On the Number of Lattice Triangulations*

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For n a positive integer, we consider triangulations of the $n \times n$ lattice set $\{0, 1, 2, \ldots, n\}^2$, i.e. crossing-free straight line embedded geometric graphs on this point set—thus with $(n+1)^2$ vertices, $3n^2 + 2n$ edges and $2n^2$ triangular faces.

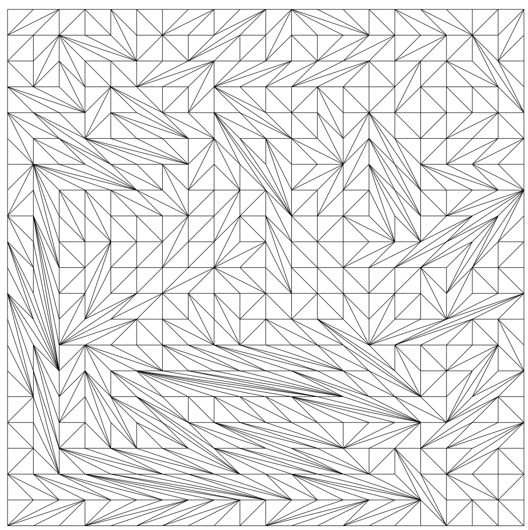

Figure 1: A triangulation of the 20×20 lattice.

Extending a previous argument by Emile Anclin [1], we show that the number of triangulations of the $n \times n$ lattice is at most $O(6.86^{n^2})$, improving on the previous bounds of $O(64^{n^2})$ and $O(8^{n^2})$ in [4] and [1], respectively. It compares to a lower bound of $\Omega(4.15^n)$ given in [2].

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The flip-graph has the triangulations as vertices, and it has two triangulations adjacent if one can be obtained from the other by replacing one single edge (an edge whose incident triangles form a convex quadrilateral, called a flippable edge). We demonstrate that the flip-graph of the triangulations of the $n \times n$ lattice is an induced subgraph of the $(3n^2 - 2n)$-dimensional hypercube (no such embedding in a hypercube of smaller dimension is possible). We also show that the diameter of the flip-graph is $\Theta(n^3)$, and in a random triangulation (uniformly from all triangulations), the expected number of flippable edges is $\Theta(n^2)$ (while there exist triangulations with as few as $O(n)$ flippable edges).

The main proofs are based on particular binary encodings of lattice triangulations which readily yield the respective results.

References

[1] E. E. Anclin, An upper bound for the number of planar lattice triangulations, *J. Combinat. Theory, Ser. A* **103** (2003), 383–386.

[2] V. Kaibel, G. Ziegler, Counting lattice triangulations, *British Combinatorial Surveys* (C.D. Wensley, ed.), Cambridge University Press, 2003.

[3] J. Matoušek, P. Valtr, E. Welzl, On two encodings of lattice triangulations, manuscript (2006).

[4] S. Yu. Orevkov, Asymptotic number of triangulations in $\mathbf{Z}^2$, *J. Combinat. Theory, Ser. A* **86** (1999), 200–203.