Results on k-Sets and j-Facets via Continuous Motion*

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Abstract

Let P be a set of n points in $\mathbb{R}^d$ in general position, i.e., no $i+1$ points on a common $(i-1)$-flat, $1 \leq i \leq d$. A *k-set of P* is a set S of k points in P that can be separated from $P \setminus S$ by a hyperplane. A *j-facet of P* is an oriented $(d-1)$-simplex spanned by d points in P which has exactly j points from P on the positive side of its affine hull.

If P is a planar point set and n is even, a *halving edge* is an undirected edge between two points, such that the connecting line has the same number of points on either side. The number of $(n/2)$-sets is twice the number of halving edges. Inspired by Dey's recent proof of a new bound on the number of k-sets we show that

$$C + \sum_{p \in P} \binom{(\deg p + 1)/2}{2} = \binom{n/2}{2}$$

where $\deg p$ is the number of halving edges incident to point p and C is the number of crossing pairs of halving edges. The identity allows us, among other things, to determine the maximum number of halving edges in a set of 12 points. An analogous identity holds for j-facets.

For P in $\mathbb{R}^3$ we show that for $j \leq n/4 - 2$ the number of $(\leq j)$-facets (i.e., i-facets with $0 \leq i \leq j$) is maximized for sets in convex position, where this number is known to be

$$(j+1)(j+2)n - 2(j+1)(j+2)(j+3)/3\,.$$

For $k \leq n/4 - 1$, $k^2 n - k(k-1)(2k+5)/3$ is the tight upper bound for the number of $(\leq k)$-sets (i.e., i-sets with $1 \leq i \leq k$).

Finally we discuss the relation between the vector of numbers of k-sets, $k = 1, \ldots, n-1$ and the vector of numbers of j-facets, $j = 0, \ldots, n-d$ for a given point set. In the plane the number of k-sets equals the number of $(k-1)$-facets. In $\mathbb{R}^3$ the k-set vector determines the j-facet vector (and vice versa) by a linear relation. There is no such relation in $\mathbb{R}^d$ for d exceeding 3.

These results can be obtained by arguments via continuous motion of one point set to another while observing certain quantities related to k-sets and j-facets. For the relation between k-sets and j-facets in $\mathbb{R}^3$, we give a more direct argument via so-called k-set polytopes.

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1 Introduction and basics

Let P be a set of n points in $\mathbb{R}^d$ in general position, i.e., no $i+1$ points on a common $(i-1)$-flat, $1 \leq i \leq d$.

k-Sets. A *k-set of P* is a set S of k points in P that can be strictly separated from $P \setminus S$ by a hyperplane. We denote by $e_k(P)$ the number of k-sets of P, and by $E_k(P)$ the number of $(\leq k)$-sets, i.e., $E_k(P) := \sum_{i=1}^{k} e_i(P)$. If P is understood, we write e_k and E_k short for $e_k(P)$ and $E_k(P)$, respectively.

In the plane, an upper bound of $O(n\sqrt{k})$ on e_k was given in [ELSS] (see also [Lo]), where also sets with $e_k = \Omega(n \log(k+1))$ were described (see [EW] for an alternative construction). After the improvement of the upper bound to $O(n\sqrt{k}/\log^*(k+1))$ in [PSS] in 1989, Dey recently provided a further substantial improvement of the bound to $O(n\sqrt[3]{k})$ (building upon considerations in [AAS]). In $\mathbb{R}^3$, the best upper bound is $O(nk^{5/3})$ from [AACS], for general k, and from [DE], for $k = \lfloor n/2 \rfloor$ (improving on bounds in [BFL, ACEGSW, Ep, AAS]). In $\mathbb{R}^d$ bounds of the form $O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - \varepsilon_d})$ for some small $\varepsilon_d > 0$ have been obtained recently in [AACS] (see [ZV, AAS] for previous steps).

The situation is much better understood for $(\leq k)$-sets, where [CS] provide an asymptotically tight bound of $O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$ (this bound is attained by points on the moment curve), and in the plane there is even a tight upper bound of kn for $k < n/2$ [AG, Pe].

For a more complete account of the history of the problem, for related notions and applications in computational geometry, see the survey [AW]. Remarkably, almost all algorithmic applications and also almost all proofs of bounds proceed actually via a different notion (and its dual) which is described next.

j-Facets. A *j-facet of P* is an oriented $(d-1)$-simplex spanned by d points in P which has exactly j points from P on the positive side of its affine hull.

We denote by $g_j(P)$ the number of j-facets of P, and by $G_j(P)$ the number of $(\leq j)$-facets, i.e., $G_j(P) := \sum_{i=0}^{j} g_i(P)$. If P is understood, we write g_j and G_j short for $g_j(P)$ and $G_j(P)$, respectively. The term "facet" is justified by the fact that the 0-facets of P are exactly the facets of the polytope conv$P$. There is a relationship between the maximum number of k-sets and the maximum number of $(k \pm d)$-facets of sets of n points, although this relation has never really been worked out carefully. In addition, the reader should be aware of ambiguities concerning the notion of j-facets in the literature.

Results. In Section 2, we give an identity for the planar case concerning the number of crossings between j-facets (which we call *j-edges* in the plane) and the sequence of numbers of j-edges incident to points $p \in P$. In this extended abstract we restrict the proof to halving edges—these are the undirected versions of $(n/2-1)$-facets under the assumption that n is even; see Theorem 1. The discovery of the relations was inspired by Dey's recent proof of the new $O(n\sqrt[3]{k})$ bound for planar k-sets. In fact, this bound follows also directly from the identities via a known lemma on the number of edges of graphs which can be embedded in the plane with few crossings ([ACNS, Lei]).

Section 3 provides exact upper bounds on the number of $(\leq j)$-facets and the number of $(\leq k)$-sets in $\mathbb{R}^3$, provided j and k is not too large (roughly $n/4$); see Theorem 2.

Finally, in Section 4 we show that the vectors $\vec{e} = (e_k)_{k=1}^{n-1}$ and $\vec{g} = (g_j)_{j=0}^{n-d}$ determine each other in $\mathbb{R}^3$ by a linear relation (for $d = 2$ this is simple); see Theorem 3. In fact, this allows us to infer the tight bound on $(\leq k)$-sets directly from the bound on $(\leq j)$-facets, so we have to provide a proof for the latter only.

Back to j-facets. A sequence $(p_1, \ldots p_d)$ of d linearly independent points in $\mathbb{R}^d$ partitions the space into two open halfspaces and a hyperplane (the affine hull of $\{p_1, \ldots p_d\}$): Points p_{d+1} for which the sign of the determinant of the matrix with rows $(p_i\ 1)$, $i = 1, \ldots d+1$, is positive, points for which the sign is negative, and points for which the sign is 0. We denote this sign by $\text{sgn}(p_1, \ldots p_d, p_{d+1})$. So a j-facet can be specified by a sequence $(p_1, \ldots p_d)$ of d points in P such that $\text{sgn}(p_1, \ldots p_d, p) > 0$ for exactly j points p in P. In fact, we will use the notation $[p_1, \ldots p_d]$ for j-facets, meaning all permutations of $(p_1, \ldots p_d)$ which can be obtained from that sequence via an even number of transpositions of adjacent elements (since this will not change the sign of the determinant with a $d+1$-th point p). An odd number of transpositions in this sequence will change a j-facet into an $(n-j-d)$-facet.

For our proofs we want to make explicit that the structure of j-facets of $P = \{p_1, \ldots p_n\}$ is completely determined by the $\binom{n}{d+1}$ signs $\text{sgn}(p_{i_1}, \ldots p_{i_{d+1}})$, $1 \leq i_1 < \ldots < i_{d+1} \leq n$. If we move the points in P, then no j-facet will change its index (i.e., the value j), unless one of these signs changes. If the sign changes for exactly one such tuple τ, then only j-facets composed of d of the points in τ change their index, either to $(j-1)$ or to $(j+1)$.

Moving around. In our proofs we consider the changes in $\vec{g}$ while the point set moves continuously. We will assume that during the motion the set stays in general position, except for a finite number of discrete instances, where the sign of exactly one $(d+1)$-tuple changes.

Of course, this paradigm is not new for the analysis of configurations in combinatorial geometry. Tverberg's original proof of his famous generalization of Radon's theorem is a prominent example [Tv]. Recently, continuous motions were used also in the context of k-sets by Gullikson and Hole [GH].

2 Planar identities

Let us briefly recapitulate the set-up for this section. We are given a set P of n points in the plane, n even, such that no three points lie on a common line. A *halving edge* is an undirected edge between two points, such that the connecting line has the same number of points on either side; sometimes, when we refer to such an edge we mean the straight line segment connecting its endpoints. Two halving edges *cross*, if their segments intersect in a single point in their relative interiors.

Theorem 1

$$C + \sum_{p \in P} \binom{(\deg p + 1)/2}{2} = \binom{n/2}{2} \tag{1}$$

where $\deg p$ *is the number of halving edges incident to* p *(this number is always odd), and* C *is the number of pairwise crossings of halving edges.*

Implications. Before we proceed to the proof, let us exhibit three implications of this identity, most prominently Tamal Dey's recent bound on the number of halving edges (which relates to k-sets by the fact that the number of $(n/2)$-sets is twice the number of halving edges).

Corollary 1 ([De]) *The number of halving edges of* P *is bounded by* $O(n^{4/3})$.

This bound is a direct consequence of the inequality $C < n^2/8$ which follows from Identity (1) and the fact that a graph with n vertices embedded in the plane with c edge crossings cannot have more than $O(\max(n, \sqrt[3]{cn^2}))$ edges ([ACNS, Lei], see also [PT, PA]).

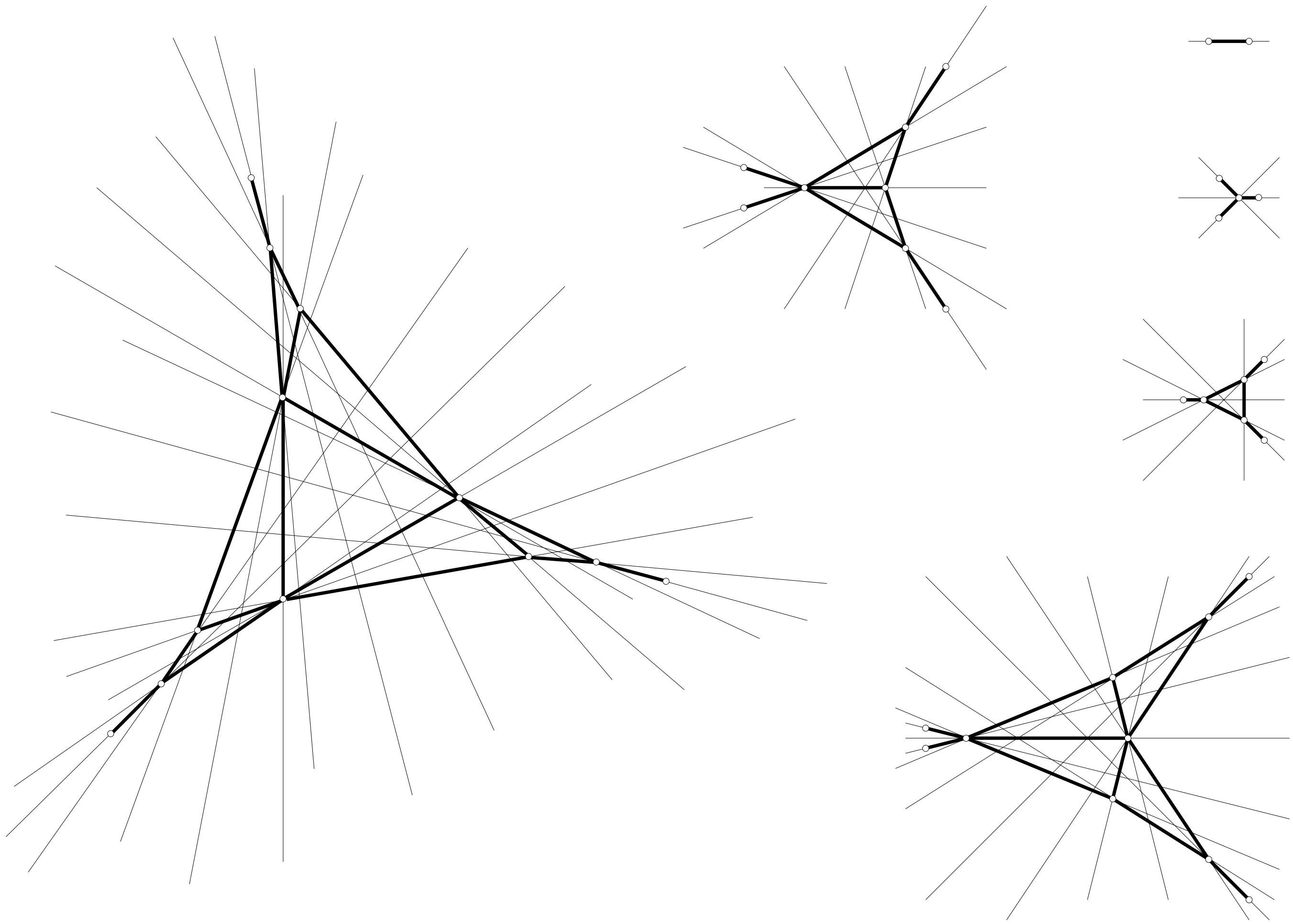

Figure 1: Point configurations which maximize the number of halving edges.

Corollary 2 *If h_n, n even, denotes the maximum possible number of halving edges of n points in the plane, then $h_2 = 1$, $h_4 = 3$, $h_6 = 6$, $h_8 = 9$, $h_{10} = 13$, and $h_{12} = 18$.*

The numbers for $n \leq 10$ were previously known (see e.g., [Fe], where h_{10} was determined by a computer-aided enumeration of all possible combinatorial configurations of 10 points in the plane). Figure 1 displays configurations which realize the quantities claimed in the corollary. All upper bounds can be readily derived from the inequality $\sum_{p \in P} \binom{(\deg p+1)/2}{2} \leq \binom{n/2}{2}$ implied by Identity (1), in conjunction with the facts—implied by Lemma 1 below—that every point is incident to an odd number of halving edges, and that there are at least three points incident to exactly one halving edge (extreme points must satisfy this condition). For example for 12, this implies that observing these constraints the sum $\sum_{p \in P} \deg p$ is maximized for the "degree sequence" $(1, 1, 1, 3, 3, 3, 3, 3, 3, 5, 5, 5)$.

Corollary 3 ([PS]) *P allows a perfect* cross-matching *(a partition into edges such that any pair of such edges crosses), iff P has exactly $n/2$ halving edges.*

First observe that in a perfect cross-matching, every edge is halving. Identity (1) tells us, that if there are already $\binom{n/2}{2}$ crossings among halving edges, then $\deg p = 1$ for all $p \in P$, and thus there cannot be more halving edges beyond those in the perfect cross-matching. On the other hand, if there are $n/2$ halving edges, then $\deg p = 1$ for all p (since $\deg p$ has to be at least 1 in any case), and thus there must be $\binom{n/2}{2}$ crossings. That is, the $n/2$ halving edges pairwise cross and form a perfect cross-matching.

Next we prepare two ingredients for the proof of Theorem 1.

Lovász' Lemma. Let ℓ be a line through point $p \in P$ which misses all other points in P. Then there is a unique side of ℓ which contains the majority of points from $P \setminus \{p\}$. Call this side the *large side* of ℓ, and the other open halfplane determined by ℓ the *small side* of ℓ.

Lemma 1 ([Lo]) *If a line ℓ contains a unique point p in P, and there are x halving edges incident to p emanating into the small side of ℓ, then there are $x + 1$ halving edges emanating into the large side of ℓ.*

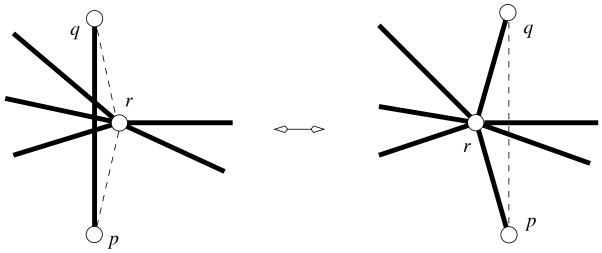

Figure 2: Local changes during mutation.

The lemma can be proven by rotating a line λ about point p starting in position ℓ until it coincides with ℓ again. The halving edges incident to p are encountered in alternation on the large and small side of ℓ, starting and ending on the large side.

In fact, the lemma completely characterizes the graph of halving edges of a point set. Simple implications of the lemma which we have mentioned before are that the number of halving edges incident to a point in P is always odd, and that there is exactly one halving edge incident to each extreme point of P.

Mutations while moving. Recall from the introduction, that if we start moving the points in P, the graph of halving edges will not change unless a triple (p, q, r) of points becomes collinear and changes its orientation. Even then, the graph of halving edges will not change except for edges on $\{p, q, r\}$. Following the terminology of oriented matroids (cf. [BLSWZ]), we call such a change of orientation a *mutation*.

Let us investigate such a mutation on three points $\{p, q, r\}$. We assume that this is the only mutation that occurred (i.e., there is no other simultaneous mutation), and that the points stayed disjoint when they passed though collinearity. First we consider only the case when $\{p, q\}$ is a halving edge before mutation, and that r lies on the segment connecting p and q at the moment of collinearity (see Figure 2). Hence, $\{q, r\}$ and $\{r, p\}$ are not halving edges before mutation. After mutation, $\{p, q\}$ is not halving, but $\{q, r\}$ and $\{r, p\}$ are. That is, the number of halving edges increased by one, and no degree in the graph of halving edges changed except for point r, whose degree increased by 2.

What happened to crossings of halving edges? If we ignore edges incident to r, then nothing changes. Crossings with the edge $\{p, q\}$ are replaced by crossings with $\{q, r\}$ or $\{r, p\}$ after mutation. As for the edges incident to r, let ℓ be a line through r parallel to the segment connecting p and q. The halfplane of ℓ containing $\{p, q\}$ is the large side of ℓ, before and after mutation. If x is the number of halving edges incident to r emanating into the large side of ℓ before mutation, then these edges were responsible for x crossings with $\{p, q\}$. These crossings disappear after mutation. No new crossings appear.

Let $\deg r$ and $\deg' r$ denote the number of halving edges incident to r before and after mutation, respectively. Note that $\deg' r = \deg r + 2$ and $x = (\deg r + 1)/2$. Let C and C' denote the number of crossings of halving edges before and after mutation, respectively. We have $C' = C - x$, and so

$$C + \binom{(\deg r + 1)/2}{2} = C + \binom{x}{2} =$$

$$C - x + \binom{x+1}{2} = C' + \binom{(\deg' r + 1)/2}{2}$$

which proves that the validity of Identity (1) is not affected by the mutation, since no degree other than $\deg r$ changes during mutation.

Now recall that we assumed that $\{p, q\}$ was a halving edge before mutation. However, the mutation described, and its inverse, are the only types of mutations affecting the graph of halving edges and its crossings.

Proof of Theorem 1. First observe that for all even n there is a set P of n points which satisfies Identity (1). The vertices of a regular n-gon, or the vertices of a regular $(n-1)$-gon together with its center are easy examples. Now it remains to use the fact that any two sets of n points in general position can be continuously transformed into each other in such a way that the points remain pairwise distinct, they never have more than one triple of points collinear, and such a collinearity occurs only finitely often. ◻

Other identities. A simple algebraic manipulation allows us to rewrite Identity (1) as

$$8C + \sum_{p \in P} (\deg p)^2 = n(n-1) \, .$$

Let $\deg_j p$ denote the number of j-edges[1] emanating from point p (which equals the number of incoming j-edges). Let C_j denote the number of crossings of j-edges, and for $i \neq j$, let $C_{i,j}$ denote the number of crossings between i-edges and j-edges. Then (reading G_{-1} as 0) we have

$$C_j + \sum_{p \in P} \binom{\deg_j p}{2} = G_{j-1} \, ,$$

for $0 \leq j \leq n/2 - 1$, and

$$C_{i,j} + \sum_{p \in P} \deg_i p (\deg_j p - 1) = 2G_{i-1} \, ,$$

for $0 \leq i < j \leq n/2 - 1$. (The latter identity allows improvements of previous bounds in [We] on $\sum_{k \in K} e_k$, for $K \subset \{1, 2, \ldots n\}$.) Proofs follow from an analysis of mutations as in the proof given here, and will be given in the full version of the paper.

[1]Recall, that we use "j-edges" for j-facets in the plane.

3 $(\leq j)$-facets and $(\leq k)$-sets in $\mathbb{R}^3$

Theorem 2 *Let P be a set of n points in general position in $\mathbb{R}^3$. Then*

$$G_j \leq (j+1)(j+2)n - 2(j+1)(j+2)(j+3)/3$$

for $j \leq n/4 - 2$, and

$$E_k \leq k^2 n - k(k-1)(2k+5)/3$$

for $k \leq n/4 - 1$. Both bounds are attained for sets in convex position.

Recall that in the plane a tight bound of $E_k \leq kn$ and $G_j \leq (j+1)n$ is known for $k < n/2$ and $j < n/2 - 1$ ([AG, Pe]). In $\mathbb{R}^3$, the number of j-facets is $2(j+1)(n-j-2)$ for every set of n points in convex position ([Lee, CS, Sh]), and by Theorem 3 below this implies that the number of k-sets is $n + 2(n-k-1)(k-1)$ (see also [GH]). The theorem above quotes the resulting numbers of $(\leq j)$-facets and $(\leq k)$-sets for point sets in convex position. In our proof we will show that we can always move a point set in $\mathbb{R}^3$ into convex position while the numbers G_j, $j \leq n/4 - 2$, never decrease. This gives the result claimed for $(\leq j)$-sets. Theorem 3 below yields $E_k = (G_{k-1} + G_{k-2})/2 + 2k$, so the bound for $(\leq k)$-sets can be easily obtained, too.

So we consider a set P of $n \geq 3$ points in $\mathbb{R}^3$, and analyze the effect of mutations of P on the vectors $\vec{g}$ and $\vec{G}$.

A mutation is the situation that four points become coplanar and change their sign as discussed in the introduction. More formally, a *mutation* is a triple (P^-, P^0, P^+), where $P^- = (p_1^-, \ldots, p_n^-)$, $P^0 = (p_1^0, \ldots, p_n^0)$, and $P^+ = (p_1^+, \ldots, p_n^+)$ are ordered point sets, P^- and P^+ in general position, such that: (i) $\mathrm{sgn}(p_1^-, \ldots, p_4^-) < 0$, $\mathrm{sgn}(p_1^0, \ldots, p_4^0) = 0$, and $\mathrm{sgn}(p_1^+, \ldots, p_4^+) > 0$. (ii) For $0 \leq a < b < c < d \leq n$, $(a,b,c,d) \neq (1,2,3,4)$, we have

$$\mathrm{sgn}(p_a^-, p_b^-, p_c^-, p_d^-) = \mathrm{sgn}(p_a^0, p_b^0, p_c^0, p_d^0)$$
$$= \mathrm{sgn}(p_a^+, p_b^+, p_c^+, p_d^+)\,.$$

(iii) There is a continuous motion from P^- to P^0 and from P^0 to P^+ with all intermediate stages in general position.

If $[p_1^-, p_2^-, p_3^-]$ is a j-facet in P^-, then we call j the *index* of the mutation.

Convex mutation. We call a mutation a *convex mutation*, if the sequence $(p_1^0, \ldots, p_4^0)$ forms a convex quadrilateral in its plane (referring to the notation as set above); see Figure 3. This scenario is characterized by the fact that for all points $p \in \mathbb{R}^3$, p is on same side of all oriented facets $[p_1^0, p_2^0, p_3^0]$, $[p_1^0, p_2^0, p_4^0]$, $[p_1^0, p_3^0, p_4^0]$, and $[p_2^0, p_3^0, p_4^0]$. In fact, because of (ii) and (iii) above, this is true for all points in $P^+ \setminus \{p_1^+, \ldots, p_4^+\}$, even if we replace the "0" by "+" in the superscripts.

It follows that $[p_1^-, p_2^-, p_3^-]$ is a j-facet that turns into a $(j+1)$-facet $[p_1^+, p_2^+, p_3^+]$, since it "gained" p_4^+ on its positive side, $[p_1, p_3, p_4]$ turns from a j- to a $(j+1)$-facet, and $[p_1, p_2, p_4]$ and $[p_2, p_3, p_4]$ turn from $(j+1)$- to j-facet. Hence, vector $\vec{g}$ (and thus $\vec{G}$) does not change during a convex mutation.

Mutation through triangle. We call a mutation a *mutation through triangle*, if p_4^0 is in the convex hull of $\{p_1^0, p_2^0, p_3^0\}$; see Figure 4. This is characterized by the fact that for all points $p \in \mathbb{R}^3$, p is on the same side of all oriented facets $[p_1^0, p_2^0, p_3^0]$, $[p_1^0, p_2^0, p_4^0]$, $[p_1^0, p_4^0, p_3^0]$, and $[p_2^0, p_3^0, p_4^0]$.

Hence, if $[p_1^-, p_2^-, p_3^-]$ is a j-facet, then $[p_1, p_2, p_3]$ turns from a j-facet to a $(j+1)$-facet, while $[p_1, p_2, p_4]$, $[p_1, p_4, p_3]$, and $[p_2, p_3, p_4]$ turn from $(j+1)$-facets to j-facets. That is, the number of j-facets increases by two, while the number of $(j+1)$-facets decreases by the same amount. There is also a mirrored change in the number of $(n-j-3)-$ and $(n-j-4)$-facets, which may lead to interferences if j is close to $n/2$. This is taken care of in the following lemma.

Lemma 2 *For a mutation through triangle with index j, with the four numbers $j, j+1, n-j-3$ and $n-j-4$ distinct, we have*

$$\begin{aligned} g_{n-j-3}(P^+) &= g_j(P^+) \\ &= g_j(P^-) + 2 = g_{n-j-3}(P^-) + 2\,, \quad \textit{and} \\ g_{n-j-4}(P^+) &= g_{j+1}(P^+) \\ &= g_{j+1}(P^-) - 2 = g_{n-j-4}(P^-) - 2\,. \end{aligned}$$

If $j + 1 = n - j - 4$, then

$$\begin{aligned} g_{n-j-3}(P^+) &= g_j(P^+) \\ &= g_j(P^-) + 2 = g_{n-j-3}(P^-) + 2\,, \quad \textit{and} \\ g_{j+1}(P^+) &= g_{j+1}(P^-) - 4\,. \end{aligned}$$

If $j = n - j - 3$, then

$$\begin{aligned} g_j(P^+) &= g_j(P^-) + 4\,, \quad \textit{and} \\ g_{n-j-4}(P^+) &= g_{j+1}(P^+) \\ &= g_{j+1}(P^-) - 2 = g_{n-j-4}(P^-) - 2\,. \end{aligned}$$

If $j \neq n - j - 4$, then

$$\begin{aligned} G_j(P^+) &= G_j(P^-) + 2\,, \quad \textit{and} \\ G_{n-j-4}(P^+) &= G_{n-j-4}(P^-) - 2\,. \end{aligned}$$

No other changes than those indicated above occur in $\vec{g}$ and $\vec{G}$. In particular, no changes occur if $j = n - j - 4$. ◻

REMARK. The lemma can be readily generalized to a description of changes of the vectors $\vec{e}$ and $\vec{E}$, and to dimensions $d > 3$. Note that there is one type of mutations in the plane and there are two types ("convex" and "through triangle") in $\mathbb{R}^3$. In $\mathbb{R}^d$ such mutations can be classified by a parameter m, $1 \leq m \leq \lceil d/2 \rceil$. Details follow in the full version.

We extract the implication of Lemma 2 relevant for our purposes.

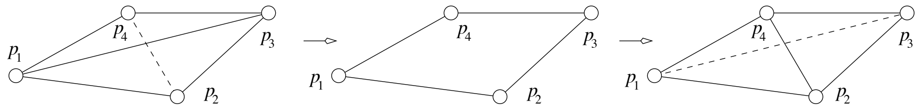

Figure 3: Convex mutation (point of view on the positive side of $[p_1^-, p_2^-, p_3^-]$).

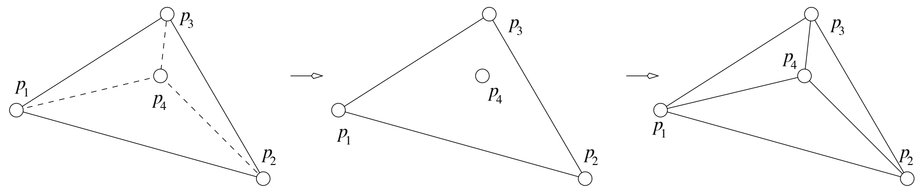

Figure 4: Through-triangle-mutation (seen from the positive side of $[p_1^-, p_2^-, p_3^-]$).

Corollary 4 *In a mutation through triangle with index j no entry in the vector $(G_0, G_1, \ldots, G_{n-j-5})$ decreases.* □

Proof of Theorem 2. We show now that we can continuously transform every point set into a set in convex position such that G_J decreases for no $J \leq n/4-2$. We know that it suffices to ensure that the index j of each mutation through triangle satisfies $n-j-5 > n/4-2$, or equivalently, $j+3 < 3n/4$. For that purpose let us recall that every point set P has a *centerpoint* c (not necessarily in P) with the property that any open halfspace that misses c contains at most $3n/4$ points from P (a consequence of Helly's Theorem, cf. [Ed]). First we observe that if c is a point in P, then for $j \leq n/4-2$, neither c is on the positive side of a j-facet nor c participates in a j-facet (in both cases we can find open halfspaces disjoint from c which contain more than $3n/4$ points). That is, for $J \leq n/4-2$, $G_J(P \setminus \{c\}) = G_J(P)$ and we can apply induction to prove the theorem (starting with a set of 4 points).

Hence, we restrict ourselves to the case that c is not in P, and without loss of generality we assume that the origin o is a centerpoint of P. For a real number $\lambda > 0$, we define $P(\lambda) := \{\mu p | p \in P, \mu = \min(1, \lambda/||p||)\}$. Note that for $\lambda_0 = \max_{p \in P} ||p||$, $P(\lambda_0) = P$, and for $\lambda_1 = \min_{p \in P} ||p||$, $P(\lambda_1)$ lies on a sphere of radius λ_1, and thus it is in convex position. The motion of $P(\lambda)$ for λ from λ_0 to λ_1 can be visualized by a sphere initially containing P, and then contracting the sphere while dragging points towards to the origin as soon as they appear on the sphere. Throughout the whole process, the origin stays a centerpoint of the moving point set.

We can always perturb the set P (without changing j-facets) in such a way that no two points lie on a common line with c, and during the whole motion the set is either in general position or there is a unique 4-tuple of points which is coplanar (and in general position in its plane). As we have learned, the mutations where the coplanar points are in convex position do not affect $\vec{G}$. Now consider the case when a point p moves through a triangle determined by three points q, r, s. Point p is still in the interior of the contracting sphere, otherwise it could not be be in the convex hull of three other points. In fact, it is easily seen that p is on the same side of the triangle spanned by $\{q, r, s\}$ as the origin before mutation, and on the opposite side after mutation. Now we have to recall the definition of a mutation through triangle and its index j. This index was determined by the number of points on the side of the triangle opposite to the point which is about to move through the triangle. Hence, the index of our mutation is the number j of points on the side of qrs opposite to p before mutation, i.e., opposite to the origin. Consequently, when p becomes coplanar with q, r, s, there is an open halfspace that misses the origin which contains these j points and p, q, r, s. Because of the centerpoint property of the origin, $j+4 \leq 3n/4$, or $j+3 < 3n/4$ as it was necessary to prove for the monotonicity of G_J, $J \leq n/4-2$. This concludes the proof of the theorem. □

4 Relation between k-sets and j-facets

Theorem 3 *For a set P of n points in $\mathbb{R}^3$ we have*

$$\begin{aligned} e_1 &= g_0/2+2, \quad e_{n-1} = g_{n-3}/2+2, \quad \textit{and} \\ e_k &= (g_{k-1}+g_{k-2})/2+2 \;\; \textit{for} \;\; 2 \leq k \leq n-2\,. \end{aligned}$$

Of course, we can also deduce how $\vec{g}$ determines $\vec{e}$. The corresponding relation of $e_k = g_{k-1}$ in the plane can be easily proved. In $\mathbb{R}^d$, $d > 3$, $\vec{g}$ does not determine $\vec{e}$ (see full version). For the proof of the theorem, we could simply observe the changes in $\vec{g}$ and $\vec{e}$ under continuous motion (see remark after Lemma 2). Such a proof provides some insight why such a relation does not generalize to $\mathbb{R}^d$, $d > 3$. We give here an alternative proof.

k-Set polytope. We employ the notion of a k-set polytope from [EVW]. Given a set P of n points and $1 \leq k \leq n-1$, the *k-set polytope of P* is

$$\mathcal{Q}_k = \operatorname{conv}\left\{\sum_{p \in T} p \mid T \in \binom{P}{k}\right\},$$

where $\binom{P}{k}$ denotes the set of all subsets of P of cardinality k. The vertices of $\mathcal{Q}_k$ are in one-to-one correspondence to the k-sets of P. We briefly recapitulate the argument. A k-set S can be separated from $P \setminus S$ by a hyperplane. That is, there is a vector c and a real number b such that $\langle c, p \rangle > b$ for $p \in S$ and $\langle c, p \rangle < b$ for $p \in P \setminus S$. Clearly, this implies that $\langle c, \sum_{p \in S} p \rangle$ exceeds $\langle c, \sum_{p \in T} p \rangle$ for all T with $S \neq T \in \binom{P}{k}$ and so $\sum_{p \in S} p$ is a vertex of $\mathcal{Q}_k$.

On the other hand, if $\sum_{p \in S} p$ is a vertex of $\mathcal{Q}_k$, then there is a vector c such that $\langle c, \sum_{p \in S} p \rangle$ exceeds $\langle c, \sum_{p \in T} p \rangle$ for all T with $S \neq T \in \binom{P}{k}$. Set $b' = \min_{p \in S} \langle c, p \rangle$. If $\langle c, q \rangle \geq b'$ for $q \in P \setminus S$, then $\langle c, \sum_{p \in S} p \rangle < \langle c, \sum_{p \in S \setminus \{p'\} \cup \{q\}} p \rangle$, for p' a point in S with $\langle c, p' \rangle = b'$; a contradiction. Hence $\langle c, q \rangle < b'$ for all $q \in P \setminus S$ and S can be separated from $P \setminus S$ by a hyperplane with normal vector c.

Proof of Theorem 3. So far we have not referred to the dimension of P. Now let us assume that P is a set in general position in $\mathbb{R}^3$. The 1-sets of P are the vertices of conv$P$ and the 0-facets of P are the facets of conv$P$. Hence, $g_0 = 2e_1 - 4$ follows directly from Euler's relation, and $e_1 = g_0/2 + 2$ and $e_{n-1} = g_{n-3}/2 + 2$ hold.

For $2 \leq k \leq n-2$ we want to show that the set of $(k-1)$- and $(k-2)$-facets of P are in one-to-one correspondence to the facets of $\mathcal{Q}_k$ which will entail the remaining relations.

Let $[q_1, q_2, q_3]$ be a $(k-1)$-facet of P with U the set of $k-1$ points on its positive side. Let c be a vector and b be a real number such that $\langle c, q_1 \rangle = \langle c, q_2 \rangle = \langle c, q_3 \rangle = b$, $\langle c, p \rangle > b$ for $p \in U$, and $\langle c, p \rangle < b$ for $p \in P \setminus (U \cup \{q_1, q_2, q_3\})$. It follows that $S_i = U \cup \{q_i\}, i = 1, 2, 3$ are the only sets T in $\binom{P}{k}$ which maximize $\langle c, \sum_{p \in T} p \rangle$. Consequently, the points $\sum_{p \in S_i} p$ are vertices of a triangular facet of $\mathcal{Q}_k$. Similarly, if $[q_1, q_2, q_3]$ is a $(k-2)$-facet of P with U the set of $k-2$ points on its positive side, then the sets $S_i = (U \cup \{q_1, q_2, q_3\}) \setminus \{q_i\}$ give rise to a facet of $\mathcal{Q}_k$. A reverse argument shows that this mapping from $(k-1)$- and $(k-2)$-facets of P to facets of $\mathcal{Q}_k$ is already a bijection. In particular, all facets of $\mathcal{Q}_k$ are triangular, and $g_{k-1} + g_{k-2} = 2e_k - 4$ is implied by Euler's relation. $\square$

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