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An improved local search algorithm for 3-SAT

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Abstract

We slightly improve the pruning technique presented in Dantsin et. al. (2002) to obtain an $\mathcal{O}^*(1.473^n)$ algorithm for 3-SAT.

Key words: exact algorithm, local search, 3-SAT
MSC2000: 68Q25

1 Introduction

An instance of 3-SAT is a boolean formula φ in n variables $x_1, \ldots, x_n$, defined as the conjunction of a set $\mathcal{C}$ of disjunctive clauses of length at most 3. Satisfiability of φ can be tested in a straightforward manner in time

$$\mathcal{O}\left(2^n \cdot n^3\right) = \mathcal{O}^*(2^n).$$

Here, as usual, we use the $\mathcal{O}^*$-notation to indicate that polynomial factors are suppressed.

During the last years so-called *exact algorithms* have been designed solving 3-SAT in time $\mathcal{O}^*(\alpha^n)$ with $\alpha < 2$, see Schoening [3] for an overview. The currently fastest randomized algorithms run in time $\mathcal{O}^*(1.3302^n)$ (see Hofmeister, Schoening, Schuler and Watanabe [2]) and the fastest deterministic algorithm (see Dantsin et. al. [1]) takes $\mathcal{O}^*(1.481^n)$. We slightly improve the pruning technique used in Dantsin et. al. [1] to obtain a running time of $\mathcal{O}^*(1.473^n)$.

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2 Local search

Let φ be an instance of 3-SAT given by a set $\mathcal{C}$ of clauses in variables $x_1, \ldots, x_n$. For $a \in \{0,1\}^n$ let $B_r(a) \subseteq \{0,1\}^n$ denote the set of 0-1 vectors with Hamming distance at most r from a. The currently fastest algorithms for 3-SAT are based on *local search*: First, a *covering code* of suitable *radius* $r \leq n$ is constructed, i.e. a set $A \subseteq \{0,1\}^n$ such that

$$\{0,1\}^n = \bigcup_{a \in A} B_r(a)$$

holds. Next we search for a truth assignment for φ in each $B_r(a)$, $a \in A$, separately. To make our paper self-contained, we briefly describe the basic idea for constructing a covering code and (to some extent) the local search within a given $B_r(a)$ as presented in Dantsin et. al. [1].

Covering codes. As $B_r := B_r(0)$ contains exactly

$$V(n,r) = \sum_{i=1}^{r} \binom{n}{i}$$

elements, a covering code $A \subseteq \{0,1\}^n$ of radius $r \leq n$ must necessarily satisfy

$$|A| \geq \frac{2^n}{V(n,r)}.$$

Covering codes of approximately this size indeed exist and can be constructed randomly: Choose

$$t = \frac{n2^n}{V(n,r)}$$

elements from $\{0,1\}^n$ uniformly at random, resulting in a set $A \subseteq \{0,1\}^n$ of size $|A| \leq t$. The probability that a particular $a^* \in \{0,1\}^n$ is *not* covered by any $B_r(a)$, $a \in A$ is at most

$$P[a^* \text{ not covered}] = \left(1 - \frac{V(n,r)}{2^n}\right)^t \leq \mathrm{e}^{-n},$$

using $1 + x \leq \mathrm{e}^x$ for $x \in \mathbb{R}$. So the probability that A is *not* a covering code is at most $2^n \mathrm{e}^{-n}$, which tends to 0 as $n \to \infty$.

This procedure can be de-randomized by taking in each step a new code word $a \in \{0,1\}^n$ that is best possible in the sense that it covers as many as possible of the yet uncovered elements in $\{0,1\}^n$. Note, however, that this *greedy construction* takes $\mathcal{O}^*(2^n)$ per step and thus almost $\mathcal{O}(2^{2n}) = \mathcal{O}^*(4^n)$ in total (which is far too slow). Dantsin et. al. [1] therefore propose the following. Let

$K \in \mathbb{N}$ be a constant and assume w.l.o.g. that $n = Kn_0$ and $r = Kr$. Then construct a covering code $A_0 \subseteq \{0,1\}^{n_0}$ in time $\mathcal{O}(4^{n_0}) = \mathcal{O}^*\left(\sqrt[K]{4}^n\right)$ and take

$$A = \underbrace{A_0 \times \ldots \times A_0}_{K \text{ times}}$$

as a covering code for $\{0,1\}^n$. Proceeding this way, the time needed for constructing the covering code becomes negligible.

Local search. Assume we want to search for a truth assignment for φ in $B_r(a) \subseteq \{0,1\}^n$. We may assume w.l.o.g. that $a = 0$, i.e., we search in $B_r = B_r(0)$. (Interchange x_i with $\overline{x}_i$ if necessary.) If $a = 0$ is not a truth assignment for φ, there must exist a *false clause*, i.e. a clause $C \in \mathcal{C}$ that is false under $a = 0$, say $C = (x_i \vee x_{i'} \vee x_{i''})$. It then suffices to search for a truth assignment in $B_{r-1} \subseteq \{0,1\}^{n-1}$ w.r.t. each of the formulae

$$\varphi_1 = \varphi[x_i = 1], \ \varphi_2 = \varphi[x_{i'} = 1] \ \text{ and } \ \varphi_3 = \varphi[x_{i''} = 1],$$

obtained by fixing a variable as indicated in brackets. If necessary, we may even fix in addition some variables to zero, e.g., define $\varphi_1 := \varphi[x_i = 1], \varphi_2 := \varphi[x_{i'} = 1, x_i = 0]$ and $\varphi_3 := \varphi[x_{i''} = 1, x_i = 0, x_{i'} = 0]$.

Continuing this way, our search can be described by a *search tree* T_r, constructed by *branching on false clauses* (one false clause per node), as indicated in figure 1.

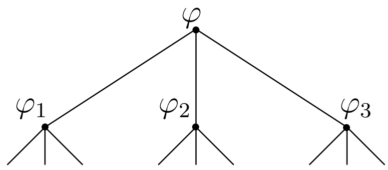

Fig. 1. The search tree T_r

Needless to say that we never branch to formulas $\varphi' = \varphi[x_i = 1, \ldots]$ that are obviously non-satisfiable because they contain an empty (non-satisfiable) clause. (For example, if $(\overline{x}_i) \in \mathcal{C}$, we would only branch to φ_2 and φ_3 in figure 1.) We denote the number of leaves of T_r by $|T_r|$ and refer to it as the *size* of T_r. Clearly,

$$|T_r| \leq 3^r \tag{1}$$

holds, an immediate consequence of the recursion $|T_r| \leq 3|T_{r-1}|$ (see figure 1). In case φ contains a false 2-clause $C \in \mathcal{C}$, then branching on C would yield $|T_r| \leq 2|T_{r-1}|$.

As pointed out in Dantsin et. al. [1], this simple argument already gives an $\mathcal{O}^*\left(\sqrt[2]{3}^n\right) \approx \mathcal{O}^*(1.7321^n)$ algorithm: Take $r = \frac{n}{2}$ and search $B_r(0)$ and $B_r(1)$

separately in time $\mathcal{O}^*(3^r) = \mathcal{O}^*\left(\sqrt[2]{3}^n\right)$ each.

Smaller search trees. The trivial bound (1) on the size of the search tree can be improved by a clever branching technique, as shown in Dantsin et. al. [1]: Assume that φ contains three pairwise disjoint false clauses $C = (x_i \vee x_{i'} \vee x_{i''}), C_1 = (x_j \vee x_{j'} \vee x_{j''})$ and $C_1' = (x_k \vee x_{k'} \vee x_{k''})$ and a (true) clause $(\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$. We may then *branch along* $(\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$, i.e. first branch on C at the root node φ, then branch on C_1 at $\varphi_1 = \varphi[x_i = 1]$ and finally branch on C_1' at $\varphi_1' = \varphi_1[x_j = 1] = \varphi[x_i = 1, x_j = 1]$. The resulting search tree is indicated in figure 2.

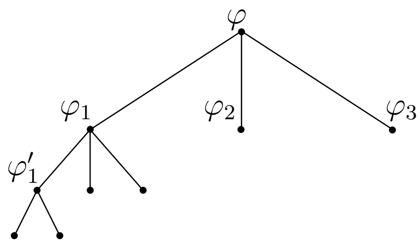

Fig. 2. Branching along $(\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$

Note that the node corresponding to φ_1' has only two descendants because $\varphi[x_i = 1, x_j = 1, x_k = 1]$ is ruled out by the clause $(\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$.

If a similar branching was possible also at φ_2 and φ_3, we would get a search tree satisfying a recursion

$$|T_r| \leq 6|T_{r-2}| + 6|T_{r-3}|. \tag{2}$$

Indeed, this is what Dantsin et. al. [1] show. Assuming inductively that $|T_k| \leq c\alpha^k$ holds for some constant $c > 0$, (2) implies that

$$|T_r| \leq \mathcal{O}(\alpha^r), \tag{3}$$

where $\alpha = \sqrt[3]{4} + \sqrt[3]{2} \approx 2.848$ is the largest root of $\alpha^3 - 6\alpha - 6 = 0$.

The main result of our paper slightly improves this bound as follows.

Theorem 1 *By branching on false clauses we can ensure that*

$$|T_r| \leq c\beta^r,$$

where $\beta = \frac{1+\sqrt{21}}{2} \approx 2.792$ *is the largest root of* $\beta^3 - 6\beta - 5 = 0$.

Running time. Let $\varrho < \frac{1}{2}$ and $r = \varrho n$. By Stirling's formula, the size of a covering code we construct is (up to a polynomial factor) bounded by

$$|A| = \mathcal{O}^*\left(\left[2\varrho^\varrho\left(1-\varrho\right)^{1-\varrho}\right]^n\right).$$

According to (3), the number of nodes in T_r is bounded by $n|T_r| = \mathcal{O}^*\left(|T_r|\right)$ and hence the total running time is thus bounded by

$$\mathcal{O}^*\left(|A||T_r|\right) = \mathcal{O}^*\left(\left[2(\alpha\varrho)^\varrho\left(1-\varrho\right)^{1-\varrho}\right]^n\right).$$

This expression is minimal for $\varrho \approx 0.26$, yielding the bound of $\mathcal{O}^*\left(1.481^n\right)$ in Dantsin et. al. [1].

Similarly, replacing α by β from Theorem 1, we obtain for $\varrho \approx 0.264$ an exact algorithm that runs in $\mathcal{O}^*\left(1.473^n\right)$.

3 Simple partial assignments

We will prove Theorem 1 by induction on $r \geq 0$. The basic idea is as follows. We first try to find a "simple truth assignment" by fixing as few as possible of the variables to $x_i = 1$ (exactly one per false clause). In case we do not succeed, we will exhibit a "good" clause to branch on.

We start by analyzing the structure of $\mathcal{C}$ and introduce some notation. Let $\mathcal{F} \subseteq \mathcal{C}$ denote the set of false clauses (at $x = 0$). We may assume w.l.o.g. that each $F \in \mathcal{F}$ is a 3-clause $F = (x_i \vee x_{i'} \vee x_{i''})$, because otherwise, as we observed already in section 2, branching on a false clause of length at most 2 yields the recursion $|T_r| \leq 2|T_{r-1}|$ and Theorem 1 follows by induction.

Secondly, we may assume that the clauses $F \subseteq \mathcal{F}$ are pairwise disjoint. Indeed, if $F = (x_i \vee x_{i'} \vee x_{i''})$ and $F' = (x_j \vee x_{j'} \vee x_{j''})$ intersect, say $x_i = x_j$, then branching on F at φ and on F' at $\varphi_2 = \varphi\left[x_{i'} = 1, x_i = 0\right]$ and $\varphi_3 = \varphi\left[x_{i''} = 1, x_i = 0, x_{i'} = 0\right]$ yields a search tree as indicated in figure 3.

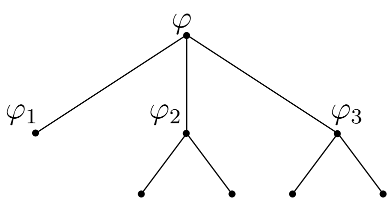

Fig. 3. Branching on intersecting clauses

The corresponding recursion is $|T_r| \leq |T_{r-1}| + 4|T_{r-2}|$ and, again, Theorem 1 follows inductively.

Thus in what follows, we may (and will) assume that φ is *regular* in the sense that $\mathcal{F}$ consists of pairwise disjoint 3-clauses. We often identify such a clause $F = (x_i \vee x_{i'} \vee x_{i''}) \in \mathcal{F}$ with its corresponding set of variables $F = \{x_i, x_{i'}, x_{i''}\}$ or with the corresponding set of elements (indices) $F = \{i, i', i''\}$. The elements i, i', i'' covered by a false clause $F \in \mathcal{F}$ are *neighbors* of each other. The elements $i \in \{1, \ldots, n\}$ covered by false clauses are called *internal* elements. We denote by $I = I_\varphi \subseteq \{1, \ldots, n\}$ the set of internal elements. The elements in $\{1, \ldots, n\} \setminus I$ are called *external*.

Recall that, as mentioned above, we first try to construct a truth assignment for φ by fixing some variable to $x_i = 1$ (one per false clause in $\mathcal{F}$). In general, fixing some variables, say $x_{i_1} = 1, \ldots, x_{i_t} = 1$, results in a new formula $\varphi' = \varphi\,[x_{i_1} = 1, \ldots, x_{i_t} = 1]$ whose clauses are obtained from the clauses in $\mathcal{C}$ by fixing $x_{i_1} = 1, \ldots, x_{i_t} = 1$ in each clause. This way each clause $C \in \mathcal{C}$ *reduces* to a corresponding clause $C' = C\,[x_{i_1} = 1, \ldots, x_{i_t} = 1] \in \mathcal{C}' = \mathcal{C}_{\varphi'}$. We say that C reduces to $C' = 1$ (a fixed *true* clause) if C contains some $x_i, i \in \{i_1, \ldots, i_t\}$. Similarly, C reduces to $C' = 0$, the empty (fixed *false*) clause if C contains only negated literals $\overline{x}_i, i \in \{i_1, \ldots, i_t\}$. Note that $C \in \mathcal{C}$ reduces to $C' \in \mathcal{F}_{\varphi'}$ if and only if all negated variables $\overline{x}_i$ in C are indexed by $i \in \{i_1, \ldots, i_t\}$.

Definition 2 (Simple partial assignment) *A* simple partial assignment *(SPA) of φ is a formula*

$$\varphi' = \varphi\,[x_{i_1} = 1, \ldots, x_{i_t} = 1]$$

that fixes at most one variable per false clause to $x_i = 1$, without creating any new false clauses, i.e., such that the following hold:

$(S1)$ $\{i_1, \ldots, i_t\} \subseteq I$

$(S2)$ $|F \cap \{i_1, \ldots, i_t\}| \leq 1$ *for each* $F \in \mathcal{F}_\varphi$

$(S3)$ $\mathcal{F}_{\varphi'} \subseteq \mathcal{F}_\varphi$.

There are certain clauses in $\mathcal{C} \setminus \mathcal{F}$ that are "irrelevant" in the sense that they never reduce to a false clause by fixing $x_{i_1} = 1, \ldots, x_{i_t} = 1$ as long as (S1) and (S2) hold: A clause $C \in \mathcal{C} \setminus \mathcal{F}$ is called *externally true* if $C = (\overline{x}_l \vee \ldots)$ with $l \in \{1, \ldots, n\} \setminus I$ being external. A clause $C \in \mathcal{C} \setminus \mathcal{F}$ is *internally true* if $C = (\overline{x}_i \vee \overline{x}_j \vee \ldots)$ with $i, j \in I$ being neighbors. Clearly, an externally and/or internally true $C \in \mathcal{C}$ reduces to a true clause $C' \in \mathcal{C}_{\varphi'}$ whenever $\varphi' = \varphi\,[x_{i_1} = 1, \ldots, x_{i_t} = 1]$ satisfies (S1) and (S2). We let $\mathcal{E} \subseteq \mathcal{C} \setminus \mathcal{F}$ denote the set of externally and/or internally true clauses.

The remaining set $\mathcal{R} = \mathcal{C} \setminus (\mathcal{F} \cup \mathcal{E})$ is called the set of *relevant* clauses. We will use these clauses to guide our search process, i.e., we will construct T_r by "branching along relevant clauses" as indicated already in section 2. We first treat the so-called "pure case", where each relevant clause contains only negated variables. This is the case where the bound (2) is tight in the approach of Dantsin et. al. [1].

4 The pure case

A regular φ is called *pure* if every $R \in \mathcal{R} = \mathcal{R}_\varphi$ contains only negated variables. Throughout this section, we assume that φ is (regular and) pure and hence so is any *SPA* φ' of φ.

We say that $R \in \mathcal{R}$ *intersects* $F = (x_i \vee x_{i'} \vee x_{i''}) \in \mathcal{F}$ if R contains one of $\overline{x}_i, \overline{x}_{i'}, \overline{x}_{i''}$. Recall that R cannot contain two of these since it would then be internally true. To motivate the following, consider an *SPA* $\varphi' = \varphi[x_i = 1]$ of φ. Any $R \in \mathcal{R}$ reduces to a true clause in φ' due to (S3). If R intersects the unique false clause $F = (x_i \vee x_{i'} \vee x_{i''})$ covering i, then either R becomes an externally true clause in φ' (namely when R contains either $\overline{x}_{i'}$ or $\overline{x}_{i''}$) or R reduces to an "even more" relevant clause $R' \in \mathcal{R}_{\varphi'}$. For example, $R = (\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$ reduces to $R' = (\overline{x}_j \vee \overline{x}_k) \in \mathcal{R}_{\varphi'}$.

Let $\varphi' = \varphi[x_{i_1} = 1, \ldots, x_{i_t} = 1]$ be an *SPA* of φ and let $F_{i_1}, \ldots, F_{i_t} \in \mathcal{F}$ be the unique clauses covering $i_1, \ldots, i_t$, resp. We say that φ' is *proper* if every $R \in \mathcal{R}$ that intersects some $F \in \{F_{i_1}, \ldots, F_{i_t}\}$ reduces to an externally true clause $R' \in \mathcal{R}_{\varphi'}$ (so R must contain some $\overline{x}_i$ with $i \in I$ being a neighbor of an element in $\{i_1, \ldots, i_t\}$).

Lemma 3 *For any two proper* SPA*'s φ' and φ'' of φ there exist a proper* SPA *$\overline{\varphi}$ with $\mathcal{F}_{\overline{\varphi}} = \mathcal{F}_{\varphi'} \cap \mathcal{F}_{\varphi''}$.*

PROOF. Let $\mathcal{F}_\varphi = \{F_1, \ldots, F_f\}$ with $F_i = (x_i \vee x_{i'} \vee x_{i''}), i = 1, \ldots, f$, and assume that, say,

$$\begin{aligned}\varphi' &= \varphi[x_1 = 1, \ldots, x_s = 1],\\ \varphi'' &= \varphi[x_{s+1} = 1, \ldots, x_t = 1, x_{j_1} = 1, \ldots, x_{j_l} = 1],\end{aligned}$$

with $j_1, \ldots, j_l$ being covered by $F_1, \ldots, F_s$. We define $\overline{\varphi}$ as

$$\overline{\varphi} = \varphi[x_1 = 1, \ldots, x_t = 1].$$

Clearly, $\overline{\varphi}$ satisfies (S1) and (S2). We verify (S3) by showing that any $R \in \mathcal{R}_\varphi$ reduces to a true clause $\overline{R} \in \mathcal{R}_{\overline{\varphi}}$. Indeed, we will show that any $R \in \mathcal{R}_\varphi$ intersecting $F_1 \cup \ldots \cup F_t$ reduces (even) to an externally true clause in $\overline{\varphi}$, thus showing at the same time that $\overline{\varphi}$ is proper.

Let $R \in \mathcal{R}_\varphi$ intersect $F_i \in \{F_1, \ldots, F_t\}$. If $i \leq s$, then R reduces to an externally true clause in φ' (since φ' is proper) and hence to an externally true clause in $\overline{\varphi}$. On the other hand, if R does not intersect $F_1 \cup \ldots \cup F_s$ (but $F_{s+1} \cup \ldots \cup F_t$), then R reduces to the same clause in $\overline{\varphi}$ as in φ''. So again, the claim follows, as φ'' is proper. □

Lemma 3 is useful in constructing proper *SPA*'s $\overline{\varphi}$ with smaller and smaller sets $\mathcal{F}_{\overline{\varphi}}$. Ideally, we would like to arrive at $\mathcal{F}_{\overline{\varphi}} = \emptyset$, in which case $\overline{\varphi}$ defines a truth assignment for φ. To describe our search process for proper *SPA*'s of φ, we introduce the notion of "b-blocking".

Definition 4 (b-blocking) *Consider a clause $R \in \mathcal{R}_\varphi$.*

(1) If $R = (\overline{x}_i \vee \ldots)$ then R 0-blocks *$i \in I$.*

(2) If $R = (\overline{x}_i \vee \ldots)$ has length at most two, then R b-blocks i for all $b \geq 0$.

(3) If $R = (\overline{x}_i \vee \ldots)$ has length three, i.e. $R = (\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$ for some $j, k \in I$ with neighbors j', j'' and k', k'', resp., then R b-blocks *i, if each of j', j'', k' and k'' is $(b-1)$-blocked by some clause in $\mathcal{R}_{\varphi[x_i=1]}$.*

We call $i \in I$ b-blocked by *$\mathcal{R}_\varphi$ if there exists some $R \in \mathcal{R}_\varphi$ (of arbitrary length) that b-blocks i.*

Example. Assume $\mathcal{F} = \mathcal{F}_\varphi$ consists of three clauses $(x_i \vee x_{i'} \vee x_{i''})$, $(x_j \vee x_{j'} \vee x_{j''})$ and $(x_k \vee x_{k'} \vee x_{k''})$. Furthermore, assume that $\mathcal{R} = \mathcal{R}_\varphi$ consists of three clauses $R = (\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$, $R' = (\overline{x}_{i'} \vee \overline{x}_{j'} \vee \overline{x}_{k'})$ and $R'' = (\overline{x}_{i''} \vee \overline{x}_{j''} \vee \overline{x}_{k''})$. Then each element in $I = I_\varphi$ is 0-blocked, but none is 1-blocked. Indeed, consider, e.g. $\varphi' = \varphi[x_i = 1]$. Then R' and R'' reduce to externally true clauses in φ'. So $\mathcal{R}_{\varphi'} = \{(\overline{x}_j \vee \overline{x}_k)\}$ and, for example, j' is not 0-blocked by $\mathcal{R}_{\varphi'}$. For this reason (see the general construction described below), it is easy to find a truth assignment for φ (e.g. by setting $x_i = 1, x_{j'} = 1, x_{k'} = 1$).

For $b \geq 0$, we let $U_b \subseteq I$ denote the set of elements $i \in I$ that are not b-blocked by $\mathcal{R}_\varphi$. We call these elements *b-unblocked* (by $\mathcal{R}_\varphi$). Let $\mathcal{U}_b \subseteq \mathcal{F}$ denote the set of false clauses $F \in \mathcal{F}$ that cover some b-unblocked $i \in I$. We also call these false clauses *b-unblocked*. By definition, we have $U_0 \subseteq U_1 \subseteq \ldots$ and also $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \ldots$

Note that we can compute the set $U_b \subseteq I$ for $b \geq 0$ along with a b-blocking clause $R \in \mathcal{R}_\varphi$ for every $i \in I \setminus U_b$ in time $\mathcal{O}\left(n^{b+3}\right)$. Indeed, for $b = 0$, it suffices to scan the $\mathcal{O}(n^3)$ clauses in $\mathcal{R} = \mathcal{R}_\varphi$.

We proceed by induction on $b \geq 0$. Thus assume $b \geq 1$ and let $i \in I$ and $\varphi' = \varphi[x_i = 1]$. By induction, the set $U'_{b-1} \subseteq I_{\varphi'}$ of elements that are $(b-1)$-unblocked by $\mathcal{R}_{\varphi'}$ can be computed in time $\mathcal{O}\left(n^{b+2}\right)$. We then check for each of the $\mathcal{O}(n^2)$ 3-clauses $R = (\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$ whether some element from $\{j', j'', k', k''\}$ is in U'_{b-1} or not. This takes (at most) $\mathcal{O}(n^2)\,\mathcal{O}(n) = \mathcal{O}(n^3)$ in total. Hence the total time needed to check $i \in I$ is $\mathcal{O}\left(n^{b+2}\right) + \mathcal{O}(n^3) = \mathcal{O}\left(n^{b+2}\right)$ and the claim follows.

The next result is crucial:

Theorem 5 *For each $b \geq 0$ there exists a proper* SPA *φ' of φ with $\mathcal{F}_{\varphi'} \subseteq \mathcal{F}_\varphi \setminus \mathcal{U}_b$.*

PROOF. By induction on $b \geq 0$. Assume first that $b = 0$. Let $F \in \mathcal{U}_0$, say $F = (x_i \vee x_{i'} \vee x_{i''})$ with $i \in U_0$. Then $\varphi' = \varphi[x_i = 1]$ is, by definition of U_0, a proper *SPA* and $\mathcal{F}_{\varphi'} = \mathcal{F} \setminus \{F\}$. The claim now follows from Lemma 3 and induction.

Next assume $b \geq 1$. Let $F = (x_i \vee x_{i'} \vee x_{i''}) \in \mathcal{U}_b$ with $i \in U_b$. As before, due to Lemma 3, it suffices to show that there is a proper *SPA* φ' of φ with $\mathcal{F}_{\varphi'} \subseteq \mathcal{F} \setminus \{F\}$. Let $\varphi_1 := \varphi[x_i = 1]$. Clearly, φ_1 is an *SPA* of φ. (Otherwise there were a clause $(\overline{x}_i) \in \mathcal{R}$. But such a clause would b-block i contradicting $i \in U_b$.) Let $U^1_{b-1} \subseteq I_{\varphi_1}$ and $\mathcal{U}^1_{b-1} \subseteq \mathcal{F}_{\varphi_1}$ denote the set of elements in I_{φ_1} resp. clauses in $\mathcal{F}_{\varphi_1}$ that are $(b-1)$-unblocked by $\mathcal{R}_{\varphi_1}$. By induction on b, there is a proper *SPA* φ'_1 of φ_1 with $\mathcal{F}_{\varphi'_1} \subseteq \mathcal{F}_{\varphi_1} \setminus \mathcal{U}^1_{b-1}$. We claim that actually φ'_1 is a proper *SPA* of φ. Clearly, φ'_1 is an *SPA* of φ (as any *SPA* of an *SPA* is an *SPA*).

To show that φ'_1 is proper, assume that

$$\varphi'_1 = \varphi_1[x_{i_1} = 1, \ldots, x_{i_t} = 1] = \varphi[x_i = 1, x_{i_1} = 1, \ldots, x_{i_t} = 1]$$

and let $F_i, F_{i_1}, \ldots, F_{i_t} \in \mathcal{F}$ denote the unique clauses in $\mathcal{F}$ covering $i, i_1, \ldots, i_t$, resp. Let $R \in \mathcal{R}_\varphi$ intersect $F_i \cup F_{i_1} \cup \ldots \cup F_{i_t}$. We are to show that R reduces to an externally true clause R'_1 in φ'_1.

Assume first that R intersects $F_i = (x_i \vee x_{i'} \vee x_{i''})$. If R contains either $\overline{x}_{i'}$ or $\overline{x}_{i''}$, the claim is obviously true. Thus assume $R = (\overline{x}_i \vee \ldots) \in \mathcal{R}$. Since $i \in U_b$, R must be a 3-clause $R = (\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$. So R reduces to $R_1 = (\overline{x}_j \vee \overline{x}_k)$ in φ_1. As $i \in U_b$, at least one neighbor of either j or k is in $\mathcal{U}^1_{b-1}$, i.e., either $F_j = (x_j \vee x_{j'} \vee x_{j''})$ or $F_k = (x_k \vee x_{k'} \vee x_{k''})$ is in $\mathcal{U}^1_{b-1} \subseteq \mathcal{F}_{\varphi_1}$. So $\mathcal{F}_{\varphi'_1} \subseteq \mathcal{F}_{\varphi_1} \setminus \mathcal{U}^1_{b-1}$ implies that φ'_1 fixes at least one variable from either F_j or F_k to 1, i.e., either F_j or F_k occurs in $\{F_{i_1}, \ldots, F_{i_t}\}$. Thus $R_1 = (\overline{x}_j \vee \overline{x}_k)$ reduces to an externally true clause R'_1 in φ'_1 (as φ'_1 is a proper *SPA* of φ) and hence so does R.

Next assume that R does not intersect F_i. Then $R \in \mathcal{R}_\varphi$ and the claim follows immediately from the fact that φ_1' is a proper *SPA* of φ_1. $\square$

Corollary 6 *If $\mathcal{U}_b = \mathcal{F}$ for some $b \geq 0$, then φ has a truth assignment that can be computed in time $\mathcal{O}\left(n^{b+3}\right)$.* $\square$

We are now ready to prove Theorem 1 in the pure case. Let $b \geq 0$ (to be specified later on) and assume there exists some $F = (x_i \vee x_{i'} \vee x_{i''}) \in \mathcal{F} \setminus \mathcal{U}_b$. (Otherwise a truth assignment exists and there is no need to construct a search tree.) We then branch on F at the root node φ of T_r, branching to $\varphi_1 = \varphi\,[x_i = 1]$, $\varphi_2 = \varphi\,[x_{i'} = 1]$ and $\varphi_3 = \varphi\,[x_{i''} = 1]$.

Since $F \notin \mathcal{U}_b$, the elements i, i' and i'' are b-blocked by $\mathcal{R}_\varphi$. Let $R \in \mathcal{R}$ b-block i. If R is a 1-clause, i.e. $R = (\overline{x}_i)$, then the subtree rooted at φ_1 is empty. If R is a 2-clause, i.e. $R = (\overline{x}_i \vee \overline{x}_j)$, then branching on $F_1 = (x_j \vee x_{j'} \vee x_{j''})$ at φ_1 yields a search tree as indicated in figure 4. Thus we obtain a recursion $|T_r| \leq 2|T_{r-1}| + 2|T_{r-2}|$ and Theorem 1 follows inductively.

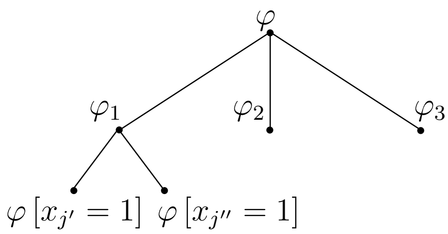

Fig. 4. When i is blocked by a 2-clause.

Hence assume that $R = (\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$ b-blocks i. In this case we obtain a search tree as in figure 2 by branching on F_1 at φ_1 and on $F_1' = (x_k \vee x_{k'} \vee x_{k''})$ at $\varphi_1' = \varphi_1\,[x_j = 1]$.

Let us denote the size of the subtree rooted at φ_1 by $|T_{r-1}^{(b)}|$ to indicate that $\varphi_1 = \varphi\,[x_i = 1]$ is obtained by fixing x_i with i being b-blocked by $\mathcal{R}_\varphi$. We thus get the recursion

$$|T_{r-1}^{(b)}| \leq 2|T_{r-2}^{(b-1)}| + 2|T_{r-3}|, \tag{4}$$

as both j' and j'' are $(b-1)$-blocked by $\mathcal{R}_{\varphi_1}$. Furthermore, of course $|T_r| \leq 3|T_{r-1}^{(b)}|$ holds, since also i' and i'' are b-blocked by $\mathcal{R}_\varphi$.

Iterating (4), we obtain for $r \geq b+2$

$$
\begin{aligned}
|T_{r-1}^{(b)}| &\le 2\left[2|T_{r-3}^{(b-2)}| + 2|T_{r-2}|\right] + 2|T_{r-3}| \\
&\vdots \\
&\le 2^b|T_{r-b-2}| + \ldots + 2|T_{r-3}| + 2^b|T_{r-b-1}^{(0)}| \\
&\le 2^b|T_{r-b-2}| + \ldots + 2|T_{r-3}| + 2^b|T_{r-b-1}|,
\end{aligned}
$$

where the last inequality follows from $|T_k^{(0)}| \le |T_k|$.

Assuming inductively that $|T_k| \le c\beta^k$ for $k < r$, we get

$$
\begin{aligned}
|T_r| &\le 3|T_r^{(b-1)}| \\
&\le 3c\beta^r \left[\frac{2^b}{\beta^{b+1}} + \sum_{k=1}^{b} \frac{2^k}{\beta^{k+2}}\right] \\
&= 3c\beta^r \left[\frac{2^b}{\beta^{b+1}} + \frac{2 - 2^{b+1}\beta^{-b}}{\beta^3 - 2\beta^2}\right].
\end{aligned}
$$

For β as in Theorem 1 and $b \ge 4$ we have for the term in the brackets

$$
\frac{2^b}{\beta^{b+1}} + \frac{2 - 2^{b+1}\beta^{-b}}{\beta^3 - 2\beta^2} < \frac{1}{3}.
$$

So $|T_r| \le c\beta^r$ follows inductively.

5 The general case

In the general case, when φ is regular, but not necessarily pure, we proceed as follows. As in section 4 we say that $i \in I$ is *blocked* by $R \in \mathcal{R}$ if $R = (\overline{x}_i \vee \ldots)$. Let $U \subseteq I$ denote the elements that are *unblocked*, i.e. not blocked by any $R \in \mathcal{R}$ and let $\mathcal{U} \subseteq \mathcal{F}$ denote the set of clauses $F \in \mathcal{F}$ that contain some $i \in U$.

If $\mathcal{F} = \mathcal{U}$, a truth assignment is easily obtained by fixing exactly one unblocked i per clause $F \in \mathcal{F}$ to $x_i = 1$. Hence assume $\mathcal{F}^* = \mathcal{F} \setminus \mathcal{U} \neq \emptyset$ in what follows and let $I^* \subseteq I$ denote the elements covered by clauses in $\mathcal{F}^*$. We distinguish two cases:

Case 1. There exists an element $i \in I^*$ that is blocked by some $R \in \mathcal{R}$ which is *not* of the form $R = (\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k)$ with $j, k \in I$.

In this case we branch on the unique clause $F \in \mathcal{F}^*$ covering i. Branching along blocking clauses as in section 4 then proves Theorem 1 inductively.

Indeed, assume that i is blocked by a clause of type $R = (\overline{x}_i \vee \overline{x}_j \vee x_k)$ with $j, k \in I$. Note that j is then covered by a clause $F_1 \neq F$ since otherwise R were internally true. We then branch on $F_1 = (x_j \vee x_{j'} \vee x_{j''})$ at $\varphi_1 = \varphi\,[x_i = 1]$ and on the false 1-clause (x_k) at $\varphi_1' = \varphi_1\,[x_j = 1]$. The resulting search tree then differs from the one in figure 2 in that one of the two subtrees of φ_1' is eliminated, yielding a recursion

$$|T_r| \leq 6|T_{r-2}| + 5|T_{r-3}|,$$

assuming the "worst case scenario", where both i' and i'' are blocked by 3-clauses with three negated variables each. In this case, Theorem 1 follows inductively (by choice of β). It is straightforward to verify that this is indeed the worst case scenario for case 1).

Case 2. All blocking clauses for elements in I^* have three negated variables each.

In this case, let $\mathcal{R}^*$ denote the set of clauses $R = (\overline{x}_i \vee \overline{x}_j \vee \overline{x}_k) \in \mathcal{R}$ with $i, j, k \in I^*$. Let φ^* denote the formula defined by the clauses $\mathcal{C}^* = \mathcal{F}^* \cup \mathcal{R}^*$. In particular, φ^* is pure. Let $\mathcal{U}_b^* \subseteq \mathcal{F}$ denote the clauses in $\mathcal{F}^*$ that are b-unblocked by $\mathcal{R}_{\varphi^*}$.

Lemma 7 *If $\mathcal{U}_b^* = \mathcal{F}^*$, then φ has a truth assignment.*

PROOF. By Theorem 5, φ^* has a proper *SPA*

$$\varphi' = \varphi^*\,[x_{i_1} = 1, \ldots, x_{i_t} = 1]$$

defining a truth assignment for φ^* (see also Corollary 6).

To define a truth assignment for φ, pick elements $j_1, \ldots, j_s \in U$, one from each clause in $\mathcal{U}$, and let

$$\overline{\varphi} = \varphi\,[x_{i_1} = 1, \ldots, x_{i_t} = 1, x_{j_1} = 1, \ldots, x_{j_s} = 1]\,.$$

We claim that $\overline{\varphi}$ defines a truth assignment for φ, i.e. that $\mathcal{F}_{\overline{\varphi}} = \emptyset$. Assume to the contrary that $R \in \mathcal{R}$ reduces to a false clause in $\overline{\varphi}$. Clearly, $R \notin \mathcal{R}^*$ must hold, since any clause in $\mathcal{R}^*$ reduces to an (externally) true clause in φ' and hence to a true clause in $\overline{\varphi}$. However, if $R \in \mathcal{R} \setminus \mathcal{R}^*$, case 2) implies that $R = (\overline{x}_i \vee \ldots)$ with $i \in I \setminus I^*$. In particular, i is blocked by R and so $i \notin \{j_1, \ldots, j_s\}$. Thus, R reduces to a true clause in $\overline{\varphi}$. $\square$

Due to Lemma 7, we may assume w.l.o.g. that $\mathcal{U}_b^* \neq \mathcal{F}^*$. Thus we may choose $F \in \mathcal{F}^* \setminus \mathcal{U}_b^*$ for branching at the root node φ of T_r and continue branching on false clauses in $\mathcal{F}^*$ along clause $\mathcal{R}^*$ as if we were searching for a truth assignment for φ^*. Theorem 1 thus follows inductively also in the general case.

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