go on to derive improved bounds on the number of crossing-free matchings of various special types: **(a)** The number of all (not necessarily perfect) crossing-free matchings is at most $O(10.43^n)$. **(b)** The number of *left-right* perfect crossing-free matchings (where the points are designated as left or as right endpoints of the matching edges) is at most $O(5.38^n)$. **(c)** The number of perfect crossing-free matchings across a line (where all the matching edges must cross a fixed halving line of the set) is at most 4^n.

Finally, we derive upper bounds for the numbers of crossing-free spanning cycles and crossing-free partitions of P in terms of the number of certain types of matchings of certain point sets P' that are constructed from P. This yields the bounds as stated above.

We summarize the state of affairs in Table 1, (including lower bounds—proofs are omitted here). In work in progress, we are currently refining a tailored analysis for spanning cycles and trees, where the bounds now stand at $O(79^n)$ and $O(296^n)$, respectively.

	tr	pm	sc	cfp
$\forall P :\le$	59 [27]	10.05	86.81	12.24
$\exists P :\ge$	8.48 [1]	3 [17]	4.64 [17]	5.23
	ma	lrpm	alpm	rdpm
$\forall P :\le$	10.43	5.38	4	9.24
$\exists P :\ge$	4	2	2	3

Table 1: Entries c in the upper bound rows stand for $O(c^n)$, and entries c in the lower bound rows for $\Omega(c^n/\mathrm{poly}(n))$, where $n := |P|$. "ma" stands for all crossing-free matchings, "lrpm" for perfect left-right crossing-free matchings, "alpm" for perfect crossing-free matchings across a line, and "rdpm" for the expected number of perfect crossing-free matchings of a set of i.i.d. points.

2 Matchings: The Setup and a Recurrence

Let P be a set of n points in the plane in general position, no three on a line, no two on a vertical line. This is no constraint when it comes to upper bounds on $\mathsf{pm}(P)$. A *crossing-free matching* M is a collection of pairwise disjoint segments whose endpoints belong to P. Each point of P is either *matched*, if it is an endpoint of a segment of M, or *isolated*, otherwise. The number of matched points is always even. If $2m$ points are matched and s points are isolated, we call M a *crossing-free m-matching* or *(m,s)-matching*. We have $n = 2m + s$.

For $m \in \mathbb{R}$ we denote by $\mathsf{ma}_m(P)$ the number of crossing-free matchings of P with m segments (this number is 0 unless $m \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$), and by $\mathsf{ma}(P)$ the number of all crossing-free matchings of P (i.e. $\mathsf{ma}(P) = \sum_m \mathsf{ma}_m(P)$). Recall $\mathsf{pm}(P) = \mathsf{ma}_{n/2}(P)$.

Let M be a crossing-free (m,s)-matching on a set P of $n = 2m + s$ points, as above. The *degree* $d(p)$ of a point $p \in P$ in M is defined as follows. It is 0 if p is isolated in M. Otherwise, if p is a left (resp., right) endpoint of a segment of M, $d(p)$ is equal to the number of visible left (resp., right) endpoints of other segments of M, plus the number of visible isolated points; "*visible*" means *vertically* visible from the *relative interior* of the segment of M that has p as an endpoint. Thus p and the other endpoint of the segment are not counted in $d(p)$. See Fig. 3.

Each left (resp., right) endpoint u in M can contribute at most 2 to the degrees of other points: 1 to each of the left (resp., right) endpoints of the segments lying vertically above and below u, if there exist such segments. Similarly, each isolated point u can contribute at most 4 to the degrees of other points: 1 to each of the endpoints of the segments lying vertically above and below u. It follows that $\sum_{p \in P} d(p) \le 4m + 4s$.

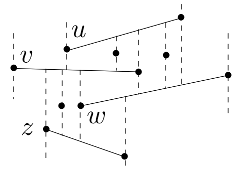


Figure 3: Degrees in a matching: $d(u) = 2$, $d(v) = 5$, $d(w) = 1$, $d(z) = 2$.

There are many segments ready for removal. The idea is to remove segments incident to points of low degree in an (m,s)-matching (points of degree at most 3 or at most 4, to be specific). We will show that there are many such points at our disposal. Then, in the next step, we show that segments with an endpoint of low degree can be reinserted in not too many ways. These two facts will be combined to derive a recurrence for the matching count.

For $i \in \mathbb{N}_0$, let $v_i = v_i(M)$ denote the number of *matched* points of P with degree i in M. Hence, $\sum_{i \ge 0} v_i = 2m$.

LEMMA 2.1. *Let $n, m, s \in \mathbb{N}_0$, with $n = 2m + s$. In every (m,s)-matching of any set of n points, we have*

$$(2.1) \quad 2n \le 4v_0 + 3v_1 + 2v_2 + v_3 + 6s \ ,$$

$$(2.2) \quad 3n \le 5v_0 + 4v_1 + 3v_2 + 2v_3 + v_4 + 7s \ .$$

Proof. Let P be the underlying point set. We have $\sum_{i \ge 0} i\, v_i = \sum_{p \in P} d(p) \le 4s + 4m = 4s + \sum_{i \ge 0} 2v_i$. Therefore, $0 \le 4s + \sum_{i \ge 0} (2 - i) v_i$. For $\kappa \in \mathbb{R}^+$, we add κ times $n = s + \sum_{i \ge 0} v_i$ to both sides to get

$$\begin{aligned} \kappa n &\le (4+\kappa)s + \textstyle\sum_{i \ge 0} (2 + \kappa - i) v_i \\ &\le (4+\kappa)s + \textstyle\sum_{0 \le i < 2+\kappa} (2 + \kappa - i) v_i \ . \end{aligned}$$

We set $\kappa = 2$ for (2.1) and $\kappa = 3$ for (2.2). ◻

There are not too many ways of inserting a segment. Fix some $p \in P$ and let M be a crossing-free matching which leaves p isolated. Now we match p with some other isolated point such that the overall matching continues to be crossing-free. For $i \in \mathbb{N}_0$, let $h_i = h_i(p, P, M)$ be the number of ways that can be done so that p has degree i after its insertion.

LEMMA 2.2.

$$(2.3) \quad 4h_0 + 3h_1 + 2h_2 + h_3 \le 24 \ ,$$

$$(2.4) \quad 5h_0 + 4h_1 + 3h_2 + 2h_3 + h_4 \le 48 \ .$$

Proof. Let $\ell_i = \ell_i(p, P, M)$ be the number of ways we can match the point p as a left endpoint of degree i. First, we claim that $\ell_0 \in \{0, 1\}$.

To show this, form the *vertical decomposition* of M by drawing a vertical segment up and down from each (matched or isolated) point of $P \setminus \{p\}$, and extend these segments until they meet an edge of M, or else, all the way to infinity; see Fig. 4. We call these vertical segments *walls* in order to distinguish them from the segments in the matching.

We obtain a decomposition of the plane into vertical trapezoids. Let τ be the trapezoid containing p (assuming general position, p lies in the interior of τ). See Fig. 4.

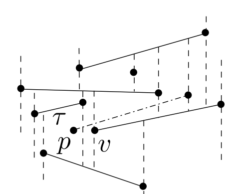


Figure 4: Inserting a segment at p; $d(p) = 1$ after insertion.

We move from τ to the right through vertical walls to adjacent trapezoids until we reach a vertical wall that is determined by a point v that is either a left endpoint or an isolated point (if at all—we may make our way to infinity when p cannot be matched as a left endpoint to any point, in which case $\ell_i = 0$ for all i).

Note that up to that point there was always a unique choice for the next trapezoid to enter. Every crossing-free segment with p as its left endpoint will have to go through all of these trapezoids. It connects either to v (which can happen only if v is isolated), or crosses the vertical wall up or down from v. The former case yields a segment that gives p degree 0. In the latter case, v will contribute 1 to the degree of p. So pv, if an option, is the only possible segment that lets p have degree 0 as a left endpoint. (pv will not be an option when it crosses some segment, or when v is a left endpoint.) We will return to this set-up when we consider degrees ≥ 1, in which case v acts as a *bifurcation point*. Before doing so, we first introduce a function f. It maps every nonnegative real vector $(\lambda_0, \lambda_1, \ldots, \lambda_k)$ of arbitrary length $k+1 \in \mathbb{N}$ to the maximum possible value the expression

$$\lambda_0 \ell_0 + \lambda_1 \ell_1 + \cdots + \lambda_k \ell_k \tag{2.5}$$

can attain (for any isolated point in any matching of any finite point set *of any size*). We have already shown that $f(\lambda) \leq \lambda$ for $\lambda \in \mathbb{R}_0^+$. We claim that for all $(\lambda_0, \lambda_1, \ldots, \lambda_k) \in (\mathbb{R}_0^+)^{k+1}$, with $k \geq 1$, we have

$$f(\lambda_0, \lambda_1, \ldots, \lambda_k) \leq \max \begin{cases} \lambda_0 + f(\lambda_1, \ldots, \lambda_k), \\ 2f(\lambda_1, \ldots, \lambda_k). \end{cases} \tag{2.6}$$

Assuming (2.6) has been established, we can conclude that $f(1) \leq 1$, $f(2,1) \leq 3$, $f(3,2,1) \leq 6$, and $f(4,3,2,1) \leq 12$; that is, $4\ell_0 + 3\ell_1 + 2\ell_2 + \ell_3 \leq 12$ and the first inequality of the lemma follows, since the same bound clearly holds for the case when p is a right endpoint. The second inequality follows similarly from $f(5,4,3,2,1) \leq 24$.

It remains to prove (2.6). Consider a constellation with a point p that realizes the value of $f(\lambda_0, \lambda_1, \ldots, \lambda_k)$. We return to the set-up from above, where we have traced a unique sequence of trapezoids from p to the right, till we encountered the first bifurcation point v (if v does not exist then all ℓ_i vanish).

Case 1: v is isolated. We know that $\lambda_0 \ell_0 \leq \lambda_0$. If we remove v from the point set, then every possible crossing-free segment emanating from p to its right has its degree decreased by 1. Therefore, $\lambda_1 \ell_1 + \cdots + \lambda_k \ell_k \leq f(\lambda_1, \ldots, \lambda_k)$, so the expression (2.5) cannot exceed $\lambda_0 + f(\lambda_1, \ldots, \lambda_k)$ in this case.

Case 2: v is a matched left endpoint. Then $\lambda_0 \ell_0 = 0$ (that is, we cannot connect p to v). Possible crossing-free segments with p as a left endpoint are discriminated according to whether they pass above or below v. We first concentrate on the segments that pass above v; we call them *relevant segments* (emanating from p). Let ℓ_i' be the number of relevant segments that give p degree i. We carefully remove isolated points from $P \setminus \{p\}$ and segments with their endpoints from the matching M (eventually also the segment of which v is a left endpoint), so that in the end all relevant segments are still available and each one, if inserted, makes the degree of p exactly 1 unit smaller than its original value (this deletion process may create new possibilities for segments from p). That will show $\lambda_1 \ell_1' + \cdots + \lambda_k \ell_k' \leq f(\lambda_1, \ldots, \lambda_k)$. The same will apply to segments that pass below v, using a symmetric argument, which gives the bound of $2f(\lambda_1, \ldots, \lambda_k)$ for (2.5) in this second case.

The removal process is performed as follows. We define a relation $\prec$ on the set whose elements are the edges of M and the singleton sets formed by the isolated points of $P \setminus \{p\}$: $a \prec b$ if a point $a' \in a$ is vertically visible from a point $b' \in b$, with a' below b'. As is well known (cf. [15, Lemma 11.4]), $\prec$ is acyclic. Let $\prec^+$ denote the transitive closure of $\prec$, and let $\prec^*$ denote the transitive reflexive closure of $\prec$.

Let e be the segment with v as its left endpoint, and consider a minimal element a with $a \prec^+ e$. Such an element exists, unless e itself is a minimal element with respect to $\prec$.

a is a singleton: So it consists of an isolated point; with abuse of notation we also denote by a the isolated point itself. a cannot be a point to which p can connect with a relevant edge. Indeed, if this were the case, we add that edge $e' = pa$ and modify $\prec$ to include e' too; more precisely, any pair in $\prec$ that involves a is replaced by a corresponding pair involving e', and new pairs involving e' are added (clearly, the relation remains acyclic and all pairs related under $\prec^+$ continue to be so related after e' is included and replaces a). See Fig. 5(a). We have $e \prec e'$ (since, by assumption, the left endpoint v of e is vertically visible below e'), and $e' \prec^+ e$ (since the right endpoint a of e' satisfies $a \prec^+ e$)—a contradiction. With a similar reasoning we can rule out the possibility that a contributes to the degree of p when matched via a relevant edge pq. Indeed, if this were the case, let e'' be the segment directly above a, which is the first link in the chain that gives $a \prec^+ e$, i.e., $a \prec e'' \prec^* e$ (e'' must exist since $a \prec^+ e$). After adding pq with a contributing to its degree, we have either $a \prec pq$ and $pq \prec e''$ (see Fig. 5(b)), or we have $pq \prec a$ (see Fig. 5(c)). In the former case, we have $a \prec pq \prec e'' \prec^* e \prec pq$—contradicting the acyclicity of $\prec$. In the latter case, we have $pq \prec a \prec^+ e \prec pq$, again a contradiction. So if we remove a, then all relevant edges from p remain in the game and the degree of each of them (i.e., the degree of p that the edge induces when inserted) does not change.

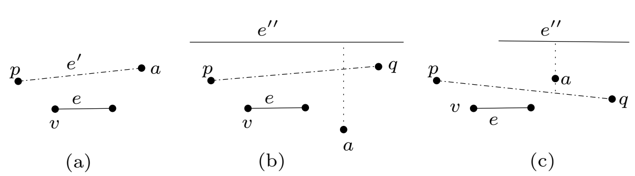


Figure 5: (a) The point a cannot be connected to p via a relevant edge. (b,c) a cannot contribute from below (in (b)) or from above (in (c)) to the degree of p when a relevant edge pq is inserted.

a is an edge: It cannot obstruct any isolated point or left endpoint below it from contributing to the degree of a relevant edge pq above v (because a is minimal with respect to $\prec$). If a obstructs a contribution to a relevant edge pq from above, then we add pq, thus $pq \prec a$ which, together with $a \prec^+ e$ and $e \prec pq$, contradicts the acyclicity of $\prec$ (Fig. 6). Again, we can remove a without any changes to relevant possible edges from p.

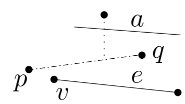


Figure 6: Edge a cannot obstruct a point from contributing from above to the degree of p when a relevant edge pq is inserted.

We keep successively removing elements until e is minimal with respect to $\prec$. Note that so far all the relevant edges from p are still possible, and the degree of p that any of them induces when inserted has not changed. Now we remove e with its endpoints. This cannot clear the way for any new contribution to the degree of a relevant edge. In fact, any such degree decreases by exactly 1 because v disappears. The claim is shown, and the proof of the lemma is completed. ◻

Deriving a recurrence.

LEMMA 2.3. *Let $n, m \in \mathbb{N}_0$, such that $m \le \frac{n}{2}$ and $s := n - 2m$. For every set P of n points, we have*

$$\mathsf{ma}_m(P) \le \begin{cases} \frac{12(s+2)}{n-3s}\,\mathsf{ma}_{m-1}(P) & \text{if } s < \frac{n}{3}, \text{ and} \\ \frac{16(s+2)}{n-7s/3}\,\mathsf{ma}_{m-1}(P) & \text{if } s < \frac{3n}{7}. \end{cases}$$

Let us note right away that the first inequality supersedes the second for $s < \frac{n}{5}$ (i.e. $m > \frac{2n}{5}$), while the second one is superior for $s > \frac{n}{5}$.

Proof. Fix the set P, and let $\mathcal{X}$ and $\mathcal{Y}$ be the sets of all crossing-free m-matchings and $(m-1)$-matchings, respectively, in P.

Let us concentrate on the first inequality. We define an edge-labeled bipartite graph $\mathcal{G}$ on $\mathcal{X} \dot\cup \mathcal{Y}$ as follows: Given an m-matching M, if p is an endpoint of a segment $e \in M$ and $d(p) \le 3$, then we connect $M \in \mathcal{X}$ to the $(m-1)$-matching $M \setminus \{e\} \in \mathcal{Y}$ with an edge labeled $(p, d(p))$; $d(p)$ is the *degree label* of the edge. Note that M and $M \setminus \{e\}$ can be connected by two (differently labeled) edges, if both endpoints of e have degree at most 3.

For $0 \le i \le 3$, let x_i denote the number of edges in $\mathcal{G}$ with degree label i. We have

$$(2n-6s)\underbrace{|\mathcal{X}|}_{\mathsf{ma}_m(P)} \le 4x_0 + 3x_1 + 2x_2 + x_3 \le 24(s+2)\underbrace{|\mathcal{Y}|}_{\mathsf{ma}_{m-1}(P)}.$$

The first inequality is a consequence of inequality (2.1) of Lemma 2.1. The second inequality is implied by inequality (2.3) in Lemma 2.2, as follows. For a fixed $(m-1)$-matching M' in P, consider an edge of $\mathcal{G}$ that is incident to M' and is labeled by (p, i) (if there is such an edge). Then p must be one of the $s+2$ isolated points of P (with respect to M'), and there is a way to connect p to another isolated point in a crossing-free manner, so that p has degree i in the new matching. Hence, the contribution by p and M' to the sum $4x_0+3x_1+2x_2+x_3$ is at most 24, by inequality (2.3) in Lemma 2.2, and the right inequality follows. The combination of both inequalities yields the first inequality the lemma.

By considering endpoints up to degree 4 (instead of 3), we get the second inequality. ◻

For $m, n \in \mathbb{N}_0$, let $\mathsf{ma}_m(n)$ be the maximum number of crossing-free m-matchings a set of n points can have.

LEMMA 2.4. *For $s, m, n \in \mathbb{N}_0$, with $n = 2m + s$,*

$$\mathsf{ma}_0(0) = 1,$$

$$\mathsf{ma}_m(n) \le \begin{cases} \frac{n}{s}\,\mathsf{ma}_m(n-1), & \text{for } s \ge 1, \\ \frac{12(s+2)}{n-3s}\,\mathsf{ma}_{m-1}(n), & \text{for } s < \frac{n}{3}, \\ \frac{16(s+2)}{n-7s/3}\,\mathsf{ma}_{m-1}(n), & \text{for } s < \frac{3n}{7}. \end{cases}$$

Proof. $\mathsf{ma}_0(0) = 1$ is trivial.

The first of the three inequalities is implied by $s \cdot \mathsf{ma}_m(P) = \sum_{p \in P} \mathsf{ma}_m(P \setminus \{p\}) \le n \cdot \mathsf{ma}_m(n-1)$, for any set P of n points. The second and third inequalities follow from Lemma 2.3. ◻

3 Solving a Recurrence

We derive an upper bound for a function $G \equiv G_{\lambda,\mu} : \mathbb{N}_0^2 \to \mathbb{R}^+$, for a pair of parameters $\lambda, \mu \in \mathbb{R}^+$, $\mu \ge 1$, which satisfies (with $s := n - 2m$)

$$G(0,0) = 1,$$

$$(3.7)\quad G(m,n) \le \begin{cases} \frac{n}{s}\,G(m, n-1), & \text{for } s \ge 1, \\ \frac{\lambda(s+2)}{n-\mu s}\,G(m-1, n), & \text{for } s < \frac{n}{\mu}. \end{cases}$$

The recurrence in Lemma 2.4 implies that an upper bound on $G_{12,3}(m,n)$ serves also as an upper bound for $\mathsf{ma}_m(n)$, and the same holds for $G_{16,7/3}(m,n)$. We will see how to best combine the two parameter pairs, to obtain even better bounds for $\mathsf{ma}_m(n)$. Later, we will encounter other instances of this recurrence, with other values of λ and μ.

We divide by $\lambda^m \mu^{n-m}$. Then (3.7) becomes

$$\frac{G(m,n)}{\lambda^m \mu^{n-m}} \le \begin{cases} \frac{n}{\mu s}\,\frac{G(m,n-1)}{\lambda^m \mu^{n-1-m}}, & \text{for } s \ge 1, \\ \frac{\mu(s+2)}{n-\mu s}\,\frac{G(m-1,n)}{\lambda^{m-1}\mu^{n-m+1}}, & \text{for } s < \frac{n}{\mu}. \end{cases}$$

We set $H(m,n) = H_\mu(m,n) := \frac{G(m,n)}{\lambda^m \mu^{n-m}}$. Therefore, still with the convention $s := n-2m$ and the assumption $\mu \ge 1$, we have (note independence of λ)

$$H(0,0) = 1,$$

$$(3.8)\quad H(m,n) \le \begin{cases} \frac{n}{\mu s}\,H(m, n-1), & \text{for } s \ge 1, \\ \frac{\mu(s+2)}{n-\mu s}\,H(m-1, n), & \text{for } s < \frac{n}{\mu}. \end{cases}$$

LEMMA 3.1. *Let $m, n \in \mathbb{N}_0$, with $m \le \frac{n}{2}$. Then $H(m,n) \le \binom{n}{m}$.*

Proof. $H(0,0) = 1 \le \binom{0}{0}$ forms the basis of a proof by induction on n and m. For all $n \in \mathbb{N}_0$, $H(0,n) \le \mu^{-n} \le 1 = \binom{n}{0}$ follows, since $\mu \ge 1$.

Let $1 \le m \le \frac{n}{2}$. If $m \le n - \mu s$ then $s \le \frac{n-m}{\mu} < \frac{n}{\mu}$. Hence, the second inequality in (3.8) can be applied, after which the first inequality can be applied. Hence,

$$\begin{aligned} H(m,n) &\le \tfrac{\mu(s+2)}{n-\mu s}\, H(m-1,n) \\ &\le \tfrac{\mu(s+2)}{n-\mu s}\, \tfrac{n}{\mu(s+2)}\, H(m-1,n-1) \\ &\le \tfrac{n}{m}\binom{n-1}{m-1} = \binom{n}{m} . \end{aligned}$$

Otherwise, $m > n - \mu s$ holds, which ensures $\mu s > n - m \ge 0$, i.e., $s \ge 1$. We can therefore employ the first inequality of (3.8), and obtain
$H(m,n) \le \frac{n}{\mu s}\, H(m, n-1) < \frac{n}{n-m}\binom{n-1}{m} = \binom{n}{m}$. ∎

By expanding along the first inequality for a while before employing Lemma 3.1, we get

$$\begin{aligned} H(m,n) &\le \tfrac{n}{\mu s} \cdots \tfrac{n-k+1}{\mu(s-k+1)}\, H(m, n-k) \\ &\le \tfrac{1}{\mu^k}\left(\textstyle\prod_{i=0}^{k-1} \tfrac{n-i}{s-i}\right)\binom{n-k}{m} \\ &= \tfrac{1}{\mu^k}\, \tfrac{\binom{n}{k}}{\binom{s}{k}}\binom{n-k}{m} \qquad (3.9) \\ &= \tfrac{1}{\mu^k}\, \tfrac{\binom{2m}{m}}{\binom{n-m-k}{m}}\binom{n}{2m} , \quad \text{for } \mathbb{N}_0 \ni k \le s. \qquad (3.10) \end{aligned}$$

When we stop this unwinding of the recurrence, we could have alternatively proceeded one more step, and upper bound $H(m, n-k)$ by $\frac{n-k}{\mu(s-k)}\binom{n-k-1}{m}$, provided $k < s$. As long as this expression is smaller than $\binom{n-k}{m}$, we should indeed have expanded further. That is, we expand as long as

$$\begin{aligned} &\tfrac{n-k}{\mu(s-k)}\binom{n-k-1}{m} < \binom{n-k}{m} \\ \Leftrightarrow \quad & k < \tfrac{\mu s + m - n}{\mu - 1} = n - m\left(\tfrac{2\mu-1}{\mu-1}\right) = n - \tfrac{m}{\rho}, \end{aligned}$$

for $\rho := \frac{\mu-1}{2\mu-1}$. That is, the best choice of k in (3.9) is

$$k = \left\lceil n - \tfrac{m}{\rho} \right\rceil = n - \left\lfloor \tfrac{m}{\rho} \right\rfloor . \qquad (3.11)$$

In fact, if this suggested value of k is negative (or if $\rho = 0$), we should not expand at all. Instead, we try to expand along the second inequality of (3.8), to get

$$H(m,n) \le \frac{\binom{\frac{s}{2}+k}{k}}{\binom{\frac{n}{2\mu}-\frac{s}{2}}{k}}\binom{n}{m-k}, \qquad (3.12)$$

for $\mathbb{N}_0 \ni k < \frac{n}{2\mu} - \frac{s}{2} + 1 = m - \frac{\mu-1}{2\mu}\, n + 1$; we employ here the usual generalization of binomial coefficients $\binom{a}{k}$ to $a \in \mathbb{R}$, namely, $\binom{a}{k} := \frac{a(a-1)\cdots(a-k+1)}{k!}$.

Rather than optimizing the value of k at which we stop the unwinding of the second recurrence inequality of (3.8), we approximate it by

$$k = \left\lceil m - \tfrac{\mu-1}{2\mu-1}\, n \right\rceil = m - \lfloor \rho n \rfloor , \qquad (3.13)$$

and note that it lies in the allowed range, provided it is positive.

When $\frac{m}{n} = \rho$, both values suggested for k in (3.11) and (3.13) are 0, which indicates that we have to content ourselves with the bound $\binom{n}{m}$ from Lemma 3.1. Otherwise, it is clear which way to expand, since $\frac{m}{n} < \rho \;\Rightarrow\; n - \left\lfloor \frac{m}{\rho} \right\rfloor \ge 0$ and $\frac{m}{n} > \rho \;\Rightarrow\; m - \lfloor \rho n \rfloor \ge 0$. We are now ready for an improved bound. For that we substitute k in (3.9) according to (3.11), and in (3.12) according to (3.13).

LEMMA 3.2. *Let $m, n \in \mathbb{N}_0$, where $2m \le n$, and set $\rho := \frac{\mu-1}{2\mu-1}$. If $\frac{m}{n} \le \rho$, then*

$$H_\mu(m,n) \le \frac{1}{\mu^{n-\lfloor m/\rho \rfloor}}\, \frac{\binom{n}{n-\lfloor m/\rho \rfloor}}{\binom{n-2m}{n-\lfloor m/\rho \rfloor}}\binom{\lfloor m/\rho \rfloor}{m}$$

and for $\frac{m}{n} > \rho$, we have

$$H_\mu(m,n) \le \frac{\binom{\frac{n}{2}-\lfloor \rho n \rfloor}{m-\lfloor \rho n \rfloor}}{\binom{m-\frac{n}{2}(1-\frac{1}{\mu})}{m-\lfloor \rho n \rfloor}}\binom{n}{\lfloor \rho n \rfloor} .$$

Thus, $G_{\lambda,\mu}(m,n) \le \overline{G}_{\lambda,\mu}(m,n)$ with

$$\overline{G}_{\lambda,\mu}(m,n) := \begin{cases} \lambda^m \mu^{\lfloor m/\rho \rfloor - m}\, \dfrac{\binom{n}{n-\lfloor m/\rho \rfloor}}{\binom{n-2m}{n-\lfloor m/\rho \rfloor}} \dbinom{\lfloor m/\rho \rfloor}{m} , \\ \qquad \text{for } \frac{m}{n} \le \rho, \text{ and} \\[2ex] \lambda^m \mu^{n-m}\, \dfrac{\binom{\frac{n}{2}-\lfloor \rho n \rfloor}{m-\lfloor \rho n \rfloor}}{\binom{m-\frac{n}{2}(1-\frac{1}{\mu})}{m-\lfloor \rho n \rfloor}} \dbinom{n}{\lfloor \rho n \rfloor} , \\ \qquad \text{for } \frac{m}{n} > \rho. \end{cases}$$

Next we work out a number of properties of the upper bound $\overline{G}_{\lambda,\mu}$.

Estimates up to a polynomial factor. In the following derivations, we sometimes use "$\approx_n$" to denote equality up to a polynomial factor in n.

We will frequently use the following estimate (implied by Stirling's formula, cf. [22, Chapter 10, Corollary 9]) $\binom{\alpha n}{\lceil \beta n \rceil} \approx_n \binom{\alpha n}{\lfloor \beta n \rfloor} \approx_n \left(\frac{\alpha^\alpha}{\beta^\beta(\alpha-\beta)^{\alpha-\beta}}\right)^n$, for $\alpha, \beta \in \mathbb{R}$, $\alpha \ge \beta \ge 0$.

Big m. We note that for $\frac{m-1}{n} \ge \rho$

$$\overline{G}_{\lambda,\mu}(m,n) = \tfrac{\lambda(s+2)}{n-\mu s}\; \overline{G}_{\lambda,\mu}(m-1,n).$$

(with $s := n - 2m$). Since $\frac{\lambda(s+2)}{n-\mu s} < 1 \Leftrightarrow s < \frac{n-2\lambda}{\lambda+\mu} \Leftrightarrow m > \frac{(\lambda+\mu-1)n+2\lambda}{2(\lambda+\mu)}$, the function $\overline{G}_{\lambda,\mu}(m,n)$ maximizes for integers m in the range $\rho n \le m \le \frac{n}{2}$ at

$$m^* := \left\lfloor \tfrac{(\lambda+\mu-1)n+2\lambda}{2(\lambda+\mu)} \right\rfloor = \left\lfloor \tfrac{n}{2} - \tfrac{n-2\lambda}{2(\lambda+\mu)} \right\rfloor , \qquad (3.14)$$

unless this value is not in the provided range. However, $m^* \le \frac{n}{2}$ unless n is very small ($n < 2\lambda$). And $m^* \ge \rho n$ unless $\lambda < \mu - 1$.

Small m. With the identity indicated in (3.10) we have, for $\frac{m}{n} \le \rho$, that $\overline{G}$ can also be written as

$$\text{(3.15)} \quad \overline{G}_{\lambda,\mu}(m,n) \;=\; \lambda^m \mu^{\lfloor m/\rho \rfloor - m} \frac{\binom{2m}{m}}{\binom{\lfloor m/\rho \rfloor - m}{m}} \binom{n}{2m}$$

$$\approx_m \;(4\lambda(\mu-1))^m \binom{n}{2m} \;.$$

This bound peaks (up to an additive constant) at $m^{**} := \left\lfloor \frac{\sqrt{\lambda(\mu-1)}}{1+2\sqrt{\lambda(\mu-1)}}\, n \right\rfloor$. Note that $m^{**} \le \rho n$ for $\lambda \le \mu - 1$.

We summarize, that $\overline{G}_{\lambda,\mu}(m,n)$ attains its maximum—up to a poly(n)-factor—over m at

$$\text{(3.16)} \qquad m = \begin{cases} m^{**} & \text{if } \lambda \le \mu - 1, \text{ and} \\ m^{*} & \text{otherwise.} \end{cases}$$

In all applications in this paper we have $\lambda > \mu - 1$, so the peak occurs at m^*.

4 Matching Bounds

4.1 Perfect Matchings For perfect matchings we consider the case where n is even, $m = \frac{n}{2}$, and $s = 0$. We note that in this case $m/n = 1/2 > \rho$, for any value of μ. Hence, the second bound of Lemma 3.2 applies. We first calculate $\frac{n}{2} - \frac{n}{2}(1 - \frac{1}{\mu}) = \frac{1}{2\mu}\, n$, and $\frac{n}{2} - \lfloor \rho n \rfloor = \left\lceil \frac{n}{2} - \frac{\mu-1}{2\mu-1}\, n \right\rceil = \left\lceil \frac{1}{2(2\mu-1)}\, n \right\rceil$. Hence,

$$\overline{G}_{\lambda,\mu}\left(\frac{n}{2}, n\right) = (\lambda\mu)^{n/2} \binom{\frac{1}{2\mu}\, n}{\left\lceil \frac{1}{2(2\mu-1)}\, n \right\rceil}^{-1} \binom{n}{\left\lfloor \frac{\mu-1}{2\mu-1}\, n \right\rfloor}$$

$$\approx_n (\lambda\mu)^{n/2} \left(\frac{\left(\frac{1}{2(2\mu-1)}\right)^{\frac{1}{2(2\mu-1)}} \left(\frac{\mu-1}{2\mu(2\mu-1)}\right)^{\frac{\mu-1}{2\mu(2\mu-1)}}}{\left(\frac{1}{2\mu}\right)^{\frac{1}{2\mu}} \left(\frac{\mu-1}{2\mu-1}\right)^{\frac{\mu-1}{2\mu-1}} \left(\frac{\mu}{2\mu-1}\right)^{\frac{\mu}{2\mu-1}}} \right)^n$$

$$= \left(\lambda^{\frac{1}{2}} (\mu-1)^{-\frac{\mu-1}{2\mu}} (2\mu-1)^{\frac{2\mu-1}{2\mu}} \right)^n \;.$$

Substituting $(\lambda,\mu) = (12,3)$ and $(16,\frac{7}{3})$, as suggested by Lemma 2.4, we obtain the following upper bounds for the number of crossing-free perfect matchings:

$$\overline{G}_{12,3}\left(\tfrac{n}{2}, n\right) \;\approx_n\; \left(2^{\frac{2}{3}} \cdot 3^{\frac{1}{2}} \cdot 5^{\frac{5}{6}}\right)^n = O(10.5129^n) \;,$$

$$\overline{G}_{16,\frac{7}{3}}\left(\tfrac{n}{2}, n\right) \;\approx_n\; \left(2^{\frac{10}{7}} \cdot 3^{-\frac{1}{2}} \cdot 11^{\frac{11}{14}}\right)^n = O(10.2264^n) \;.$$

While the second bound is obviously superior, we remember that the recurrence with $(\lambda,\mu) = (12,3)$ is better for $m > \frac{2n}{5}$ (or $s < \frac{n}{5}$). This observation leads to the following better bound for P a set of n points and for $k = \lfloor \frac{n}{2} - \frac{2n}{5} \rfloor = \lfloor \frac{n}{10} \rfloor$, where we expand as in the first inequality of Lemma 2.3.

$$\begin{aligned} \mathsf{pm}(P) \;&\le\; \left(\prod_{i=0}^{k-1} \frac{12(2i+2)}{n-6i}\right) \mathsf{ma}_{n/2-k}(P) \\ &\le\; 4^k \binom{n/6}{k}^{-1} \overline{G}_{16,7/3}(n/2-k, n) \\ &\approx_n \left(2^{20/21}\, 3^{-2/7}\, 5^{1/21}\, 11^{11/14}\right)^n = O(10.0438^n). \end{aligned}$$

Perfect versus all matchings. Recall from Lemma 2.3 that $\mathsf{ma}_m(P) \le \frac{12(s+2)}{n-3s} \mathsf{ma}_{m-1}(P)$. Note that $\frac{12(s+2)}{n-3s} < 1$ for $m > \frac{7n}{15} + \frac{4}{5}$ (and in this range the factor $\frac{12(s+2)}{n-3s}$ is smaller than the alternative offered in Lemma 2.3). That is, there are always fewer perfect matchings than there are $\lfloor \frac{7n}{15} + \frac{4}{5} \rfloor$-matchings. More specifically, for sets P with $n := |P|$ even, and for $k = \frac{n}{2} - \lfloor \frac{7n}{15} + \frac{4}{5} \rfloor = \lceil \frac{n}{30} - \frac{4}{5} \rceil$, we have

$$\begin{aligned} \mathsf{pm}(P) \;&=\; \mathsf{ma}_{n/2}(P) \le \prod_{i=0}^{k-1} \frac{12(2i+2)}{n-6i}\; \mathsf{ma}_{n/2-k}(P) \\ &=\; \left(\tfrac{12\cdot 2}{6}\right)^k \binom{n/6}{k}^{-1} \mathsf{ma}_{n/2-k}(P) \\ &\approx_n\; 4^{n/30} \left(\left(\tfrac{1}{5}\right)^{1/5} \left(\tfrac{4}{5}\right)^{4/5}\right)^{n/6} \mathsf{ma}_{\lfloor \frac{7n}{15} + \frac{4}{5} \rfloor}(P) \\ &=\; \left(2^{1/3}\, 5^{-1/6}\right)^n \mathsf{ma}_{\lfloor \frac{7n}{15} + \frac{4}{5} \rfloor}(P)\,. \end{aligned}$$

Therefore, $\mathsf{pm}(P) \le \left(2^{1/3}\, 5^{-1/6}\right)^n \mathsf{ma}(P)\text{poly}(n) = O(0.9635^n)\, \mathsf{ma}(P)$, i.e. in every point set there are exponentially (in n) more crossing-free matchings than there are crossing-free perfect matchings.

4.2 All Matchings Our considerations in the derivation of the bound for perfect matchings imply the following upper bound for matchings with m segments.

$$\mathsf{ma}_m(P) \le \begin{cases} \overline{G}_{16,7/3}(m,n)\;, & m \le \frac{2n}{5}, \\ \overline{G}_{12,3}(m,n)\; \frac{\overline{G}_{16,7/3}(\frac{2n}{5},n)}{\overline{G}_{12,3}(\frac{2n}{5},n)}\;, & \text{otherwise.} \end{cases}$$

To determine where this expression maximizes, we note that $\overline{G}_{16,7/3}$ does not peak in its "small m"-range ($m \le \frac{4}{11}$) since $16 > \frac{7}{3} - 1$ (recall (3.16)). In the "big m"-range, it peaks at roughly $\frac{26n}{55}$ (see (3.14)), which exceeds $\frac{2}{5}$. Therefore, the maximum occurs when $\overline{G}_{12,3}$ comes into play, which peaks at roughly $\frac{7n}{15}$. For that value the upper bound evaluates to $\approx_n$ $(2^{13/21}\, 3^{-2/7}\, 5^{3/14}\, 11^{11/14})^n = O(10.4244^n)$. Summing up

THEOREM 4.1. *For P a set of n points in the plane*
(1) $\mathsf{pm}(P) \le \left(2^{20/21}\, 3^{-2/7}\, 5^{1/21}\, 11^{11/14}\right)^n \text{poly}(n) = O(10.0438^n)$.
(2) $\mathsf{pm}(P) \le \left(2^{1/3}\, 5^{-1/6}\right)^n \mathsf{ma}(P)\text{poly}(n) = O(0.9635^n)\, \mathsf{ma}(P)$.
(3) $\mathsf{ma}(P) \le \left(2^{13/21}\, 3^{-2/7}\, 5^{3/14}\, 11^{11/14}\right)^n \text{poly}(n) = O(10.4244^n)$.

4.3 Random Point Sets Let P be any set of $N \in \mathbb{N}$ points in the plane, no three on a line, and let $r \in \mathbb{N}$ with $r \le N$. If R is a subset of P chosen uniformly at random from $\binom{P}{r}$, then, for $\lambda = 16$, $\mu = \frac{7}{3}$, and provided $m \le \frac{\mu-1}{2\mu-1} N = \frac{4}{11} N$, and $r \ge 2m$, we have, using (3.15),

$$\mathbf{E}[\mathsf{ma}_m(R)] = \frac{1}{\binom{N}{r}} \sum_{R \in \binom{P}{r}} \mathsf{ma}_m(R) = \frac{\binom{N-2m}{r-2m}}{\binom{N}{r}} \mathsf{ma}_m(P)$$

$$\leq (4\lambda(\mu-1))^m \frac{\binom{N}{2m}\binom{N-2m}{r-2m}}{\binom{N}{r}} \mathrm{poly}(m)$$

$$\approx_m (4\lambda(\mu-1))^m \binom{r}{2m} = \left(2^8\, 3^{-1}\right)^m \binom{r}{2m} .$$

We see that if we sample r points from a large enough set, then the expected number of crossing-free matchings observes for all m the upper bound derived for the range of small m.

Suppose now that, for n even, we sample n i.i.d. points from an arbitrary distribution, for which we only require that two sampled points coincide with probability 0. Then we can first sample a set P of $N > \frac{11}{8}n$ points, and then choose a subset of size n uniformly at random from the family of all subsets of this size. We obtain a set R of n i.i.d. points from the given distribution. If P is in general position, by the argument above the expected number of perfect crossing-free matchings is at most $\approx_n (2^8\, 3^{-1})^{n/2}$. If P exhibits collinearities, we perform a small perturbation yielding a set $\tilde{P}$ and the subset $\tilde{R}$. Now the bound applies to $\tilde{R}$, and also to R since a sufficiently small perturbation cannot decrease the number of crossing-free perfect matchings.

Theorem 4.2. *For any distribution in the plane for which two sampled points coincide with probability* 0*, the expected number of crossing-free perfect matchings of n i.i.d. points is at most* $\left(2^4\, 3^{-1/2}\right)^n \mathrm{poly}(n) = O(9.2377^n)$.

4.4 Left-Right Perfect Matchings Here we assume that P is partitioned into two disjoint subsets L, R and consider bipartite matchings in $L \times R$ such that, for each edge of the matching, its left endpoint belongs to L and its right endpoint to R.

We modify the definition of the degrees of the points: If $p \in L$ is a matched to a point in R, then $d(p)$ is equal to the number of left endpoints plus the number of *right-labeled* isolated points that are vertically visible from (the relative interior of) e. A symmetric definition holds for right endpoints. (Intuitively, a right-labeled isolated point q has to contribute only to the degrees of left-labeled points, because, when we insert a right endpoint, it cannot connect to q, and it does not matter whether its incident edge passes above or below q; that is, q does not cause any bifurcation in the ways in which p can be connected.) Since isolated points contribute now only 2 to degrees of endpoints, we have $\sum_{p\in P} d(p) \leq 4m + 2s$. The analysis further improves, because when we reinsert a point $p \in L$, say, the corresponding numbers h_i must be equal to ℓ_i, since p can only be the left endpoint of a matching edge. A similar improvement holds for points $q \in R$. Hence, we can bound the sum $4h_0 + 3h_1 + 2h_2 + h_3$ by 12, rather than 24; similarly, we have $5h_0 + 4h_1 + 3h_2 + 2h_3 + h_4 \leq 24$. That is, we have for (λ, μ) the pairs $(6, 2)$ and $(8, \frac{5}{3})$ available. We infer a bound of $\left(\prod_{i=0}^{k-1} \frac{6(2i+2)}{n-4i}\right) \overline{G}_{8,5/3}(\frac{n}{2} - k, n)$, for $k = \lfloor \frac{n}{6} \rfloor$, implying

Theorem 4.3. *Let P be a set of n points in the plane and assume that the points are classified as left endpoints or right endpoints. The number of left-right perfect crossing-free matchings in P is at most* $\left(2^{7/10}\, 3^{-3/20}\, 7^{7/10}\right)^n \mathrm{poly}(n) = O(5.3793^n)$.

4.5 Matchings Across a Line Consider next the special case of crossing-free bipartite perfect matchings between two sets of $\frac{n}{2}$ points each that are separated by a line. Here we can obtain an upper bound that is smaller than the one in Theorem 4.3.

Theorem 4.4. *Let n be an even integer. The number of crossing-free perfect bipartite matchings between two separated sets of $\frac{n}{2}$ points each in the plane is at most ${C_{n/2}}^2 < 4^n$; (C_m is the mth Catalan number).*

Proof. Let L and R be the given separated sets. Without loss of generality, take the separating line λ to be the y-axis, and assume that the points of L lie to the left of λ and the points of R lie to its right. Let M be a crossing-free perfect bipartite matching in $L \times R$. For each edge e of M, let e_L (resp., e_R) denote the portion of e to the left (resp., right) of λ, and refer to them as the *left half-edge* and the *right half-edge* of e, respectively. We will obtain an upper bound for the number of combinatorially different ways to draw the left half-edges of a crossing-free perfect matching in $L \times R$. The same bound will apply symmetrically to the right half-edges, and the final bound will be the square of this bound.

In more detail, we ignore R, and consider collections S of $\frac{n}{2}$ pairwise disjoint segments, each connecting a point of L to some point on λ, so that each point of L is incident to exactly one segment. For each segment in S, we label its λ-endpoint by the point of L to which it is connected. The increasing y-order of the λ-endpoints of the segments thus defines a permutation of L, and our goal is to bound the number of different permutations that can be generated in this way. (In general, this is a *strict* upper bound on the quantity we seek.)

We obtain this bound in the following recursive manner. Write $m := |L| = \frac{n}{2}$. Sort the points of L from left to right (we may assume that there are no ties—they can be eliminated by a slight rotation of λ), and let $p_1, p_2, \ldots, p_m$ denote the points in this order. Consider the half-edge e_1 emanating from the leftmost point p_1. Any other point p_j lies either above or below

e_1. By rotating e_1 about p_1, we see that there are at most m (exactly m, if we assume general position) ways to split $\{p_2, \ldots, p_m\}$ into a subset L_1^+ of points that lie above e_1 and a complementary subset L_1^- of points that lie below e_1, where in the i-th split, $|L_1^+| = i-1$ and $|L_1^-| = m - i$. Note that, in any crossing-free perfect bipartite matching that has e_1 as a left half-edge incident to p_1, all the points of L_1^+ (resp., of L_1^-) must be incident to half-edges that terminate on λ *above* (resp., *below*) the λ-endpoint of e_1.

Hence, after having fixed i, we can proceed to bound recursively and separately the number of permutations induced by L_1^+, and the number of those induced by L_1^-. In other words, denoting by $\Pi(m)$ the maximum possible number of different permutations induced in this way by a set L of m points (in general position), we get the recurrence $\Pi(m) \le \sum_{i=1}^{m} \Pi(i-1)\Pi(m-i)$, for $m \ge 1$, where $\Pi(0) = 1$. However, this is the recurrence that (with equality) defines the Catalan numbers, so we conclude that $\Pi(m) \le C_m$.

A (probably weak) upper bound for the number of crossing-free perfect bipartite matchings in $L \times R$ is thus ${C_m}^2$. Indeed, for any permutation π_L of L and any permutation π_R of R, there is at most one crossing-free perfect bipartite matching in $L \times R$ that induces both permutations. Namely, it is the matching that connects the j-th point in π_L to the j-th point in π_R, for each $j = 1, \ldots, m$.

The asserted bound of ${C_m}^2 = {C_{n/2}}^2 < 4^n$ follows. ◽

5 Two Implications

Spanning Cycles

THEOREM 5.1. *Let P be a set of n points in the plane. Then the number of crossing-free spanning cycles satisfies* $\mathsf{sc}(P) \le (2^{7/5}\, 3^{7/10}\, 7^{7/5})^n \mathrm{poly}(n) = O(86.8089^n)$.

Proof. We construct (from P) a new set P' of $2n$ points by creating two copies p^+, p^- of each $p \in P$, and by placing these copies co-vertically close to the original location of p, with p^+ above p^-. Let π be a cycle in P. We map π to a perfect matching in P': For each $p \in P$, let q, r be its neighbors in π. (i) If both q, r lie to the left of p, with the edge qp lying above rp, we connect p^+ to either q^+ or q^-, and connect p^- to either r^+ or r^- (the actual choices will be determined at q and r by similar rules). (ii) The same rule applies when both q, r lie to the right of p. (iii) If q lies to the left of p and r lie to the right of p, then we connect p^+ to either q^+ or q^-, and connect p^- to either r^+ or r^-. Clearly, the resulting graph π^* is a crossing-free perfect matching in P', assuming general position of the points of P, if we draw each pair p^+, p^- sufficiently close to each other.

We assign to each point $p \in P$ a label that depends on π. A point whose two neighbors in π lie to its left is labeled as a *right point*, a point whose two neighbors in π lie to its right is labeled as a *left point*, and a point having one neighbor in π to its right and one to its left is labeled as a *middle point*. We assign the cycle π to the pair (π^*, λ), where π^* is the resulting perfect matching on P' and λ is the labeling of P, as just defined.

Each pair (π^*, λ) can be realized by at most one cycle π in P, by merging each pair p^+, p^- back into the original point p. (in general, the resulting graph is a collection of pairwise disjoint cycles.) It therefore suffices to bound the number of such pairs (π^*, λ).

A given labeling λ of P uniquely classifies each point of P' as being either a left point of an edge of the matching or a right endpoint of such an edge. Hence, the number of crossing-free perfect matchings π' on P' that respect this left-right assignment is at most $(2^{7/10}\, 3^{-3/20}\, 7^{7/10})^{2n}\mathrm{poly}(n)$. The number of labelings of P is 3^n. Hence, the number of crossing-free cycles in P is at most $(2^{7/5}\, 3^{7/10}\, 7^{7/5})^n \mathrm{poly}(n)$, as asserted. ◽

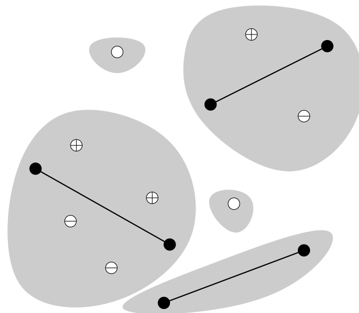
Clearly, it follows from the proof that the bound holds for the number of crossing-free spanning paths as well, and also for the number of cycle covers (or path covers) of P.

Crossing-free Partitions For a bound on $\mathsf{cfp}(P)$, we relate crossing-free partitions of a point set P to matchings. To this end, every crossing-free partition is mapped to a tuple (M, S, I^+, I^-) where (see Fig. 7) (i) M is the matching in P, whose edges connect the leftmost to the rightmost point of each set with at least two elements (such a segment is called the *spine* of its set), (ii) S is the set of all points that form singleton sets in the partition, and (iii) I^+ (resp., I^-) is the set of points in $P \setminus S$ that are neither leftmost nor the rightmost in their set, and which lie *above* (resp., *below*) the spine of their set. M is crossing-free, and the partition is uniquely determined by (M, S, I^+, I^-). Therefore, any bound on the number of such tuples will establish an bound on the number of crossing-free partitions. For every crossing-free matching M on P there are $3^{n-2|M|}$ triples (S, I^+, I^-) which form a 4-tuple with M (not all of them have to come from a crossing-free partition). Therefore $\sum_m 3^{n-2m}\mathsf{ma}_m(P)$ is a bound on the number of crossing-free partitions. Ignoring the 3^n-factor for the time being, we have to determine an upper bound on $3^{-2m}\mathsf{ma}_m(P)$, for which we employ the bound from (4.2). We observe that $3^{-2m}\overline{G}_{\lambda,\mu}(m,n) = \overline{G}_{\lambda/9,\mu}(m,n)$, and therefore

$$3^{-2m}\mathsf{ma}_m(P) \le \begin{cases} \overline{G}_{16/9,7/3}(m,n), & m \le \frac{2n}{5}, \\ \overline{G}_{4/3,3}(m,n)\,\frac{\overline{G}_{16,7/3}(\frac{2n}{5},n)}{\overline{G}_{12,3}(\frac{2n}{5},n)}, & \text{otherwise.} \end{cases}$$

Since $\frac{16}{9} \geq \frac{7}{3} - 1$ (see (3.16)) the peak will not occur in the "small m"-range of $\overline{G}_{16/9,7/3}$. In its "big m"-range, the maximum occurs at m roughly $\frac{14n}{37}$ (see (3.14)) which lies in the interval $[\frac{4}{11}, \frac{2}{5}]$. Also, $G_{4/3,3}$ peaks for $m \leq \frac{2n}{5}$ since $\frac{4}{3} \leq 3-1$ (consult (3.16)). Therefore, the bound peaks at m roughly $\frac{14n}{37}$ with the value

$$3^n\, \overline{G}_{16/9,7/3}(\lfloor \tfrac{14n}{37} \rfloor, n) \approx_n (2^{4/7}\, 3^{-1/2}\, 11^{11/14}\, 37^{3/14})^n.$$



Spines, isolated (○), top (⊕), and bottom (⊖) points.

Figure 7: Encoding a crossing-free partition.

THEOREM 5.2. *Let P be a set of n points in the plane. Then the number of crossing-free partitions satisfies* $\mathsf{cfp}(P) \leq \left(2^{4/7}\, 3^{-1/2}\, 11^{11/14}\, 37^{3/14}\right)^n \operatorname{poly}(n) = O(12.2388^n)$.

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References

[1] O. Aichholzer, Th. Hackl, C. Huemer, F. Hurtado, H. Krasser, B. Vogtenhuber, On the number of plane graphs, *Proc. 17th Annual ACM-SIAM Symp. on Discrete Algorithms* (2006), to appear.

[2] O. Aichholzer, F. Hurtado, M. Noy, On the number of triangulations every planar point set must have, *Proc. 13th Canad. Conf. Comput. Geom.* (2001), 13–16.

[3] O. Aichholzer, H. Krasser, The point-set order-type database: A collection of applications and results, *Proc. 13th Canad. Conf. Comput. Geom.* (2001), 17–20.

[4] M. Ajtai, V. Chvátal, M. M. Newborn, E. Szemerédi, Crossing-free subgraphs, *Annals Discrete Math.* **12** (1982), 9–12.

[5] S. G. Akl, A lower bound on the maximum number of crossing-free Hamiltonian cycles in a rectilinear drawing of K_n, *Ars Combinatorica* **7** (1979), 7–18.

[6] H. Alt, U. Fuchs, and K. Kriegel, On the number of simple cycles in planar graphs, *Combinat. Probab. Comput.* **8**:5 (1999), 397–405.

[7] H.W. Becker, Planar rhyme schemes, *Math. Mag.* **22** (1948-49), 23–26.

[8] P. Brass, W. Moser, J. Pach, *Research Problems in Discrete Geometry*, Springer, New York, 2005.

[9] V. Capoyleas, G. Rote, G. Woeginger, Geometric clustering, *Journal of Algorithms* **12** (1991), 341–356.

[10] V. G. Deineko, M. Hoffmann, Y. Okamoto, G. J. Woeginger, The Traveling Salesman Problem with few inner points, Proc. 10th International Computing and Combinatorics Conference, *Lecture Notes in Computer Science* **3106** (2004), 268–277.

[11] E. Demaine, Simple polygonizations, `http://theory.lcs.mit.edu/~edemaine/polygonization/` (version January 9, 2005).

[12] L. Deneen, G. Shute, Polygonizations of point sets in the plane, *Discrete Comput. Geom.* **3** (1988), 77–87.

[13] M. O. Denny, C. A. Sohler, Encoding a triangulation as a permutation of its point set, *Proc. 9th Canad. Conf. Comput. Geom.* (1997), 39–43.

[14] A. Dumitrescu, On two lower bound constructions, *Proc. 11th Canad. Conf. Comput. Geom.* (1999).

[15] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, EATCS Monographs on Theoretical Computer Science **10**, Springer-Verlag, 1987.

[16] A. Errera, *Mém. Acad. Roy. Belgique Coll.* 8^{o} (2) **11** (1931), 26 pp.

[17] A. García, M. Noy, J. Tejel, Lower bounds on the number of crossing-free subgraphs of K_N, *Comput. Geom. Theory Appl.* **16** (2000), 211–221.

[18] A. García, J. Tejel, A lower bound for the number of polygonizations of N points in the plane, *Ars Combinatorica* **49** (1998), 3–19.

[19] M. Grantson, C. Borgelt, C. Levcopoulos, Minimum weight triangulation by cutting out triangles, *Proc. 16th Ann. Int. Symp. on Algorithms and Computation* (2005), to appear.

[20] R. B. Hayward, A lower bound for the optimal crossing-free Hamiltonian cycle problem, *Discrete Comput. Geom.* **2**:4 (1987), 327–343.

[21] F. Hurtado, M. Noy, Counting triangulations of almost-convex polygons, *Ars Comb.* **45** (1997), 169–179.

[22] F. J. MacWilliams, N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland Mathematical Library **16**, 1977.

[23] P. McCabe, R. Seidel, New lower bounds for the number of straight-edge triangulations of a planar point set, *Proc. 20th Europ. Workshop Comput. Geom.* (2004).

[24] T. S. Motzkin, Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products, *Bull. Amer. Math. Soc.* **54** (1948), 352–360.

[25] M. Newborn, W. O. J. Moser, Optimal crossing-free Hamiltonian circuit drawings of the K_n, *J. Combinat. Theory, Ser. B* **29** (1980), 13–26.

[26] J. Pach and G. Tóth, Graphs drawn with few crossings per edge, *Combinatorica* **17**:3 (1997), 427–439.

[27] F. Santos, R. Seidel, A better upper bound on the number of triangulations of a planar point set, *J. Combinat. Theory, Ser. A* **102**:1 (2003), 186–193.

[28] R. Seidel, On the number of triangulations of planar point sets, *Combinatorica* **18**:2 (1998), 297–299.

[29] W.S. Smith, *Studies in Computational Geometry Motivated by Mesh Generation*, Ph.D.Thesis, Princeton University, 1989.

[30] R.P. Stanley, *Enumerative Combinatorics*, vol. **2**, Cambridge University Press, 1999.