Random Triangulations of Planar Point Sets

Micha Sharir*
School of Computer Science
Tel Aviv University, Tel Aviv 69978, Israel, and
Courant Institute of Mathematical Sciences
New York University, New York, NY 10012, USA
michas@tau.ac.il

Emo Welzl
Institut für Theoretische Informatik
ETH Zürich
CH-8092 Zürich, Switzerland
emo@inf.ethz.ch

ABSTRACT

Let S be a finite set of $n+3$ points in general position in the plane, with 3 extreme points and n interior points. We consider triangulations drawn uniformly at random from the set of all triangulations of S, and investigate the expected number, $\hat{v}_i$, of interior points of degree i in such a triangulation. We provide bounds that are linear in n on these numbers. In particular, $n/43 \le \hat{v}_3 \le (2n+3)/5$.

Moreover, we relate these results to the question about the maximum and minimum possible number of triangulations in such a set S, and show that the number of triangulations of any set of n points in the plane is at most 43^n, thereby improving on a previous bound by Santos and Seidel.

Categories and Subject Descriptors: G.2 [Discrete Mathematics]: Combinatorics—*Counting problems*

General Terms: Theory

Keywords: Random triangulations, counting, degree sequences, number of triangulations, crossing-free geometric graphs, crossing-free spanning trees, charging

1. INTRODUCTION

Given a finite point set S in the plane, a triangulation is a maximal crossing-free geometric graph on S (in a geometric graph the edges are realized by straight line segments). Here we consider random triangulations, where "random" refers to uniformly at random from the set of all triangulations of S (where S is any fixed set). We are primarily interested in the degree sequences of such random triangulations.

To be precise, we assume that S is a set of $n+3$ points in general position in the plane so that the convex hull of S is a triangle. For such a set and $i \in \mathbb{N}$, we let $\hat{v}_i = \hat{v}_i(S)$ denote the expected number of interior points of degree i in a random triangulation of S. While—for n large enough—the number of vertices of degree 3 in a triangulation may be any integer between 0 and roughly $\frac{2n}{3}$, we show that

$$\frac{n}{43} \le \hat{v}_3 \le \frac{2n+3}{5} .$$

Note that general position is essential for the lower bound. Consider the case where the n interior points lie on a common line containing one of the extreme points in S, see Fig. 1. Then there is a unique triangulation and this triangulation has one interior point of degree 3; hence, $\hat{v}_3 = 1$.

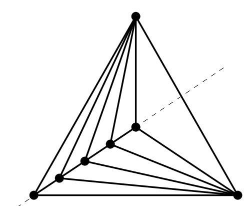

Figure 1: Unique triangulation.

We relate these results to the question about the maximum and minimum possible number of triangulations in a set of n points in the plane. We show that the number of triangulations of any such set is at most 43^n, thereby improving on a previous bound of 59^n by Santos and Seidel [17]. We can also use the upper bound on $\hat{v}_3$ to infer a lower bound of roughly 2.5^n on the number of triangulations every set of $n+3$ points in general position with triangular convex hull has. However, this is inferior to the recent $0.093 \cdot 2.63^n$-bound by McCabe and Seidel [10].

Our results use charging schemes among vertices in triangulations that heavily build on the structure imposed by edge flips on the set of all triangulations (see also the discussion of (dis-)charging below). Our approach should be regarded as a continuation of the proof by Santos and Seidel [17] for the 59^n upper bound for the number of triangulations. This connection may not be obvious in our presentation, since we deal with a different scenario, but it should become more apparent when we get as an intermediate result a lower bound of $n/59$ for $\hat{v}_3$. The two 59's are the "same"! Still, we believe that it was the new setting that allowed us to proceed further and derive a better bound for the number of triangulations.

Little seems to be known about random triangulations of (fixed) point sets, although the generation of random triangulations has raised some interest (see, e.g., [1, Section 4.3]). Moreover, it is a folklore open problem to determine the mixing rate of the Markov process that starts at some triangulation and keeps flipping a random flippable edge; see [12, 13] where this is treated for points in convex position. We are currently investigating whether our methods have anything to say about this problem. Finally, for ab-

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stract graphs (without enforced straight line embedding on a given point set), there are results about random planar graphs (here one has to discriminate between the labeled and the unlabeled case), see, e.g., [7, 9, 11].

Number of Triangulations—History. David Avis was perhaps one of the first to ask whether the maximum number of triangulations of n points in the plane is bounded by c^n for some $c > 0$, see [3, page 9]. This fact was established in 1982 by Ajtai, Chvátal, Newborn, and Szemerédi [3], who show that there are at most 10^{13n} crossing-free graphs on n points—in particular, this bound holds for triangulations.

Further developments have yielded progressively better upper bounds for the number of triangulations[1] [20, 5, 18], so far culminating in the previously mentioned 59^n bound [17] in 2003. This compares to $\Omega(8.48^n)$, the largest known number of triangulations for a set of n points, recently derived by Aichholzer et al. [2]; this improves an earlier lower bound of about 8^n (up to a polynomial factor) given by García et al. [6].

For n points in convex position, the number of triangulations is known to be C_{n-2}, where $C_m := \frac{1}{m+1}\binom{2m}{m} = \Theta(m^{-3/2}4^m)$, $m \in \mathbb{N}_0$, is the mth *Catalan number* (this is known as the Euler-Segner problem, cf. [21, page 212] for a discussion).

Other Crossing-free Graphs. Besides the intrinsic interest in obtaining bounds on the number of triangulations, they are useful for bounding the number of other kinds of crossing-free geometric graphs on a given point set, exploiting the fact that any such graph is a subgraph of some triangulation. For example, the best known upper bound on the number of crossing-free straight-edge spanning trees on a set of n points in the plane is $O((5.\dot{3}\,\tau)^n)$, if τ^n is a bound on the number of triangulations; with $\tau = 43$ this is now $O(229.\dot{3}^n)$. This follows from a result by Ribó and Rote, [14, 16], who show that any planar graph on n vertices contains at most $5.\dot{3}^n$ spanning trees. Similar results have been observed for crossing-free spanning cycles, where a bound of $O((\sqrt{6}\tau)^n) = O((2.45\,\tau)^n)$ can be obtained, as communicated by Raimund Seidel; the resulting bound of $O(105.33^n)$ still falls short of the bound of $O(86.81^n)$ for cycles given in [19], though. The total number of crossing-free planar graphs on n points is at most $2^{3n-6}\tau^n < (8\,\tau)^n$. So this is now improved to 344^n (from 472^n).

Next we mention a result and a notion, both seemingly related to what we are doing; hence, they were popping up repeatedly when presenting our result. While we want to take the opportunity to clarify in this way, a fruitful closer connection may be established in the end.

Tutte's Number of Rooted Triangulations. Let us briefly discuss a classical result from 1962 by Tutte in his *census*-series [22]. He considers so-called *rooted triangulations*, i.e., maximal planar graphs, with a fixed face with vertices a, b, and c and n additional vertices. Two such triangulations are considered to be equal if there is an isomorphism between them, which maps each of the points a, b, and c to itself, though. The number of such triangulations is easily seen to be 1 for $n = 1$ and 3 for $n = 2$. Based on an ingenious analysis employing generating functions, Tutte shows that for $n \geq 2$ the number of such triangulations is

$$\frac{2}{n(n+1)}\binom{4n+1}{n-1} = \Theta\left(\frac{1}{n^{5/2}}\;9.\overline{481}^n\right) .$$

How does this relate to the number of triangulations of given $n+3$ points? On the one hand, Tutte counts more, because there are fewer constraints: Interior points can be moved arbitrarily. On the other hand, distinct triangulations in the geometric setting may be equal in Tutte's; see Fig. 2. Thus the results are incomparable, although we cannot rule out that a connection may be established.

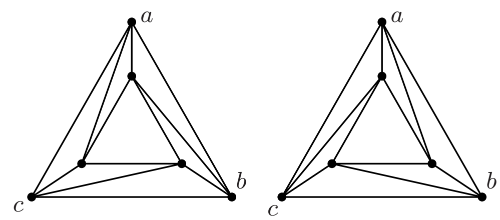


Figure 2: Distinct triangulations of a point set that are equal in Tutte's setting.

(Dis-)Charging. The notion of "charging" rings a bell in the context of planar graphs. The proof of the Four-Color-Theorem employs Heesch's idea of discharging (*Entladung*, [8]) in order to prove that certain configurations are unavoidable in a maximal planar graph, cf. [4] or a later proof in [15]. There one initially puts charge $6 - i$ on each vertex of degree i in a maximal planar graph—thus the overall charge is 12 (by Euler's formula). Now vertices of positive charge push their charge to other vertices (they discharge) without changing the overall charge. Given that a certain set of configurations L does not occur, one proves that all vertices can discharge with a nonpositive charge in the end—a contradiction and thus the configurations in L are unavoidable.

Our scheme differs in two respects. First of all we need a quantitative version. We let every vertex of degree i have a charge of $7 - i$; in this way we can make sure that the overall value in a maximal planar graph is at least n (again, using Euler's formula), or, equivalently, there is at least 1 for every vertex on the average. Secondly, the "discharging" goes across a family of planar graphs, the set of all triangulations of a given point set. We show that the charge can be redistributed so that no vertex of degree exceeding 3 has positive charge, and degree-3 vertices have charge at most 43 each. This allows us to conclude that at least $\frac{1}{43}$ of all vertices over all triangulations have degree 3.

Further Steps. We know that the "43" in the bounds is not tight for our approach, and we are currently working on a more exhaustive analysis, which seems to suggest that the best constant that the technique yields gets close to 32 (and perhaps even slightly smaller). We hope to report on this in the full version of this paper. There we also plan to provide an argument that, for all $i \geq 3$, there is a positive constant δ_i so that $\hat{v}_i \geq \delta_i n$, provided n is large enough (if $n < i-2$, there is no vertex of degree i). In the present version, we prove this statement for $i = 4$, with $\delta_i = 1/540$. Of course, improved bounds on τ imply better upper bounds on the number of other classes of crossing-free graphs, as discussed above.

[1] Interest was also motivated by the obviously related practical question (from geometric modeling [20]) of how many bits it takes to encode a triangulation of a point set.

2. DEGREES IN TRIANGULATIONS

We fix a triple H of non-collinear points in the plane, and, without further mention, restrict ourselves to finite point sets P that are contained in the convex hull of H. We say that P is in *general position*, if no three points in $P^+ := P \cup H$ are collinear (P^+ is what we used to denote by S in the introduction). Let $\mathcal{T}^+(P)$ denote the set of all triangulations of P^+. Recall that a triangulation of N points whose convex hull is a triangle has exactly $3N-6$ edges and $2N-5$ inner faces, all triangular.

Degrees in Triangulations of P. For $i \in \mathbb{N}$ and triangulation $T \in \mathcal{T}^+(P)$, we let $v_i = v_i(T)$ denote the number of points in P (not P^+) that have degree i in T. Obviously, $v_i \in \mathbb{N}_0$, $v_1 = v_2 = 0$, and $\sum_i v_i = n := |P|$. Moreover,

$$\sum_i i\, v_i \leq 6n - 5 \ , \quad \text{if } n \geq 2. \tag{1}$$

For the latter inequality, note that if d_1, d_2, and d_3 are the degrees in T of the three points of H, then

$$d_1 + d_2 + d_3 + \sum_i i\, v_i = 2(3(n+3)-6) = 6n+6 \ ,$$

and $d_1+d_2+d_3 \geq 11$, since in a maximal planar graph with at least 5 vertices all vertices have degree at least 3, and no two vertices of degree 3 are adjacent.

The vector $(v_i)_{i\in\mathbb{N}}$, however, is constrained beyond (1). For example, $v_3 \leq \frac{2n+1}{3}$, which can be seen as follows. Given $T \in \mathcal{T}^+(P)$ remove all the v_3 points from P in T that have degree 3. Note that no two such points can be adjacent in T. Therefore, the resulting graph is a triangulation T' of the remaining points ${P'}^+$, and each of its faces contains at most one point in $P \setminus P'$. So for $k := |P'| = n - v_3$, the number of points removed is at most $2(k+3)-5 = 2k+1$. Therefore, $v_3 \leq 2(n-v_3)+1$; that is, $v_3 \leq \frac{2n+1}{3}$ as claimed. In order to see that this bound of $\lfloor\frac{2n+1}{3}\rfloor$ is tight, set $k = \lceil\frac{n-1}{3}\rceil$, choose any triangulation in $\mathcal{T}^+(Q)$ for any set Q of k points, and place another $n-k = \lfloor\frac{2n+1}{3}\rfloor$ points, no two in the same face of the triangulation. This is possible by the choice of k. Connect all added points to the three vertices of their respective faces, and we are done. We summarize

$$0 \leq v_3 \leq \left\lfloor \frac{2n+1}{3} \right\rfloor \ , \tag{2}$$

which is tight except for the lower bound when n is small.

Degrees in Random Triangulations and the Number of Triangulations. For $i \in \mathbb{N}$ let

$$\hat{v}_i = \hat{v}_i(P) := \mathbb{E}(v_i(T)),$$

for T uniformly at random in $\mathcal{T}^+(P)$. Due to linearity of expectation, any linear identity or inequality in the v_i's (such as (1) or (2)) will also be satisfied by the $\hat{v}_i$'s. However, as we will show, the $\hat{v}_i$'s are significantly more constrained than the v_i's. In particular, there is a constant $\delta > 0$ such that $\hat{v}_3 \geq \delta\, n$ if $n > 0$ and the point set is in general position; recall Fig. 1 to see that general position is indeed necessary here. Before we establish this bound, let us relate it to the question about the number of triangulations. For that, let $\mathsf{tr}^+(P) := |\mathcal{T}^+(P)|$ and $\mathsf{tr}^+(n) := \max_{|P|=n} \mathsf{tr}^+(P)$.

LEMMA 2.1. *(i) If $\delta > 0$ is a real constant such that, for all $n \in \mathbb{N}$, $\hat{v}_3(P) \geq \delta\, n$ for every set P of n points in general position, then, for all $n \in \mathbb{N}_0$,*

$$\mathsf{tr}^+(n) \leq \left(\tfrac{1}{\delta}\right)^n \ .$$

(ii) If $\delta' > 0$ is a real constant and $n_0 \in \mathbb{N}$ such that, for all n, $n_0 \leq n \in \mathbb{N}$, $\hat{v}_3(P) \leq \delta'\, n$ for every set P of n points in general position, then for every set P of $n \in \mathbb{N}$ points in general position, $\mathsf{tr}^+(P) = \Omega\left(\left(\frac{1}{\delta'}\right)^n\right)$.

Proof. (i) Let P be a set of $n > 0$ points with $\mathsf{tr}^+(P) = \mathsf{tr}^+(n)$. Without loss of generality, let P be in general position (a small perturbation of a point set cannot decrease the number of triangulations).

Note that we can get triangulations of P^+ by choosing a triangulation of $P^+ \setminus \{q\}$ for some $q \in P$, and then inserting q as a vertex of degree 3 in the unique face it lands in. In fact, a triangulation $T \in \mathcal{T}^+(P)$ can be obtained in exactly $v_3(T)$ ways in this manner. In particular, if $v_3(T) = 0$, T cannot be obtained at all in this fashion. This is easily seen to imply that

$$\sum_{T\in\mathcal{T}^+(P)} v_3(T) = \sum_{q\in P} \mathsf{tr}^+(P \setminus \{q\}) \ .$$

The left-hand side of this identity equals $\hat{v}_3(P) \cdot \mathsf{tr}^+(P)$, and its right-hand side is upper bounded by $n \cdot \mathsf{tr}^+(n-1)$. Hence,

$$\mathsf{tr}^+(P) \leq \frac{n}{\hat{v}_3(P)} \cdot \mathsf{tr}^+(n-1) \leq \frac{1}{\delta} \cdot \mathsf{tr}^+(n-1)$$

(since we assume $\hat{v}_3(P) \geq \delta\, n$), and thus $\mathsf{tr}^+(n) \leq \frac{1}{\delta} \cdot \mathsf{tr}^+(n-1)$ for all $n \in \mathbb{N}$. Since $\mathsf{tr}^+(0) = 1$, the lemma follows.

(ii) Along the same lines—omitted. ◻

$\mathsf{tr}^+(n)$ is also an upper bound for the number of triangulations of an arbitrary point set S of n points, without restricting it to be contained in the convex hull of H, and without adding H to make its convex hull triangular. To see this, take S and apply an affine transformation so that it lies in the convex hull of H. This does not change the number of triangulations, and adding H cannot decrease the number of triangulations.

An Example. Suppose P lies on a convex arc in the convex hull of H as depicted in Fig. 3. Then all edges indicated there have to be present in all triangulations of P^+. What remains is to fill in a triangulation of a convex polygon with $n+2$ vertices, $n := |P|$. The number of such triangulations is C_n, thus $\mathsf{tr}^+(P) = C_n$, roughly 4^n.

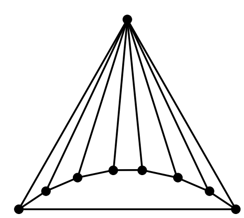


Figure 3: Points on a convex arc.

For a point in P to have degree 3, its adjacent vertices in the convex polygon have to be connected to each other, which leaves an $(n+1)$-gon to be triangulated in C_{n-1} ways. Therefore, the probability that this point has degree 3 is exactly $\frac{C_{n-1}}{C_n} = \frac{n+1}{2(2n-1)} = \frac{1}{4} + O\left(\frac{1}{n}\right)$ and $\hat{v}_3 = \frac{n}{4} + O(1)$. It is easy to show that $\hat{v}_4 = \hat{v}_3$ for these point sets, provided $n \geq 2$.

3. LOWER BOUND ON $\hat{v}_3$

The basic idea of our proof is to have each vertex of any triangulation of P charge to vertices of degree 3. If every vertex charges at least 1 and each vertex of degree 3 is charged at most c, then we know that $\hat{v}_3 \geq \frac{n}{c}$. The actual charging scheme is more involved. First, since there are triangulations that have no degree 3 vertices, the charging has to go across triangulations. Moreover, vertices will charge amounts different from 1 (even negative charges will occur). However, on average, each vertex will charge at least 1. The difficulty in the analysis will be to bound the maximum charge c to a vertex of degree 3.

Vints and Flipping. We consider the set $P \times \mathcal{T}^+(P)$ and call its elements *vints* (*v*ertex-*in*-*t*riangulation). The degree of a vint (p, T) is the degree (number of neighbors) of p in T; a vint of degree i is called an *i-vint*. The overall number of vints is obviously $n \cdot \mathsf{tr}^+(P)$, and the number of i-vints is $\hat{v}_i \cdot \mathsf{tr}^+(P)$.

We define a relation on the set of vints. If $u = (p_u, T_u)$ and $v = (p_v, T_v)$ are vints, then we say that $u \to v$ if $p_v = p_u$ and T_v can be obtained by flipping one edge incident to p_u in T_u. That is, u and v are associated with the same point but in different triangulations, and u has to be an $(i+1)$-vint and v an i-vint, for some $i \geq 3$. We denote by $\to^*$ the transitive reflexive closure of $\to$, and if $u \to^* v$, we say that *u can be flipped down to v*. Charges will go from vints to 3-vints they can be flipped down to.

The *support of a vint* u is the number of 3-vints it can be flipped down to, i.e.

$$\mathrm{supp}(u) := |\{v \mid v \text{ is 3-vint with } u \to^* v\}| .$$

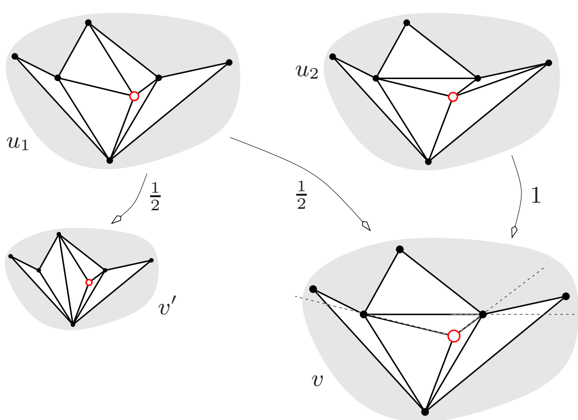


Figure 4: A 3-vint v that is charged $\mathrm{ch}_4(v) = \frac{1}{2} + 1$ by 4-vints u_1 and u_2 in the provisional charging scheme.

A Provisional Charging Scheme. Given our original plan, a natural charging scheme would let a vint u charge $\frac{1}{\mathrm{supp}(u)}$ to each 3-vint it can be flipped down to—in this way it will charge a total of 1. Let us call this the *provisional charging* scheme; see Fig. 4. Since every vint can be flipped down to some 3-vint, the charges are well-defined in this way. For technical reasons, discussed shortly, our final charging scheme will be somewhat different, multiplying the charges an i-vint makes by $7 - i$.

Let us gain some understanding of the notion of $\mathrm{supp}(u)$. Note that the removal of an interior point p and its incident edges in a triangulation T creates a star-shaped polygon (with respect to p). We call this the *hole* of the vint (p, T).

LEMMA 3.1. *For a vint u, $\mathrm{supp}(u)$ equals the number of triangulations of the hole of u. Therefore,*
(i) if u is an i-vint, $1 \leq \mathrm{supp}(u) \leq C_{i-2}$, where the upper bound attained iff the hole is convex (see Section 1 for the definition of the Catalan numbers C_m), and
(ii) if $u \to^* u'$ for vints u and u', then $\mathrm{supp}(u) \geq \mathrm{supp}(u')$.

Proof. (i) follows from the fact that a convex i-gon has C_{i-2} triangulations, which is the maximum for all i-gons. (ii) uses the fact that if $u \to u'$ then the hole of u' is contained in the hole of u, with the vertices of the former a subset of the vertices of the latter; i.e., every triangulation of the hole of u' can be extended to at least one triangulation of the hole of u. ◻

For a 3-vint v and $i \in \mathbb{N}$, we let $\mathrm{ch}_i(v)$ be the amount charged to v by i-vints in the provisional charging scheme described above.

LEMMA 3.2. *For every 3-vint v and all $i \geq 3$, we have $0 \leq \mathrm{ch}_i(v) \leq C_{i-1} - C_{i-2}$. In particular, $\mathrm{ch}_3(v) = 1$, $\mathrm{ch}_4(v) \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\}$, $\mathrm{ch}_5(v) \leq 9$, $\mathrm{ch}_6(v) \leq 28$, etc.*

Proof. It follows from an analysis in [17, Lemma 4] that the number of i-vints that can charge a 3-vint is at most $C_{i-1} - C_{i-2}$, and since a vint can charge at most 1 to a 3-vint, the bound follows.

$\mathrm{ch}_3(v) = 1$ is obvious. For the claim on $\mathrm{ch}_4(v)$ it suffices to observe that there are at most three 4-vints that can charge a given 3-vint v, and that the support of a 4-vint is either 1 or 2. The remaining numbers simply evaluate the expression $C_{i-1} - C_{i-2}$, and are given for future reference. ◻

The Actual Charging Scheme. In our provisional charging scheme, a 3-vint is charged $\sum_i \mathrm{ch}_i(v)$. We note that the bounds in Lemma 3.2 are tight (provided n is large enough compared to i). This will follow from the analysis given below, and is illustrated in Fig. 5 for the case $i = 5$ (the figure too will be better understood after the following analysis). Therefore, there is no uniform upper bound on the amount charged to individual 3-vints in the provisional scheme. For that reason, we switch to a charging where

an i-vint u charges $\frac{7-i}{\mathrm{supp}(u)}$ to each 3-vint v with $u \to^* v$.

Note that in this scheme, a 3-vint charges 4 to itself (so that sounds like bad news), but 7-vints do not charge at all and all i-vints with $i \geq 8$ charge a *negative* amount, so that is good news for the 3-vints (which want to be charged as little as possible).

There is Enough Charge for Everybody. The overall charge of an i-vint is $7-i$, so the overall charge accumulated for all vints associated with a triangulation T is exactly

$$\textstyle\sum_i (7-i) v_i(T) = \sum_i 7 v_i(T) - \sum_i i\, v_i(T) \geq 7n - 6n = n,$$

where we have used (1). So vints charge so that, on average, each gets to charge at least 1.

No 3-Vint Gets Charged too Much. For a 3-vint v, we set

$$\begin{aligned} \mathrm{charge}(v) &:= \textstyle\sum_i (7-i)\mathrm{ch}_i(v) \qquad (3) \\ &= 4\,\mathrm{ch}_3(v) + 3\,\mathrm{ch}_4(v) + 2\,\mathrm{ch}_5(v) + \mathrm{ch}_6(v) \\ &\quad -\mathrm{ch}_8(v) - 2\,\mathrm{ch}_9(v) - \cdots \end{aligned}$$

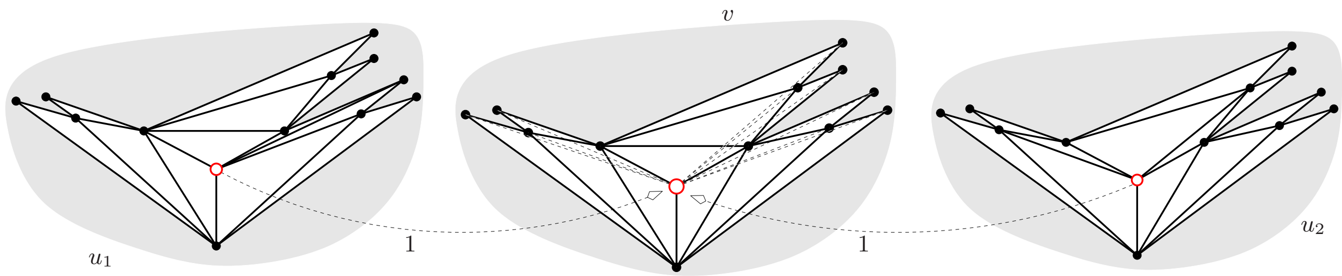


Figure 5: A 3-vint v that gets charged 1 by nine 5-vints (two of which are displayed) in the provisional charging scheme. Hence, $\mathrm{ch}_5(v) = 9$.

For an initial upper bound, we can ignore the negative terms and invoke the bounds on the $\mathrm{ch}_i(v)$'s from Lemma 3.2, to get

$$\mathrm{charge}(v) \le 4 \cdot 1 + 3 \cdot 3 + 2 \cdot 9 + 28 = 59,$$

which implies $\hat{v}_3 \ge \frac{n}{59}$, and by Lemma 2.1, this gives an upper bound of 59^n for the number of triangulations of any set of n points. This is the Santos-Seidel bound which we have derived now with ideas similar to theirs but in a different setting.

We improve on this by observing that if all $\mathrm{ch}_4(v)$, $\mathrm{ch}_5(v)$, and $\mathrm{ch}_6(v)$ are large, then the $\mathrm{ch}_i(v)$, $i \ge 8$, are large as well, and therefore $\mathrm{charge}(v)$ is not so large after all. For example, if indeed $\mathrm{ch}_4(v) = 3$, $\mathrm{ch}_5(v) = 9$, and $\mathrm{ch}_6(v) = 28$ (which is possible), then $\mathrm{charge}(v)$ is extremely small: at most -142636 (the analysis below will clarify this statement).

How do we find those vints that flip down to a given 3-vint $v = (p_v, T_v)$? Clearly, there is v itself. If an edge in a triangle incident to p_v can be flipped in T_v (such an edge cannot be incident to p_v!), then flipping such an edge yields a 4-vint $u = (p_v, T_u)$ that can be flipped down to v (by reversing the flip just made). If in the triangulation T_u there is a flippable edge that is not incident to p_v but part of a triangle incident to p_v, then we can flip this edge to get a 5-vint that can be flipped down to v, etc.

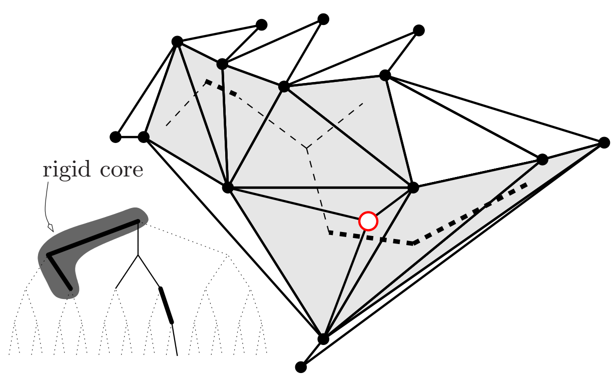


Figure 6: The flip-tree of a 3-vint. Bold edges are rigid edges.

In order to represent this structure, we associate with a 3-vint $v = (p_v, T_v)$ a *flip-tree* $\tau(v)$ as follows. The root of the tree is labeled by the pair (t_v, N_v), where t_v is the hole of v (a triangle) and N_v is the set of its three vertex points (the neighbors of p_v in T_v). All other nodes of the tree are associated with a pair (t, q), where t is a face of T_v and q is a point incident to that face (note that t_v at the root is not a face of T_v—it contains p_v and its incident faces).

(i) Every edge e of t_v gives rise to a child if this edge can be flipped in T_v. If so, this child is labeled by the triangle incident to e that is not incident to p_v, and by the point in this triangle which is not incident to e. So the root has at most three children.

(ii) Consider now a non-root node of the tree labeled by (t, q) and an edge e of t incident to q. If e is a boundary edge, no child will be obtained via e. Otherwise, let t' be the other triangle incident to e. If t' together with the triangle formed by e and p_v is a convex quadrilateral (where e can be flipped), then this gives rise to a child of (t, q) labeled by (t', q') where q' is the point of t' that is not incident to e. So a non-root node has at most two children.

Note that the union of all triangles of the nodes of any subtree of $\tau(v)$ (containing the root) form a polygon that is star-shaped with respect to p_v; this follows easily by the inductive construction of $\tau(v)$. The triangles form a triangulation of the polygon, and the subtree is actually the dual tree of this triangulation. If we retriangulate this polygon in T_v by connecting p_v to all vertices of the polygon, we get a vint that flips down to v. And we get all vints that flip down to v in this way. That is:

Lemma 3.3. *The subtrees of $\tau(v)$ containing its root are in bijective correspondence with the vints that flip down to v.*

The next step is to determine how much these vints charge to v. This depends on the number of triangulations of the holes of these vints—the fewer triangulations, the more v is charged in the provisional scheme. The analysis given here only discriminates between vints that charge 1 to v in the provisional scheme, and all other vints (which charge at most $\frac{1}{2}$ in that scheme).

We first define *rigid edges* of $\tau(v)$: An edge of the tree connects two nodes labeled by two triangles t and t' with a common edge e. If e cannot be flipped in the union of these two triangles, then we call the "dual" tree edge rigid. Beware that e may be flippable in T_v while it is not flippable in $t \cup t'$—this may happen if one of the two triangles is t_v (and thus not a triangle of T_v). Now the *rigid core*, $\tau^*(v)$, of $\tau(v)$ is defined to be the maximal subtree of $\tau(v)$ that includes the root and consists exclusively of rigid edges. $\tau^*(v)$ is non-empty, since it always contains the root of $\tau(v)$.

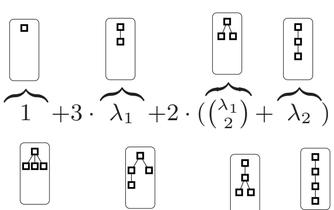
Lemma 3.4. *The subtrees of the rigid core $\tau^*(v)$ containing the root are in bijective correspondence with the vints* *u that flip down to v and provisionally charge* 1*, i.e., with* $\mathrm{supp}(u) = 1$*.*

Proof. Consider a vint u that flips down to v. We recall that $\mathrm{supp}(u) = 1$ iff the hole of u has exactly one triangulation (with u and its incident edges removed). Note that one triangulation of this polygon can be obtained by taking the set of triangles in the subtree corresponding to u. If all edges in this subtree are rigid, then none of the dual edges in the triangulation can be flipped. That is, there is only one triangulation of the hole, since the set of triangulations of a polygon is connected via edge-flips. Also, if any of the edges is not rigid, then its dual edge can be flipped, and so obviously there are at least two triangulations. ◻

Analysis of a Rigid Tree. We analyze the contribution of a rigid tree R to the charging of its vint v. In order to recapitulate the set-up, consider all subtrees of R containing the root. Each j-edge subtree corresponds to a $(j+3)$-vint and therefore charges $7-(j+3) = 4-j$ and these charges over all subtrees are summed up to the overall charge to v. We split this sum into positive charges ($j \le 3$ edges) which we accumulate in $\mathrm{contr}^+(R)$, and negative charges ($j \ge 5$ edges) collected in $\mathrm{contr}^-(R)$.

Given a rigid tree, we let λ_i, $i = 1, 2, 3$, denote the number of edges between level $i-1$ and level i in the tree (the root is at level 0, so, e.g., λ_1 is the degree of the root). We denote the number of nodes at level 1 with two children by ν_2. There are several restrictions on these parameters: $\lambda_1 \le 3$, $\lambda_2 \le 2\lambda_1$, $\lambda_3 \le 2\lambda_2$ and $\nu_2 \le \lambda_2/2$. For example, for the rigid core in Fig. 7 we have $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 0$, and $\nu_2 = 1$.

These parameters suffice to express the positive contribution $\mathrm{contr}^+(R)$ of a rigid tree R as follows.



$$\begin{aligned} & 4\cdot\overbrace{1} + 3\cdot\overbrace{\lambda_1} + 2\cdot\Big(\overbrace{\tbinom{\lambda_1}{2}} + \overbrace{\lambda_2}\Big) \\ + \; & 1\cdot\Big(\overbrace{\tbinom{\lambda_1}{3}} + \overbrace{\lambda_2(\lambda_1-1)} + \overbrace{\nu_2} + \overbrace{\lambda_3}\Big) \\ = \; & 4 + \tbinom{\lambda_1}{3} + \lambda_1^2 + 2\lambda_1 + (\lambda_1+1)\lambda_2 + \lambda_3 + \nu_2 \\ = \; & \begin{cases} 20 + 4\lambda_2 + \lambda_3 + \nu_2 & \text{if } \lambda_1 = 3, \\ 12 + 3\lambda_2 + \lambda_3 + \nu_2 & \text{if } \lambda_1 = 2, \text{ and} \\ 7 + 2\lambda_2 + \lambda_3 + \nu_2 & \text{if } \lambda_1 = 1. \end{cases} \end{aligned} \tag{4}$$

For example, if R is a complete tree of height 3 then $\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = 12$, and $\nu_2 = 3$. That makes $\mathrm{contr}^+(R) = 20 + 4\cdot 6 + 12 + 3 = 59$.

We are ready for estimates for positive and negative charges in terms of number of edges.

Lemma 3.5. *Let m be the number of edges of a rigid tree R. Then*

$$\mathrm{contr}^+(R) \le \frac{13+9m}{2}$$

and, provided R has height at most 3*,*

$$\mathrm{contr}^-(R) \le \min\{0, 14-3m\}\ .$$

Proof. Note that $\nu_2 \le \frac{\lambda_2}{2}$ and $\lambda_2 + \lambda_3 \le m - \lambda_1$.

Let us first assume that $\lambda_1 = 3$. Using (4), we get an upper bound on $\mathrm{contr}^+(R)$ of

$$20 + \frac{9}{2}\lambda_2 + \lambda_3 \le 20 + \frac{9}{2}(m-3) = \frac{13+9m}{2}\ .$$

In a similar fashion we get a bound of $\frac{10+7m}{2}$ if $\lambda_1 = 2$, and $\frac{9+5m}{2}$ if $\lambda_1 = 1$. Obviously, these latter bounds are dominated by the one with $\lambda_1 = 3$.

If $m = 5$ then the whole tree R itself charges $4-5 = -1$; no other negative charges occur. Therefore, $\mathrm{contr}^-(R) = -1$.

For the general bound on $\mathrm{contr}^-(R)$ let first $m \ge 6$. The whole tree itself charges $4-m < 0$. We get another subtree with $m-1$ edges by removing a leaf, with charge $5-m < 0$. Removing a different leaf in this tree gives another subtree with $m-1$ edges and charge $5-m$ (every tree of height at most 3 with at least 4 edges has at least 2 leaves). These three subtrees alone charge $14-3m$ as claimed. The bound also holds if $m \le 5$ (we have seen that $\mathrm{contr}^-(R) = -1$ for $m = 5$, and for $m \le 4$, $\mathrm{contr}^-(R) = 0$). ◻

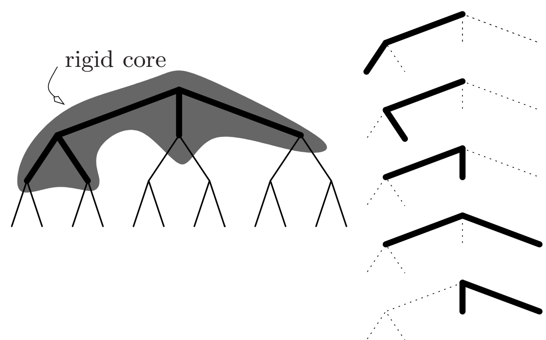


Figure 7: The rigid core that maximizes the modified charge, with the five subtrees corresponding to 5-vints that provisionally charge 1.

Analysis of the Flip-Tree of a Vint. In order to upper bound $\mathrm{charge}(v)$, we have to consider the whole flip-tree, not just its rigid core. We first restrict ourselves to vints that correspond to subtrees of $\tau(v)$ of depths at most 3. Note that in this way we do not lose any 3-, 4-, 5-, or 6-vints, i.e., no vint that charges a positive amount in the actual scheme is lost. Moreover, we let all i-vints, $i = 4, 5, 6$, whose subtree is not part of the rigid core charge $\frac{7-i}{2}$; this is an upper bound on their actual charge. Finally, we include in the charge only the negative charges that come from i-vints, $i \ge 8$, whose subtrees are part of the rigid core, and thus charge $7-i$. These modifications cannot decrease the overall charge made to the 3-vint v. We refer to the overall charge counted in this manner as the *modified charge* to v.

How large can the modified charge be? We further simplify the analysis, by assuming that our tree is complete[2] up to level 3. If not, we can extend the tree with non-rigid edges, and thus increase the modified charge (since those edges will not be used for negative charges). Now we simply have to maximize the modified charge over all possibilities of rigid cores of complete trees of height 3.

[2] "Complete" means that the root has three children, and all other non-leaf nodes have two children.

With this pessimistically modified charge, we get an upper bound on the charge of a 3-vint of

$$\begin{aligned} & \mathrm{contr}^+(R) + \frac{1}{2}\left(59 - \mathrm{contr}^+(R)\right) + \mathrm{contr}^-(R) \\ = \quad & \frac{59}{2} + \frac{\mathrm{contr}^+(R)}{2} + \mathrm{contr}^-(R) \\ \leq \quad & \frac{118+13+9m}{4} + \mathrm{contr}^-(R) \\ = \quad & \frac{131+9m}{4} + \mathrm{contr}^-(R) \ , \end{aligned}$$

where R is the rigid core of the flip-tree, and m is the number of its edges. If $m \leq 4$, then we know that $\mathrm{contr}^-(R) = 0$ and the expression is bounded by $\frac{131+36}{4} = 41.75$. If $m \geq 5$ we employ the upper bound for $\mathrm{contr}^-(R)$ which yields a bound of

$$\frac{131+9m}{4} + 14 - 3m = \frac{187-3m}{4} \leq \frac{172}{4} = 43 \ .$$

The unique rigid core (up to symmetry) that maximizes the modified charge is shown in Fig. 7. Indeed, the 3-vint is provisionally charged 1 by one 3-vint (itself), three 4-vints, five 5-vints (out of possible 9), six 6-vints (out of 28), and one 8-vint. Its modified charge is thus

$$4 \cdot 1 + 3 \cdot 3 + 2 \cdot \left(5 + \tfrac{4}{2}\right) + \left(6 + \tfrac{22}{2}\right) - 1 = 43 \ .$$

THEOREM 3.6. *$\hat{v}_3 \geq \frac{n}{43}$ for every set of n points.*

The modified charge used in the last step of the analysis has a lot of room for improvement. First, we have assumed that each 3-, 4-, 5-, and 6-vint that does not come fully from the rigid core charges $\frac{1}{2}$. However, to really charge $\frac{1}{2}$, the associated hole must have only two triangulations, and thus only one flippable edge. Any other vint charges at most $\frac{1}{3}$ to the 3-vint. One should therefore examine all rigid cores and all possible ways to attach to them non-rigid children, and count separately the number of vints with charge 1, those with charge $\frac{1}{2}$, and bound pessimistically the number of remaining positively-charging vints (which charge at most $\frac{1}{3}$). Initial exploration with this approach (using a program that has tested all possible flip trees) suggests that the bound drops to 38 (where the worst 3-vint is still the one shown in Fig. 7). A more careful analysis, that includes also vints with negative charges should decrease the bound further. Of course, the ultimate manifestation of the technique would be to test by a program (or by a careful case analysis) all possible neighborhoods (up to level 3) and calculate exactly the maximum charge possible. Even then the analysis would probably still be too weak, because one would not expect the average 3-vints to be as bad as the worst-case one.

4. MISCELLANEOUS BOUNDS

We exhibit here a number of further restrictions on the expected degree sequences $(\hat{v}_i)_{i\in\mathbb{N}}$ of finite planar point sets.

LEMMA 4.1. *For all integers $3 \leq i < j$ there is a positive integer $\delta_{i,j}$ such that $\hat{v}_i \geq \frac{\hat{v}_j}{\delta_{i,j}}$. In particular, $\hat{v}_i \geq \frac{\hat{v}_{i+1}}{i}$, $\hat{v}_i \geq \frac{2\hat{v}_{i+2}}{i(i+3)}$, $\hat{v}_3 \geq \frac{\hat{v}_i}{C_{i-1}-C_{i-2}}$, $\hat{v}_4 \geq \frac{\hat{v}_i}{C_{i-1}-2C_{i-2}}$.*

Proof. For the inequality $\hat{v}_i \geq \frac{\hat{v}_{i+1}}{i}$, we let every $(i+1)$-vint charge some i-vint it can be flipped down to. Since every vertex of degree at least 4 is incident to a flippable edge, such an i-vint is always available. Note that an i-vint can be reached at most i times in this way.

For the general inequality we observe that we can choose $\delta_{i,j} = t_{i,j-i+1}$ where $t_{i,k}$ denotes the number of binary trees with k nodes with an exceptional root of degree i; that is, the root has i potential children pointers, but not all of them need to be used (just like the binary nodes distinguish between a left and a right child, the root discriminates its children via an index in $\{1, 2, \ldots, i\}$). To see this, use a generalization of the flip-trees from the previous section, where the root now encodes the hole of the given i-vint, which has i potential children. It is known that $t_{2,k} = C_k$ (for the generic binary trees), which yields also $t_{1,k} = C_{k-1}$. $t_{i,1} = 1$, $t_{i,2} = i$, and $t_{i,3} = \binom{i}{2} + 2i = \frac{i(i+3)}{2}$ can be easily seen. The number observes the recurrence $t_{i,k} = t_{i-1,k+1} - t_{i-2,k+1}$ (proof omitted, generalizes an argument in [17]). Now the asserted values for $\delta_{i,j}$ can be readily obtained: $\delta_{i,i+1} = t_{i,2} = i$, $\delta_{i,i+2} = t_{i,3} = \frac{i(i+3)}{2}$, $\delta_{3,j} = t_{3,j-2} = t_{2,j-1} - t_{1,j-1} = C_{j-1} - C_{j-2}$, and, finally, $\delta_{4,j} = t_{4,j-3} = t_{3,j-2} - t_{2,j-2} = C_{j-1} - 2C_{j-2}$. ◻

THEOREM 4.2. *For $n \in \mathbb{N}_0$, $\hat{v}_3 \leq \frac{2n+3}{5}$.*

Proof. We apply a scheme where every 3-vint charges 3 units to vints of larger degrees or to boundary edges (there are three). We show that no vint is charged more than 2 units, and no boundary edge more than 1. This will imply that

$$3\hat{v}_3 \leq 3 + 2\sum_{j\geq 4} \hat{v}_j = 2(n - \hat{v}_3) + 3, \qquad (5)$$

which yields the asserted inequality.

Let $v = (p, T)$ be a 3-vint, and let t_v denote its hole, which is a triangle. For each edge e of t_v we do the following, depending on the properties of e; see Fig. 8.

(1) *e is a boundary edge.* Then we let v charge 1 to e; we call this a *boundary-charge*.

(2) *There is a triangle t incident to e on its other side:*

(2.1) *t forms with p a convex quadrilateral.* We can flip e to get a 4-vint (p, T') to which v charges 1; we call this a *flip-charge*.

(2.2) *t forms with p a non-convex quadrilateral.* Let a be the endpoint of e which is reflex in this quadrilateral; note that a cannot lie on the boundary, and it has to be of degree at least 4, since interior vertices of degree 3 are never adjacent (the "interior" condition is necessary only in case $n = 1$). Here v charges 1 to vint (a, T); we call this a *neighbor-charge*. Let us label such a charge with the *responsible edge e*.

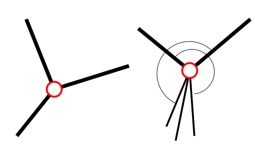


Figure 9: Separable edges.

Let $w = (q, T)$ be a vint. We call an edge ρ incident to q in T a *separable edge at w* if it can be separated from the other edges incident to q by a line through q. An equivalent condition is that the two angles between ρ and its clockwise and counterclockwise next edges (at q) sum up to more than π. In the context of the neighbor-charge as described above, the responsible edge e is separable at (a, T). We observe the easy following properties (see Fig. 9 for an illustration).

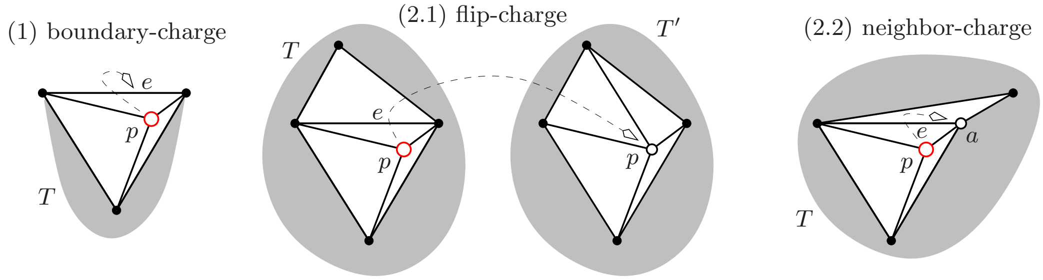


Figure 8: The various types of charges of a 3-vint in the proof of Theorem 4.2.

(S0) No edge is separable at both vints induced by its endpoints.

(S1) If w has degree 3, every edge incident to its point is separable at w (recall here that points of vints are interior).

(S2) If w has degree at least 4, at most two incident edges can be separable at w.

(S3) If w is of degree at least 4 and there are two edges separable at w, then they must be consecutive.

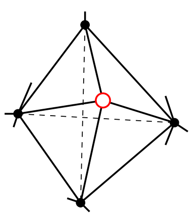


Figure 10: Only two edges incident to a 4-vint can be flippable, and no interior neighbor of a 4-vint with a convex hole can be of degree 3.

We note that the charges resulting from the three edges of a hole t_v are all different. This is obvious for boundary-charges and flip-charges. For neighbor-charges, it is impossible that vint (a, T) is charged twice, by each of its incident edges in t_v, because these two edges cannot both be separable (as follows, e.g., from (S3)).

We are now ready to show that no vint u can be charged more than twice. Consider first the case of a 4-vint $u = (p_u, T_u)$. Let h_u denote the quadrangular hole of p_u. We note that at most two edges incident to p_u are flippable: One out of each pair of opposite edges is separable at u and thus unflippable; see Fig. 10.
(a) If u receives two flip-charges, it cannot be charged as a neighbor, because in this case h_u must be convex, and then no vertex of h_u can be interior and of degree 3; see Fig. 10.
(b) u can be charged at most once as a neighbor. Indeed, if a is a vertex of h_u of degree 3, then it must be a reflex vertex of h_u, and there can be at most one such vertex.

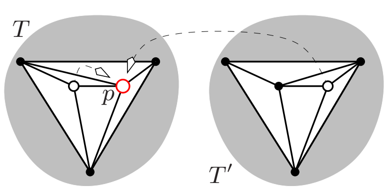


Figure 11: 4-vint (p, T) with non-convex hole is charged twice.

(a) and (b) establish the claim for 4-vints. (We note the following stronger property: If the hole of p_u is convex, then u is charged exactly twice (by edge flips). On the other hand, if the hole of p_u is non-convex then it can be charged twice, once by an edge flip and once as a neighbor, if and only if p_u and its charging neighbor are enclosed in a triangle as in Fig. 11.)

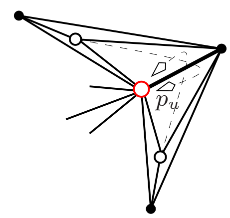


Figure 12: Neighbor-charges to vint u with one separable edge.

Consider next the case where $u = (p_u, T_u)$ is a vint of degree at least 5. Each flip-charge is to a 4-vint and therefore vints of degree at least 5 can receive neighbor-charges only. We claim that in this case p_u can be a neighbor of at most two points of degree 3 that charge it as a neighbor. Recall the ingredients necessary for such a neighbor-charge to be made to u: (i) an edge e that is separable at u, and (ii) a neighbor a of p_u that has degree 3 so that the edges e and p_ua are consecutive around p_u. Clearly, if there is only one edge separable at u then there are at most two such constellations; see Fig. 12. If there are two separable edges at u, then they have to be consecutive around p_u (recall (S3)). This rules out the possibility that any of these two edges is involved in more than one neighbor-charge, since an edge cannot be both separable at p_u and connect to a point of degree 3; see Fig. 13. ◻

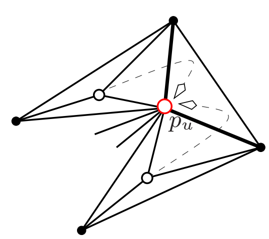


Figure 13: Neighbor-charges to vint u with two separable edges.

A weakness in the proof of Theorem 4.2 is that it "assumes" that every vint of degree at least 4 is charged exactly twice. We can show (details in the full version) that this cannot be the case, which gives a slight improvement on the result (however, still short of improving–via Lemma 2.1–the lower bound of McCabe and Seidel [10] on the number of triangulations of n points with a triangular convex hull).

We note that an upper bound on $\hat{v}_3$ smaller than $\frac{n}{4}$ is impossible by the example at the end of Section 2 (with $\hat{v}_3 = \frac{n}{4} + O(1)$).

Finally, we derive a lower bound on $\hat{v}_4$ via purely algebraic manipulations of the inequalities in Lemma 4.1 and Theorem 4.2.

Lemma 4.3. *For $n \geq 2$, $\hat{v}_4 \geq \frac{n}{540}$.*

Proof. We have a supply of upper bounds for all the other $\hat{v}_i$'s in terms of $\hat{v}_4$, due to Lemma 4.1 and Theorem 4.2, namely

$\hat{v}_3 \le \frac{2n+3}{5}$ $\quad$ $\hat{v}_5 \le (C_4 - 2C_3)\hat{v}_4 = 4\hat{v}_4$

$\hat{v}_6 \le (C_5 - 2C_4)\hat{v}_4 = 14\hat{v}_4$ $\quad$ $\hat{v}_7 \le (C_6 - 2C_5)\hat{v}_4 = 48\hat{v}_4$

$\hat{v}_8 \le (C_7 - 2C_6)\hat{v}_4 = 165\hat{v}_4$ $\quad \ldots$

Moreover, using (1), we have

$$\sum_i (9-i)\hat{v}_i \ge 9n - (6n-5) = 3n+5 \ .$$

Hence

$$\begin{aligned} 3n+5 &\le 6\hat{v}_3 + 5\hat{v}_4 + 4\hat{v}_5 + 3\hat{v}_6 + 2\hat{v}_7 + \hat{v}_8 \\ &\le \frac{6(2n+3)}{5} + \hat{v}_4 \cdot (5 + 4{\cdot}4 + 3{\cdot}14 + 2{\cdot}48 + 165) \\ &= \frac{12n+18}{5} + 324\hat{v}_4 \ , \end{aligned}$$

implying that $\hat{v}_4 \ge n/540$, as asserted. ◻

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