On the Number of Crossing-Free Configurations on Planar Points Sets

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Algorithmic and Combinatorial Geometry
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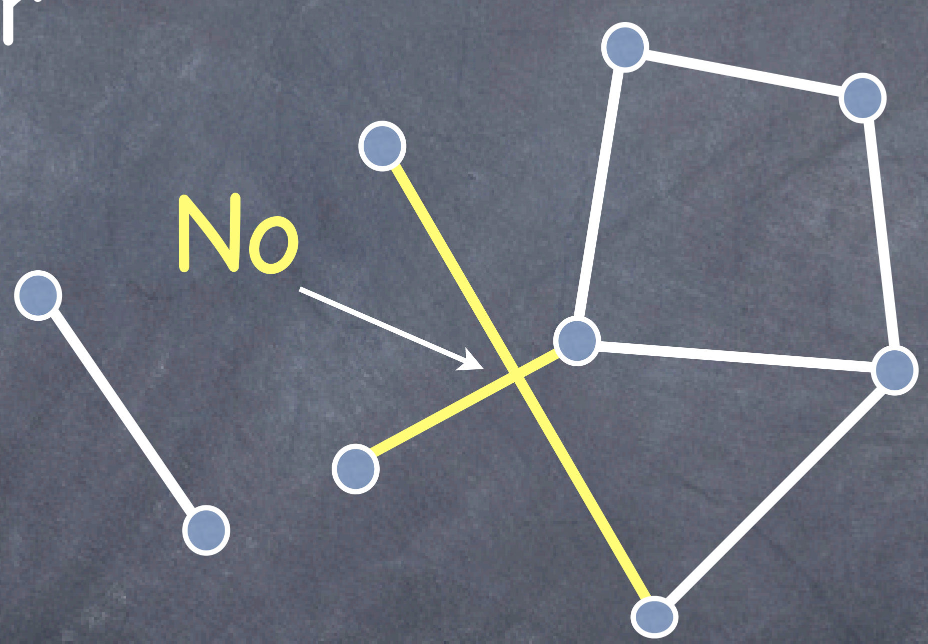
On the Number of Crossing-Free Configurations on Planar Points Sets

Counting

specific - extremal - algorithmic

Crossing-Free Geometric Graphs

P … finite planar point set

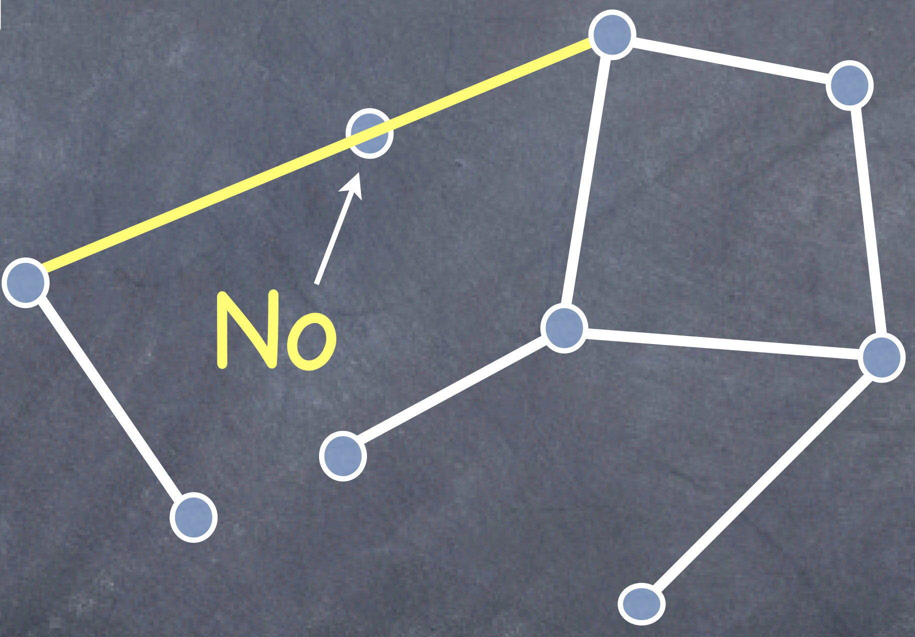


straight line edges

no crossings - no edge through point

Crossing-Free Geometric Graphs

P ... finite planar point set



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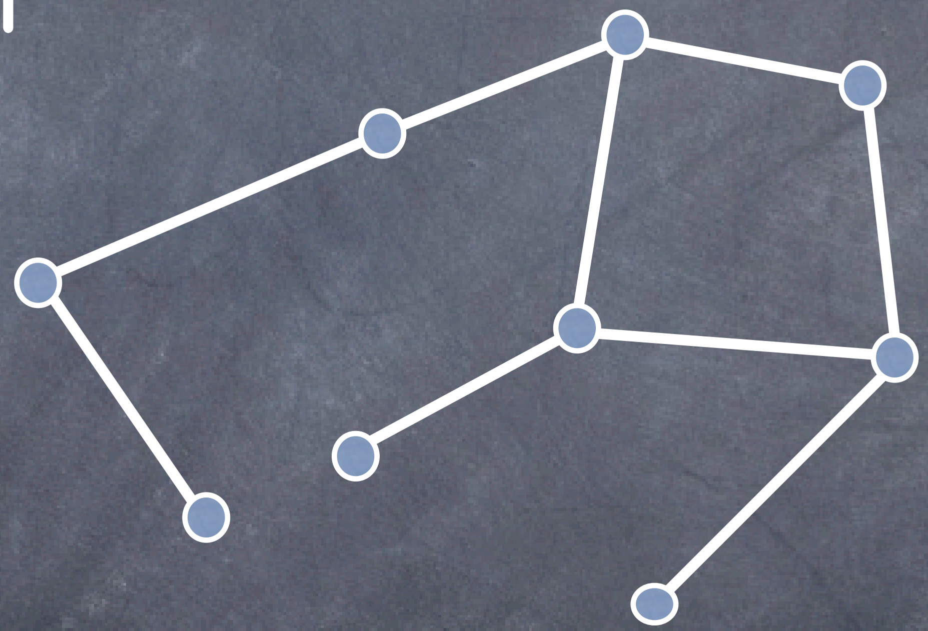

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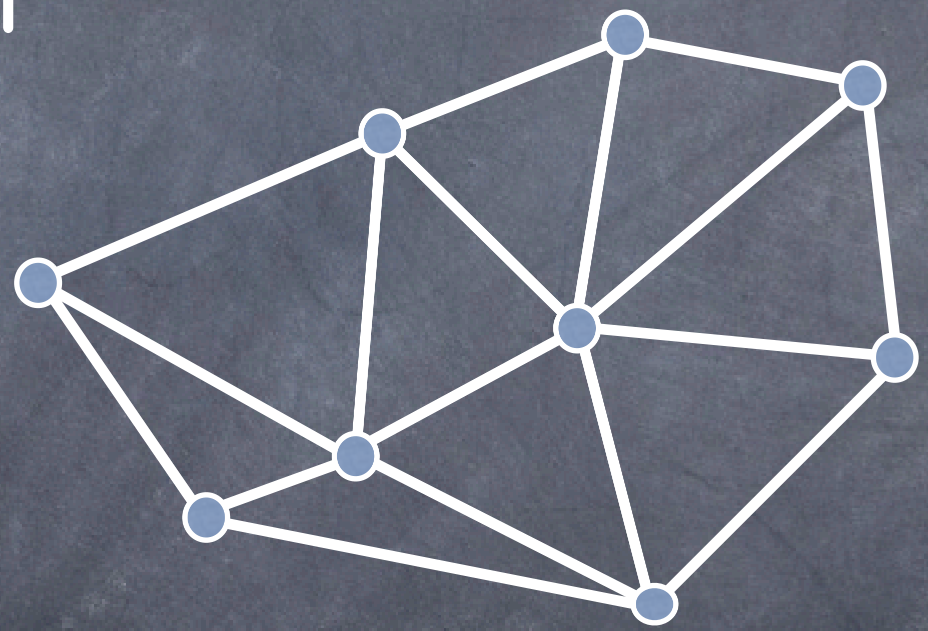

straight line edges

maximal crossing-free: triangulation

I. (Specific) Counting

The number $tr(G_{n+2})$ of triangulations of the vertices G_{n+2} of a convex (n+2)-gon satisfies

$$tr(G_{n+2}) = C_n \approx_n 4^n$$

with $$C_n := \frac{1}{n+1}\binom{2n}{n} = \Theta(\frac{1}{n^{3/2}}4^n)$$

Catalan Numbers

I. (Specific) Counting

The number $tr(G_{n+2})$ of triangulations of the vertices G_{n+2} of a convex $(n+2)$-gon satisfies

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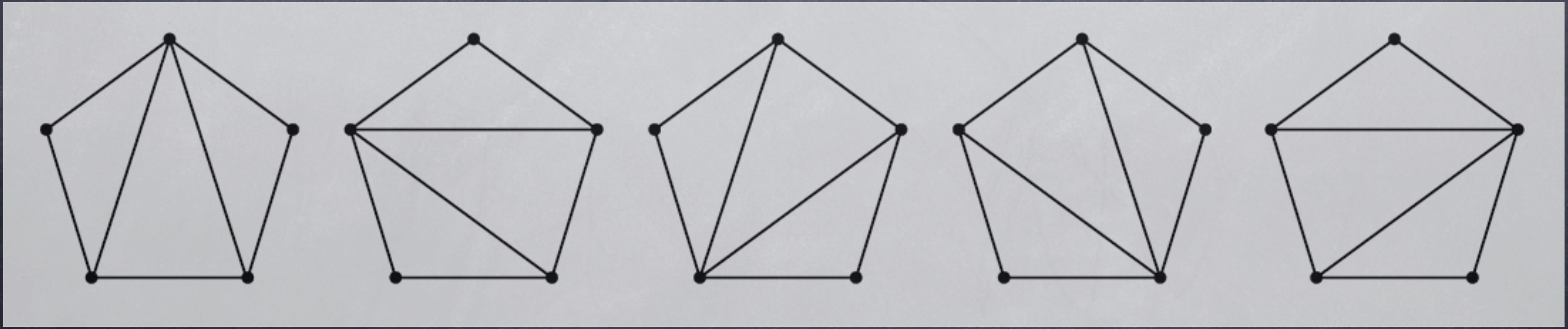
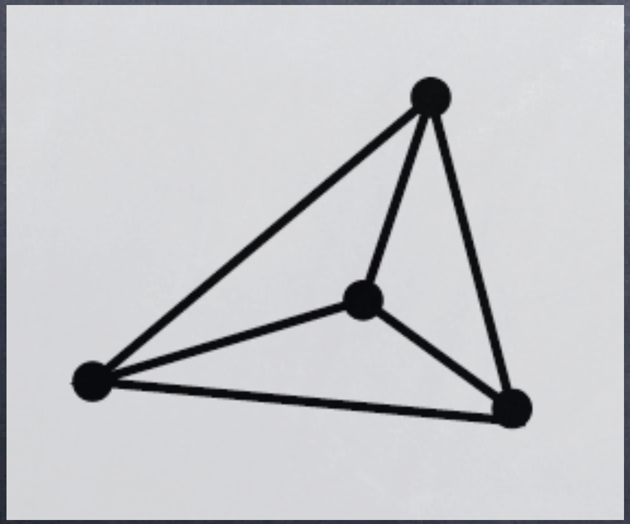
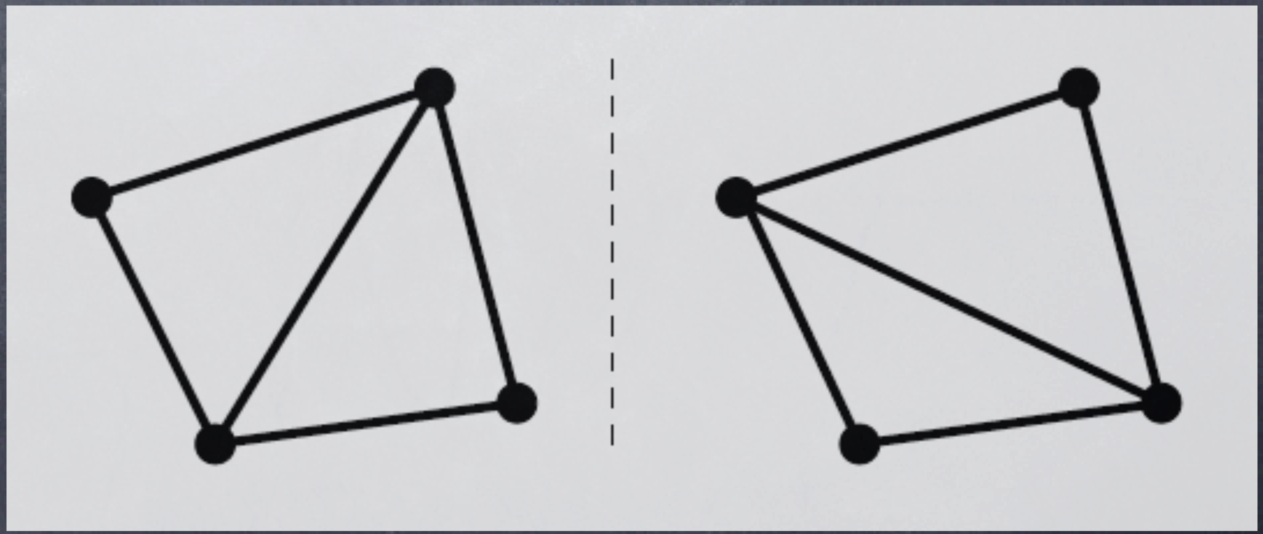


II. Extremal Counting

The number tr(P) of triangulations of a 4-element point set P satisfies

$$tr_{min}(4) = 1 \leq tr(P) \leq 2 = tr(4)$$





II. Extremal Counting

$$tr(n) := \max_{|P|=n} tr(P)$$

$$tr_{min}(n) := \min_{|P|=n} tr(P)$$

with P in general position

$$tr_{min}(n) \lesssim_n 4^n \lesssim_n tr(n)$$

III. Algorithmic Counting

The number tr(P) of triangulations of an n-element point set P can be computed in time

$$O(\mathrm{tr}(P) \cdot \mathrm{poly}(n))$$

by enumeration

I. (Specific) Counting

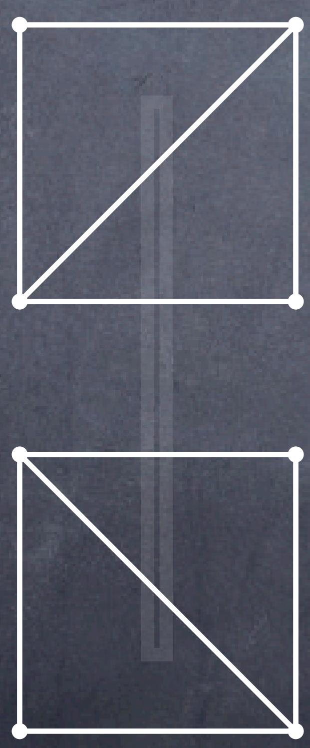
1. Triangulations of Convex Polygons

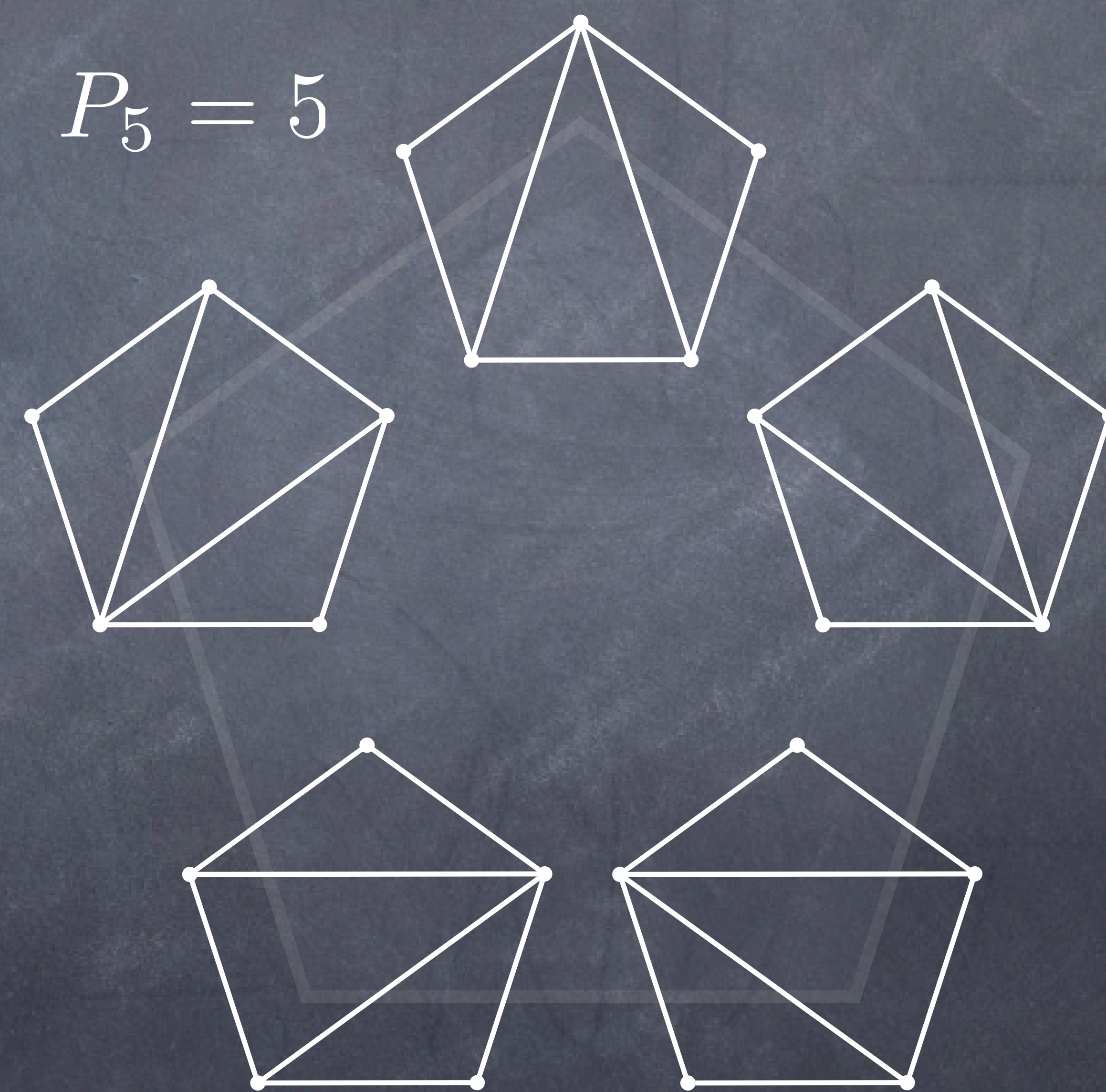
Point Sets in Convex Position

In how many ways can we triangulate a convex n-gon?

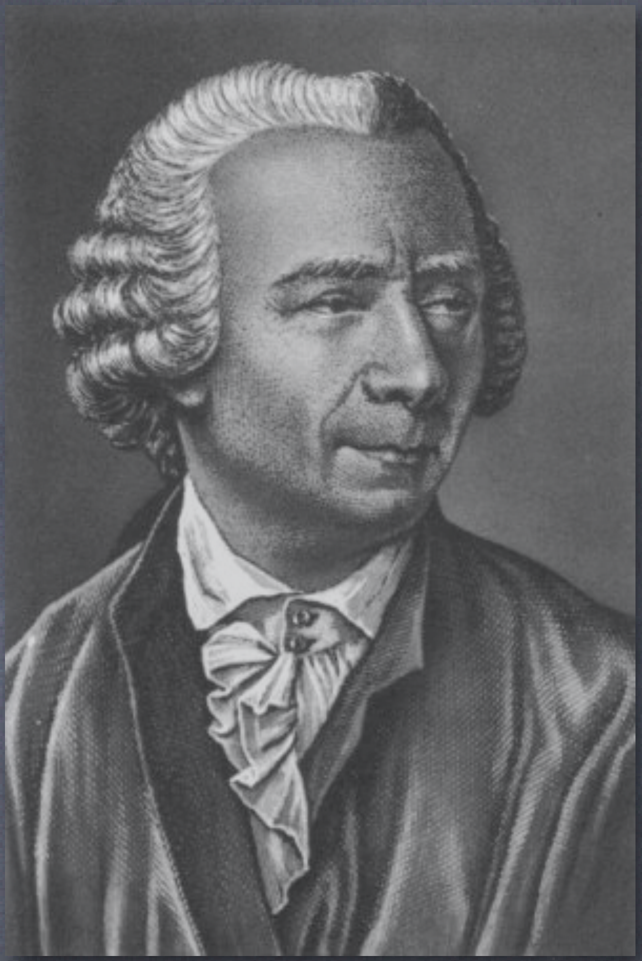
$P_n := tr(G_n)$

$\mathrm{P}_2 = 1, \quad P_3 = 1, \quad P_4 = 2, \quad P_5 = 5$





Started with a letter in 1751: Euler to Goldbach



Leonhard Euler
15.4.1707 Basel
18.9.1783 St. Petersburg

Christian Goldbach
18.3.1690 Königsberg
20.11.1764 Moscow

Euler computed these numbers
up to 10-gons ...

schehen könne. Setze ich nun die Anzahl dieser verschiedenen Arten $= x$, so habe ich per inductionem gefunden

wenn $n = 3, 4, 5, 6, 7, 8, 9, 10$

so ist $x = 1, 2, 5, 14, 42, 132, 429, 1430.$

... but he considered his method (whatever it was,
we don't know) too tedious.

die folgende leicht gefunden wird. Die Induction aber, so ich gebraucht, war ziemlich mühsam, doch zweifle ich nicht, dass diese Sach nicht sollte weit leichter entwickelt werden können. Ueber die Progression der Zahlen 1, 2, 5, 14, 1758 Segner set up the "Catalan Recurrence" and computed more numbers.

num. laterum	numerus resolutionum.
XIII	58786
XIV	208012
XV	742900
XVI	2674440
XVII	9694845
XVIII	35357670
XIX	129644790
XX	477638700
XXI	1767263190
XXII	6564120420
XXIII	24466267020
XXIV	91482563640
XXV	343059613650

With these numbers Euler saw a conjecture confirmed which he mentioned already in his letter to Goldbach:

"Ita si pro polygno n laterum numerus resolutonum sit P pro polygono sequente n+1 laterum resolutionum erit "

$$P_{n+1} := \frac{4n-6}{n} P_n$$

but he has little hope to prove that

Auctori huius schediasmatis non displicaturum esse speramus

It took 80 years until Gabriel Lamé (according to Gauss the best French mathematician of his time) proved Euler's conjecture in 1838.

Note sur une Équation aux différences finies;

Par E. CATALAN.

M. *Lamé* a démontré que l'équation

$$P_{n+1} = P_n + P_{n-1}P_3 + P_{n-2}P_4 + \ldots + P_4P_{n-2} + P_3P_{n-1} + P_n, \quad (1)$$

se ramène à l'équation linéaire très simple,

$$P_{n+2} = \frac{1}{n+1}\binom{2n}{n} \Leftarrow \quad P_{n+1} = \frac{4n-6}{n} P_n. \quad (2)$$

Admettant donc la concordance de ces deux formules, je vais chercher à en déduire quelques conséquences.

Segner
Recurrence-
Free

Proof of the Euler Recurrence

$$P_{n+1} = \frac{4n-6}{n} P_n.$$

Multiply-decorate-biject

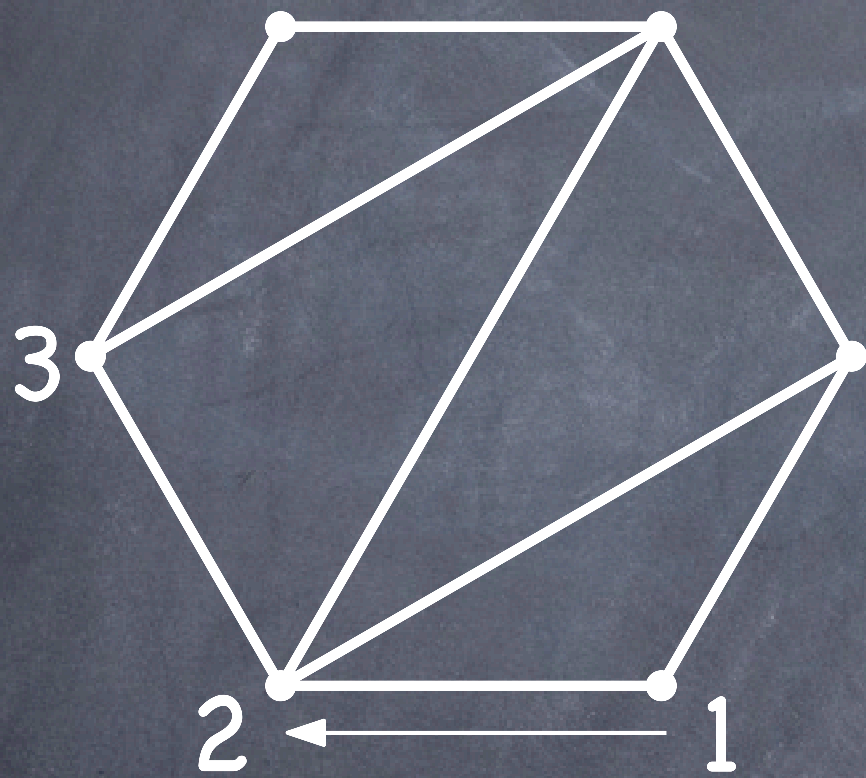
$$P_{n+1} = \frac{4n-6}{n} P_n$$

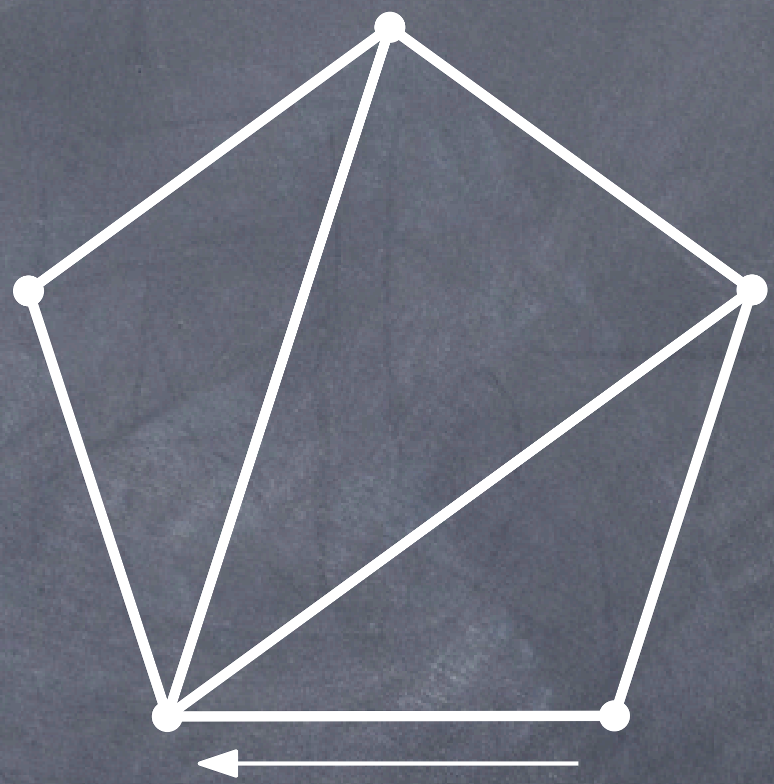
$$\Leftrightarrow$$

$$n\, P_{n+1} = 2(2n-3)\, P_n$$

Multiply-decorate-biject

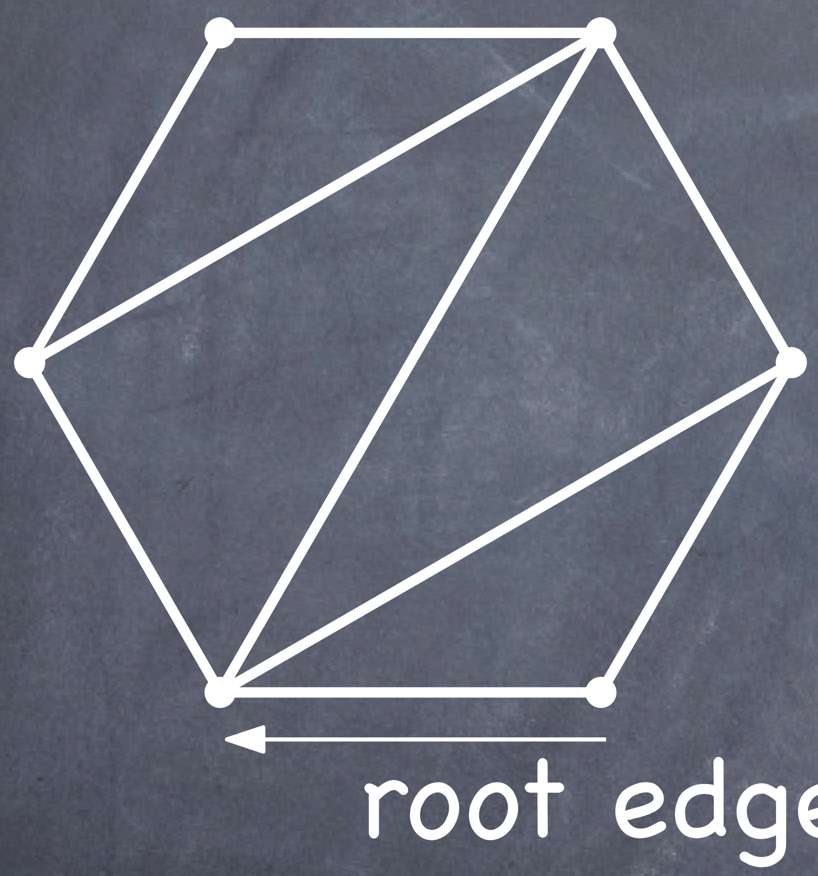
$$n\,P_{n+1} = 2(2n-3)\,P_n$$

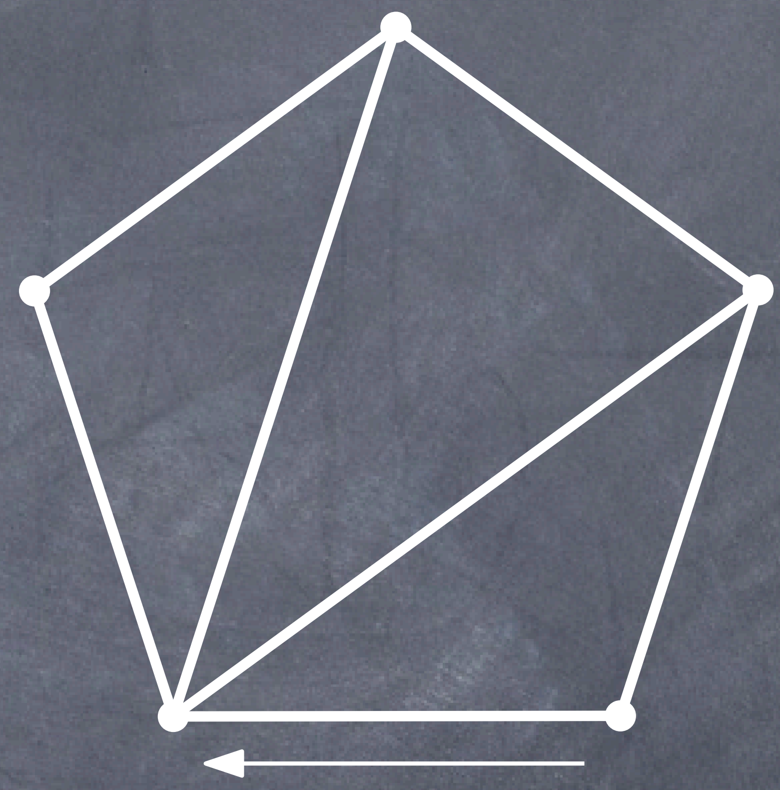




Multiply-decorate-biject

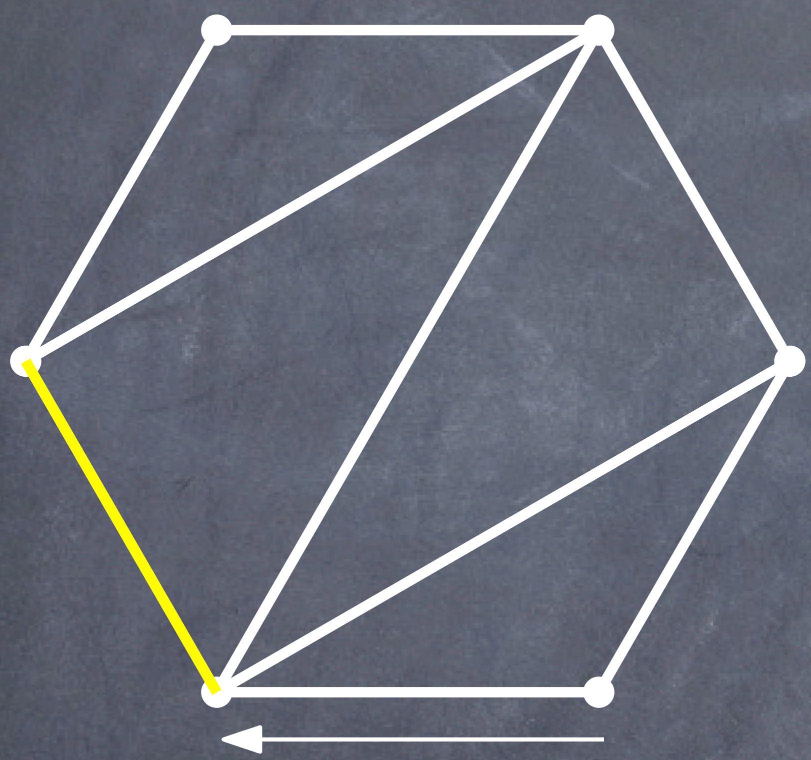
$$n\,P_{n+1} = 2(2n-3)\,P_n$$



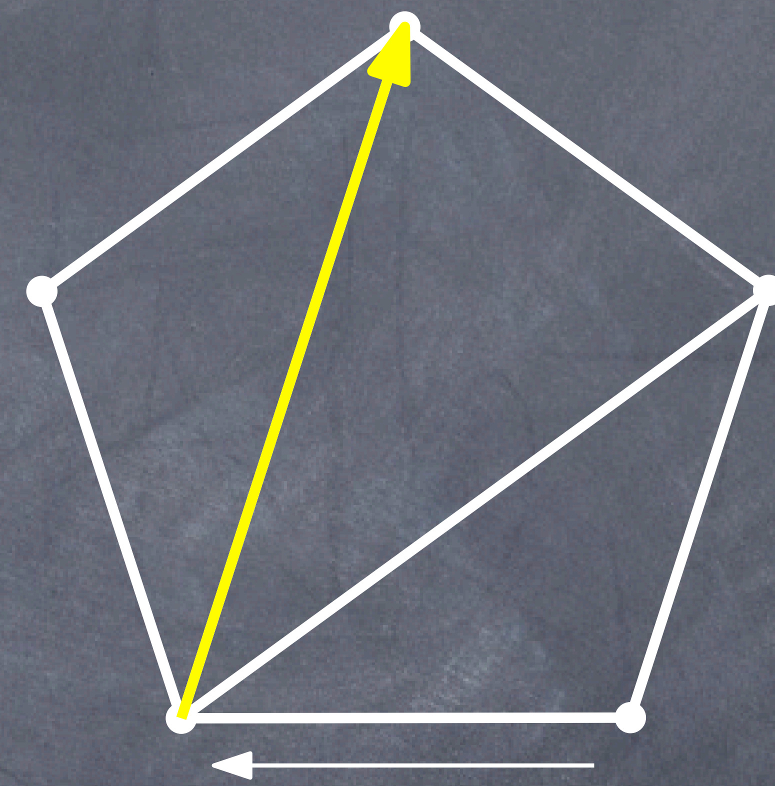


Multiply-decorate-biject

$$n\,P_{n+1} = 2(2n-3)\,P_n$$



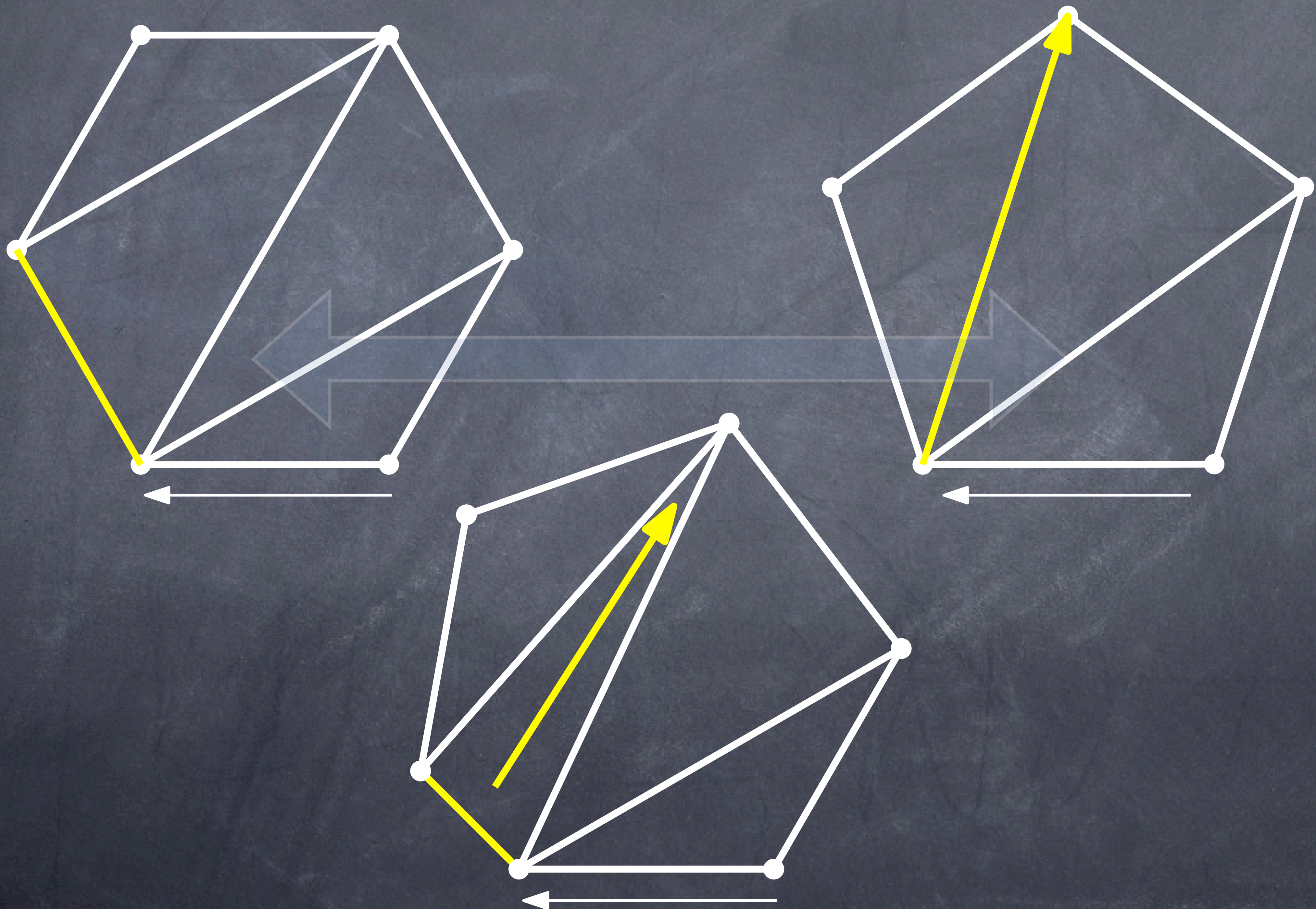
decorate boundary edge
not root edge



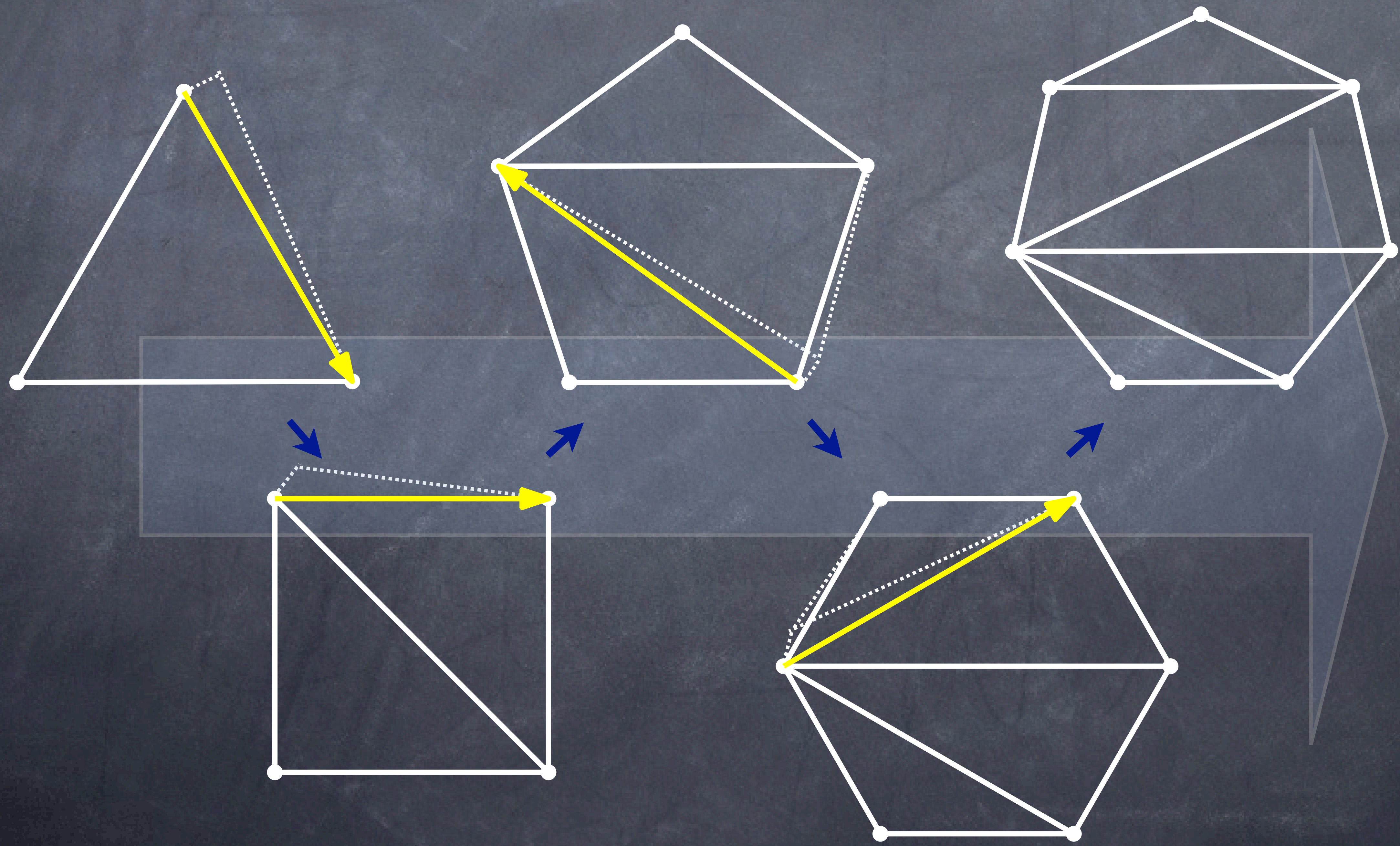
decorate any edge
and orient

Multiply-decorate-biject

$$n\, P_{n+1} = 2(2n-3)\, P_n$$



The proof of the recurrence exhibits an elegant evolution of a uniformly random triangulation: (choose random edge, orient randomly, expand)*



Why did Euler write to Goldbach about this problem?

Question may have its roots in surveying.

Why did Euler write to Goldbach about this problem?

But here is what he found quite remarkable “nicht wenig merkwürdig”

können. Ueber die Progression der Zahlen 1, 2, 5, 14, 42, 132, etc. habe ich auch diese Eigenschaft angemerket, dass $1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \text{etc.} = \frac{1 - 2a - \sqrt{(1-4a)}}{2aa}$. Also wenn $a = \frac{1}{4}$, so ist

$$1 + \frac{2}{4} + \frac{5}{4^2} + \frac{14}{4^3} + \frac{42}{4^4} + \text{etc.} = 4.$$

Euler.

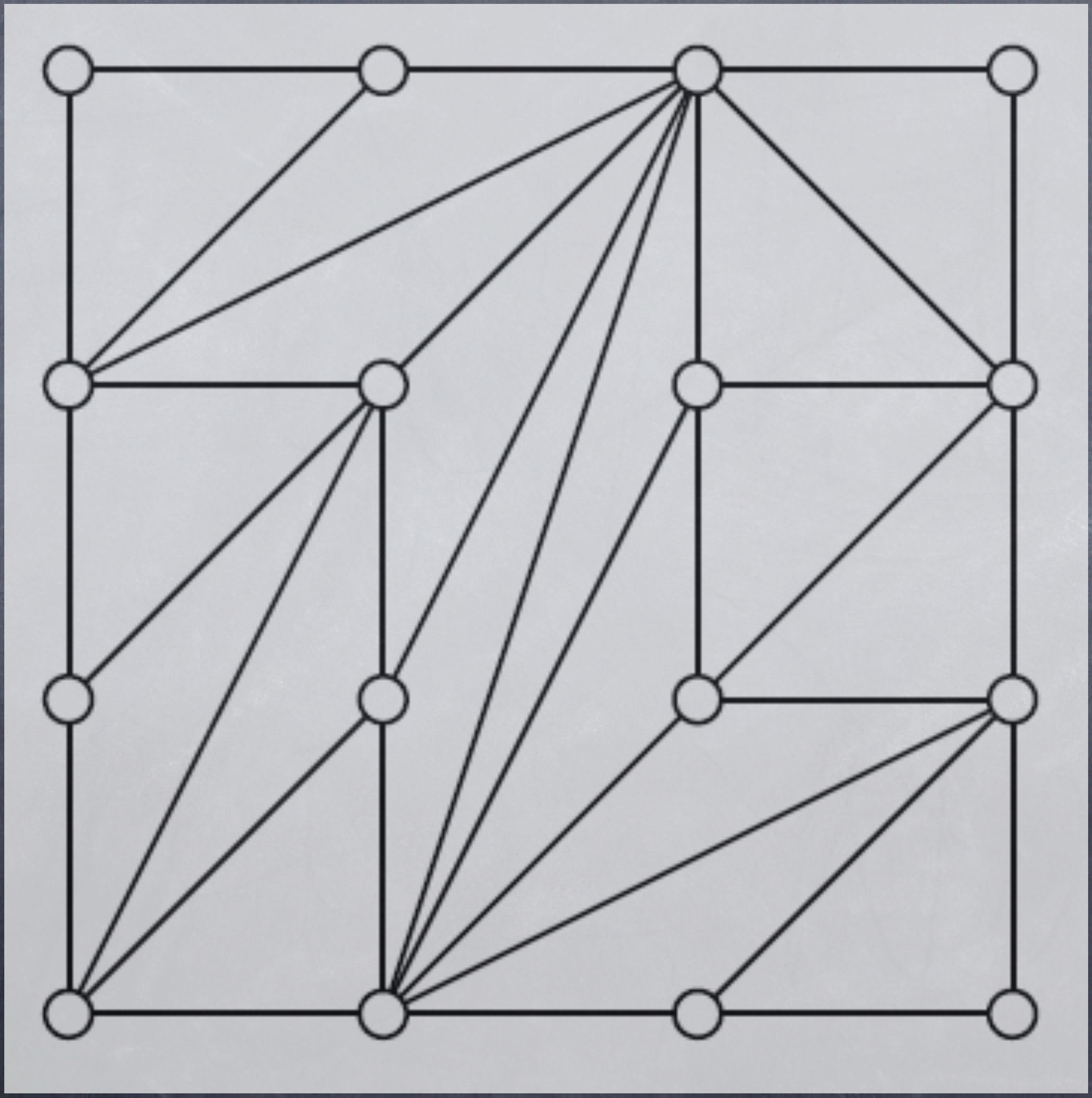
I. (Specific) Counting

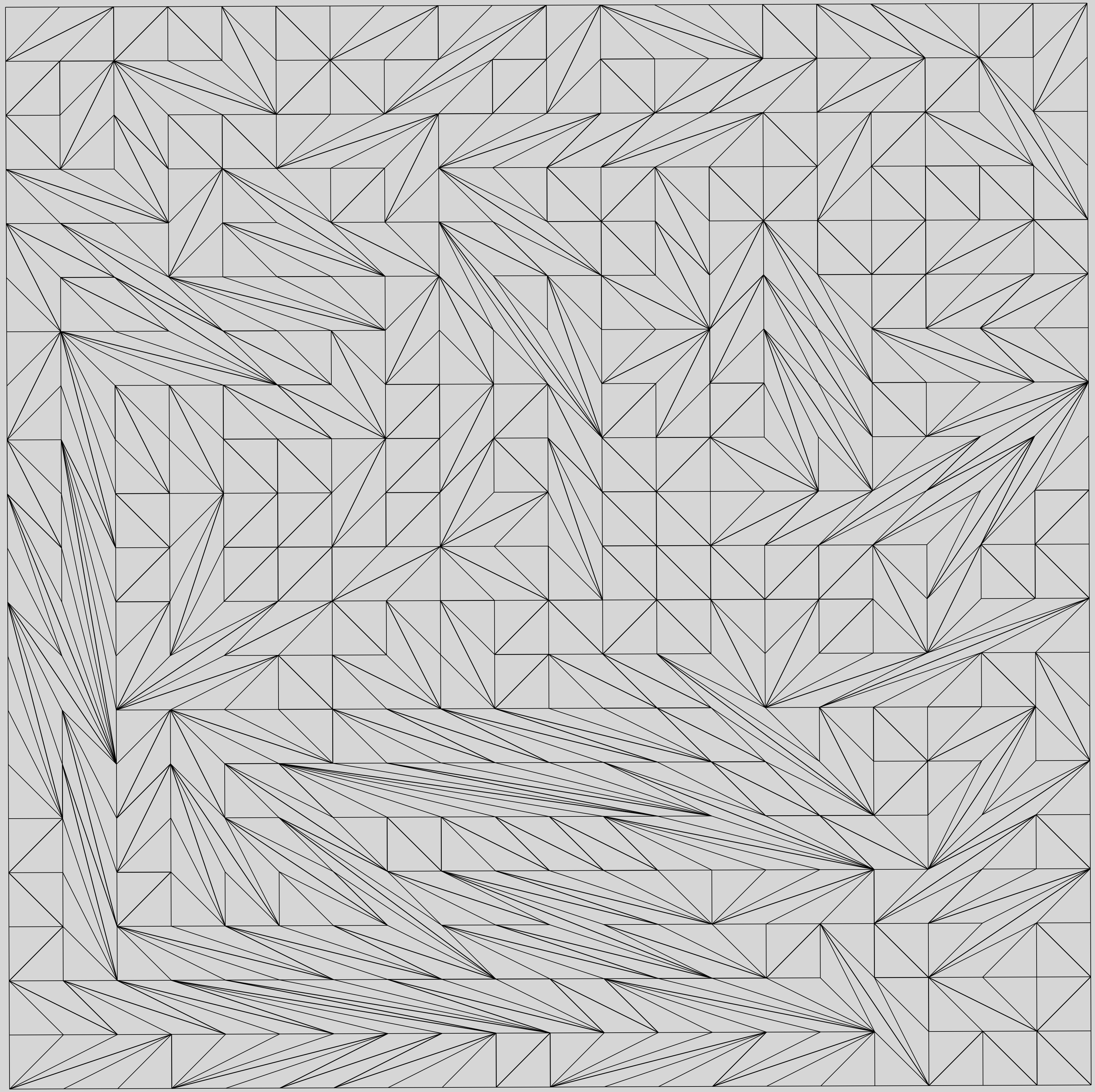
2. Lattice Triangulations

n×n Lattice: $L_{n\times n} = \{0,1,\ldots,n\}^2$

$(n+1)^2$ points

$L_{3\times 3}$





Bounds on tr($L_{n\times n}$)

- $O(64^{n^2})$ [Orevkov'99]
- $O(8^{n^2})$ [Anclin'02]
- $\Omega(4.15^{n^2})$ [Kaibel,Ziegler'02]
- $O(6.86^{n^2})$ [Matoušek,Valtr,Welzl'06]

6.86^{n^2} "=" F_{4n^2-1} (Fibonacci Number)

Open Problem

Regular Triangulations of the Lattice

Is the number of regular (i.e. liftable) triangulations of $L_{n\times n}$ exponentially smaller than $tr(L_{n\times n})$?

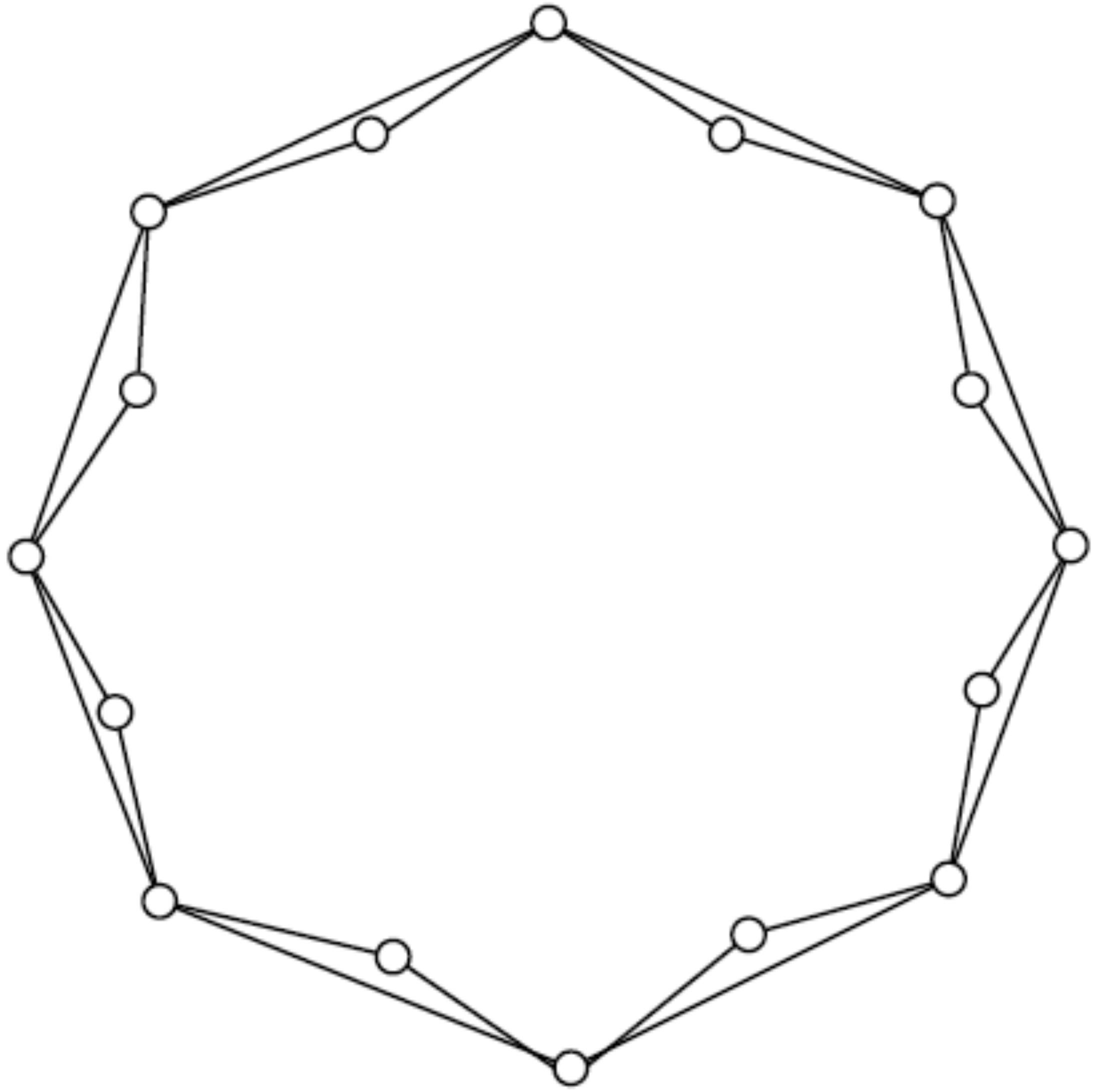
[Kaibel,Ziegler]

$$\text{reg-tr}(L_{n\times n}) \leq c^{(n+1)^2}\, tr(L_{n\times n})$$

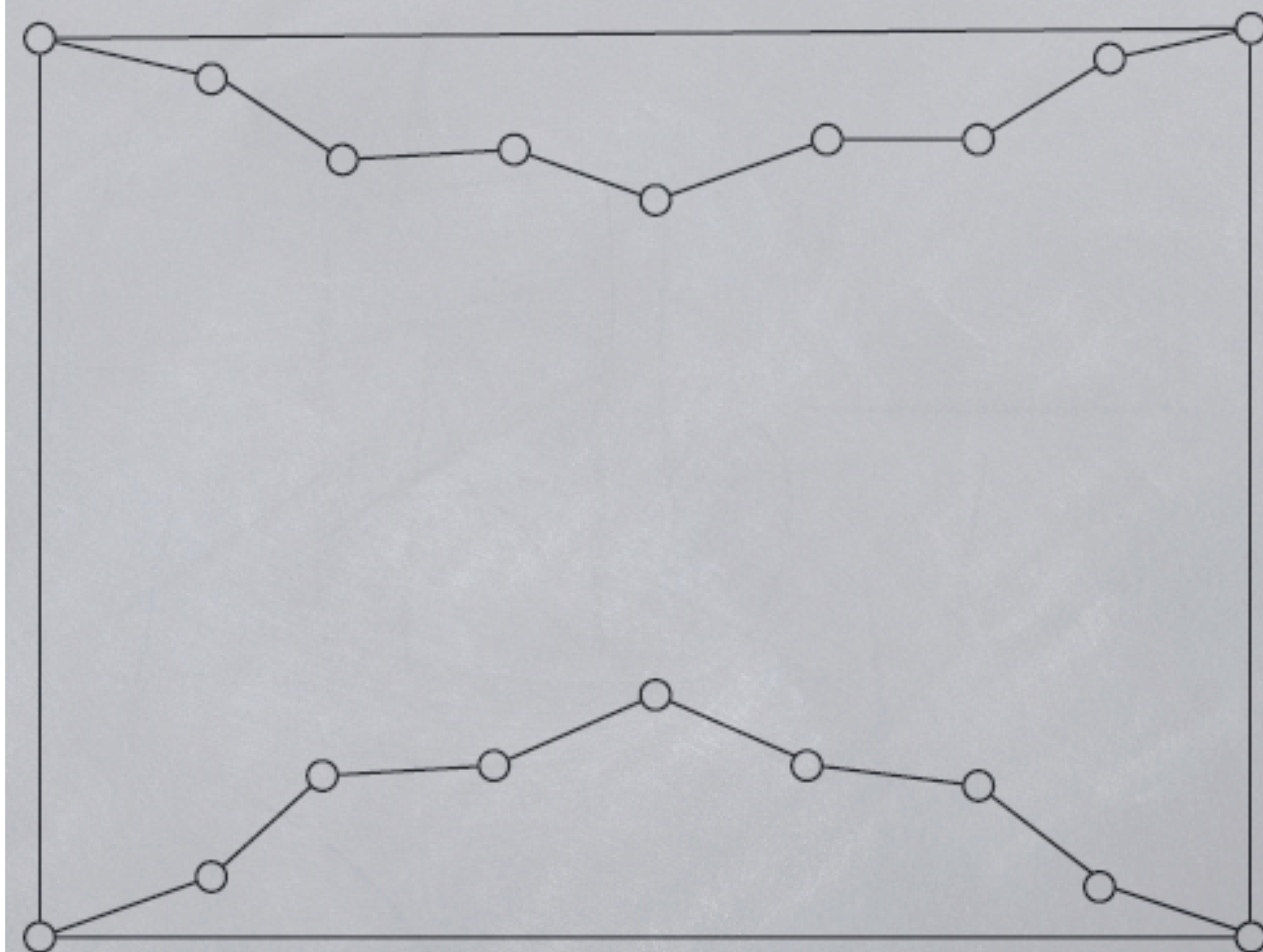
for $c<1$?

I. (Specific) Counting

3. Two more Examples



$O(3.47^n)$ triangulations
[Hurtado,Noy'97]



$\Omega(8.48^n)$ triangulations
[Aichholzer,Hackl,Krasser
Huemer,Hurtado,Vogtenhuber'06]

Convex Position is not Extremal

$tr(n) := \max_{|P|=n} tr(P)$

with P in general position

$tr_{min}(n) := \min_{|P|=n} tr(P)$

$? \lesssim_n tr_{min}(n) \lesssim_n 3.47^n$

$\approx_n 4^n$

$< tr(G_n) <$

$8.48^n \lesssim_n tr(n) \lesssim_n ?$

II. Extremal Counting

1. Number of Triangulations

Upper Bound on tr(n)

An upper bound of $n^{2(3n-6)} = 2^{O(n \log n)}$ is easy.

Encode a triangulation by listing the at most 3n-6 edges in a sequence of numbers in {1,2,...,n} of length at most 2(3n-6).

Upper Bound on tr(n)

An upper bound of $n^{2(3n-6)} = 2^{O(n \log n)}$ is easy.

Late 70's: David Avis raised the question and conjectured a bound of c^n; (see also related question by Newborn and Moser on crossing-free spanning cycles)

[Ajtai, Chvátal, Newborn, & Szemerédi`82] proved and employed the Crossing Lemma for

$tr(n) \leq 10000000000000^n$

Issue Resolved - Except for the Base Constant

$\mathrm{tr}(n)$	$\leq$	$(10^{13})^n$	[Ajtai *et al.*'82]
		$173\,000^n$	[Smith'89]
		276.8^n	[Denny,Sohler'97]
		59^n	[Santos,Seidel'03]
		43^n	[Sharir,Welzl'06]

$\mathrm{tr}(n) \leq 30^n$ [Sharir, Sheffer, Welzl`09]

$\mathrm{tr}(n) \geq 8.48^n$ [Aichholzer et al.`06]

II. Extremal Counting

2. Random Triangulations

vertices of degree 3

with link to number of triangulations

Triangular Convex Hull

Fix a set H of vertices of a triangle Δ.
For point sets $P \subseteq \Delta$, let $P^+ := P \cup H$.

$$tr^+(n) := \max_{|P|=n,\ P \subseteq \Delta} tr(P^+)$$

$tr(n) \leq tr^+(n)$, since for any set P, let Q be a scaled translate of P so that $Q \subseteq \Delta$. Then $tr(P) = tr(Q) \leq tr(Q^+)$.

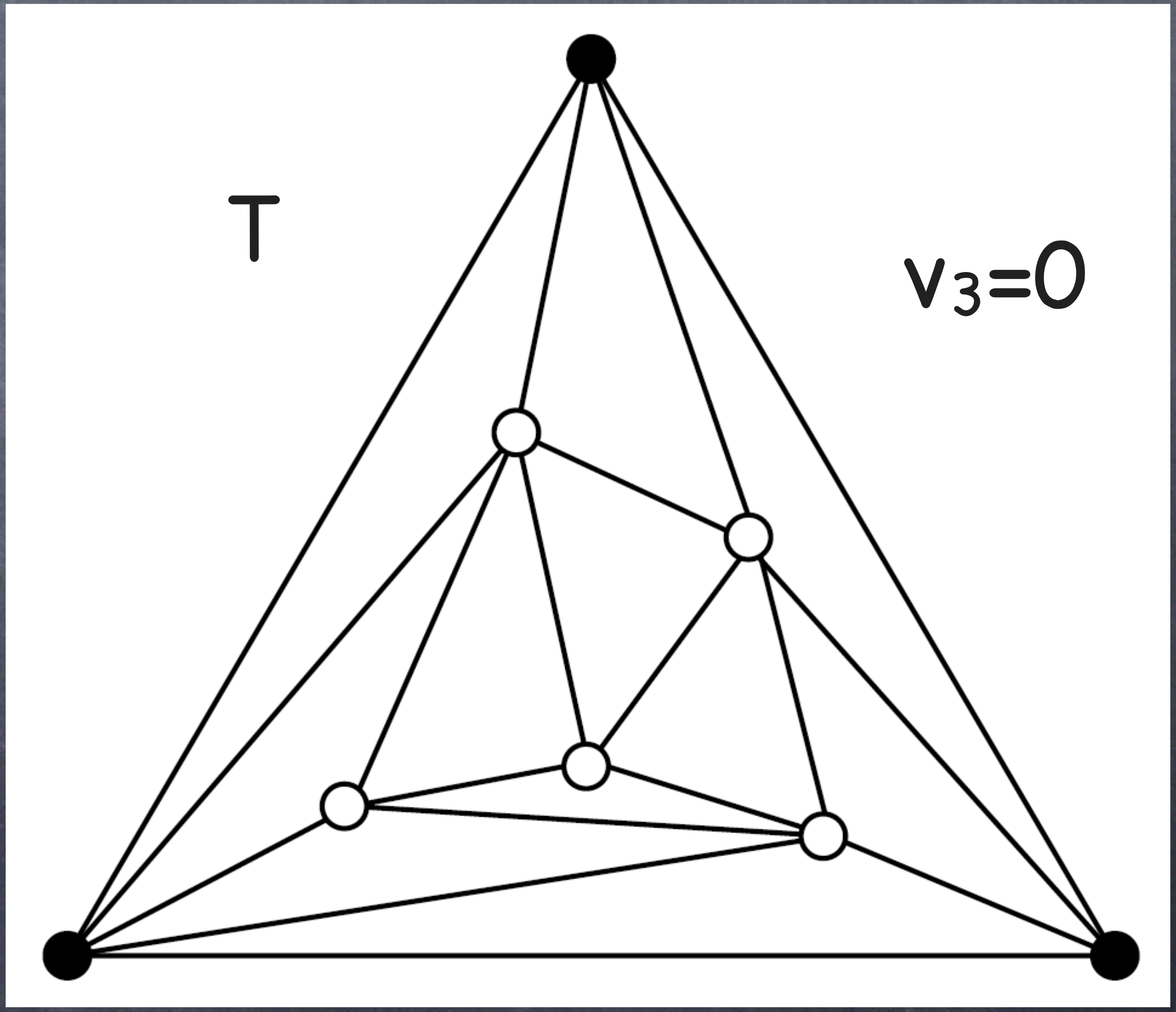
Degree 3 Vertices

$v_3 = v_3(T)$:= number of **inner** vertices of degree 3 in triangulation T of P^+.

$v_3 \geq 0$

$\hat{v}_3 = \hat{v}_3(P) := \mathbf{E}[v_3]$

with expectation over uniform distribution of all triangulations of P^+.



Degree 3 Vertices

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$\hat{v}_3 = \hat{v}_3(P) := \mathbf{E}[v_3]$

with expectation over uniform distribution of all triangulations of P^+.

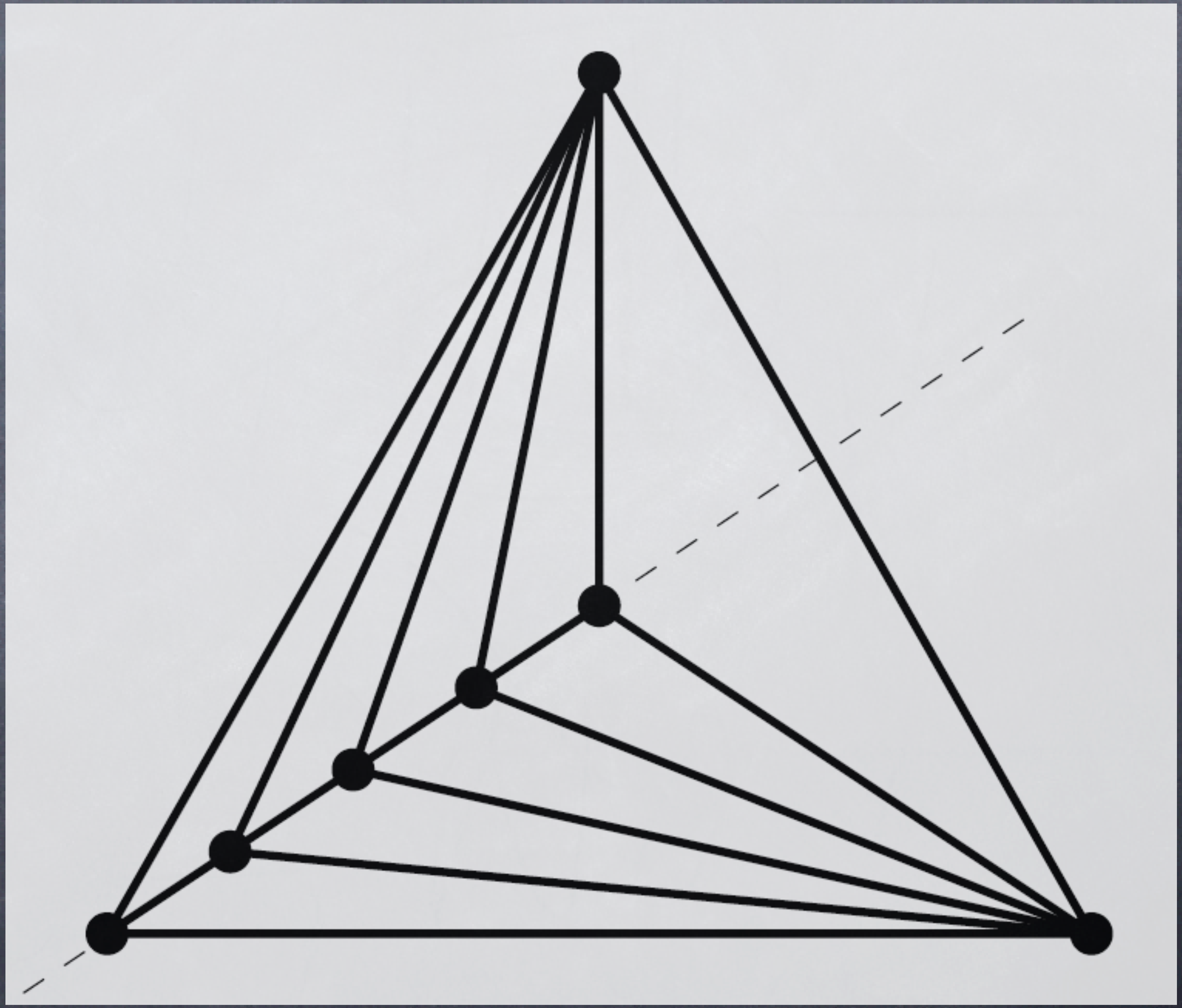
$\hat{v}_3/n =$ **Prob**[random inner vertex in a random triangulation has degree 3]

Can we separate this probability away from 0 (independently from n)?

No!

unique
triangulation with
one vertex of
degree 3

$\hat{v}_3/n = 1/n \to 0$



Yes!

Lemma: If P^+ is in general position, then

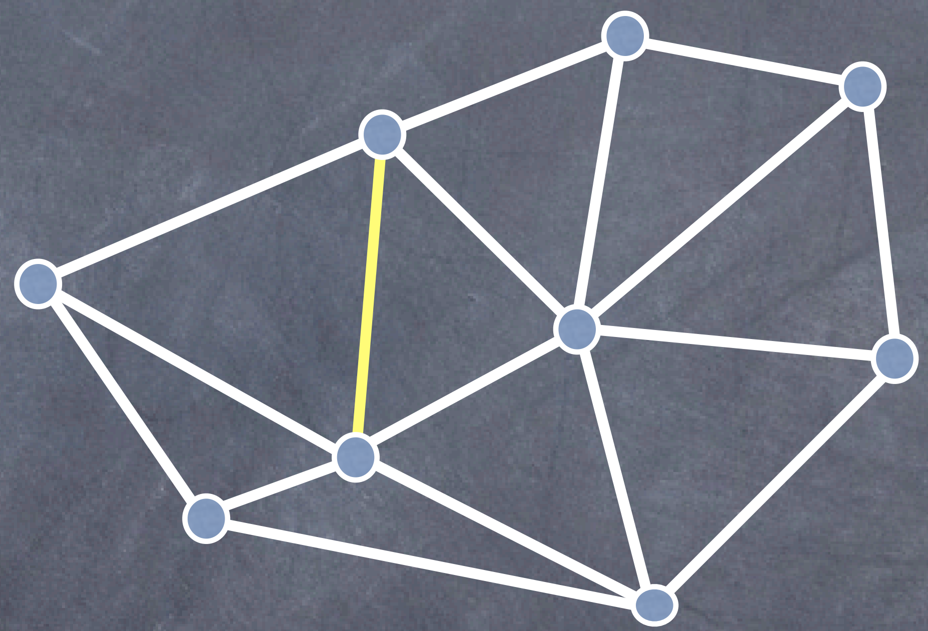
$$\hat{v}_3 \geq n/30$$

Lemma: If $\hat{v}_3 \geq \delta|P|$ for all P with P^+ in general position, then $tr^+(n) \leq (1/\delta)^n$ for all n.

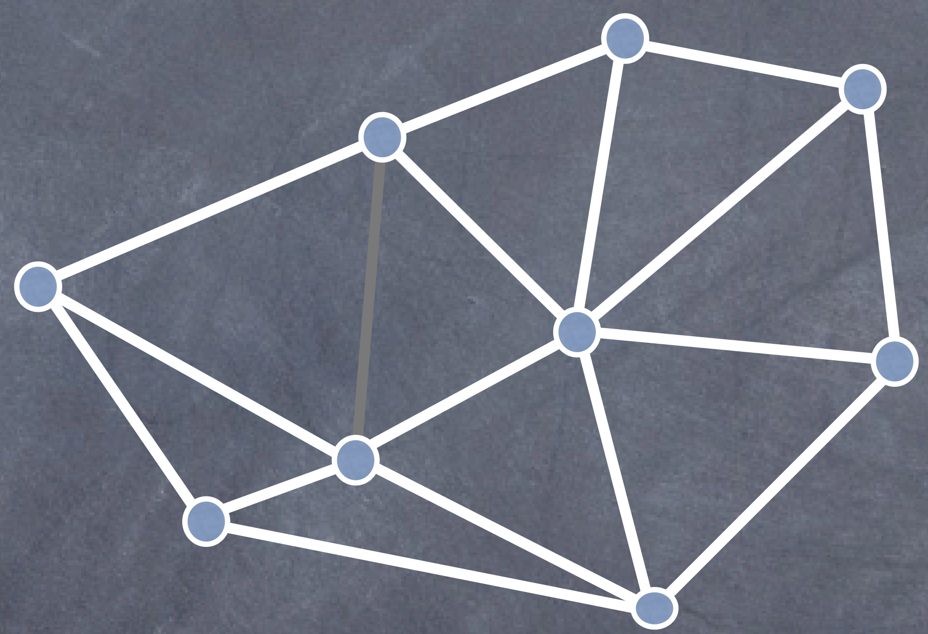
$$\Rightarrow\ tr^+(n) \leq 30^n$$

$$\Rightarrow\ tr(n) \leq 30^n$$

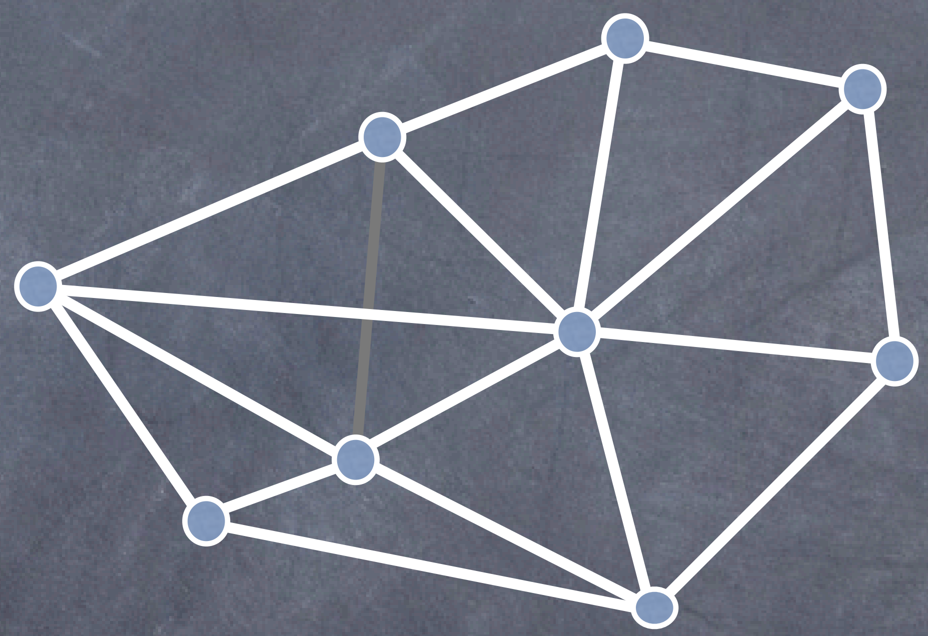
Edge Flip in Triangulation



Edge Flip in Triangulation

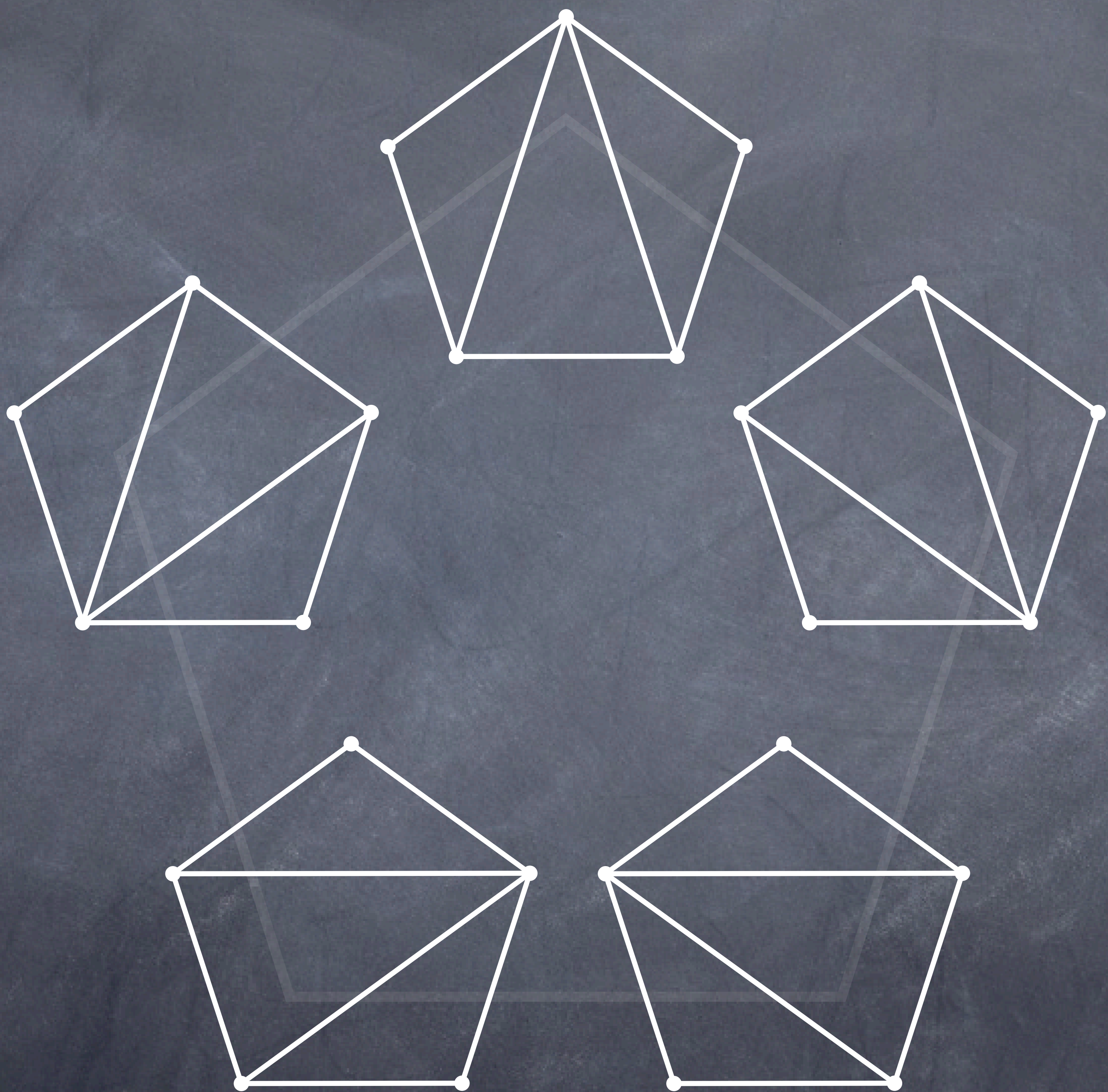


Edge Flip in Triangulation



Set of triangulations is connected via edge flips.

...



Lemma: If P^+ is in general position, then

$$\hat{v}_3 \geq n/30$$

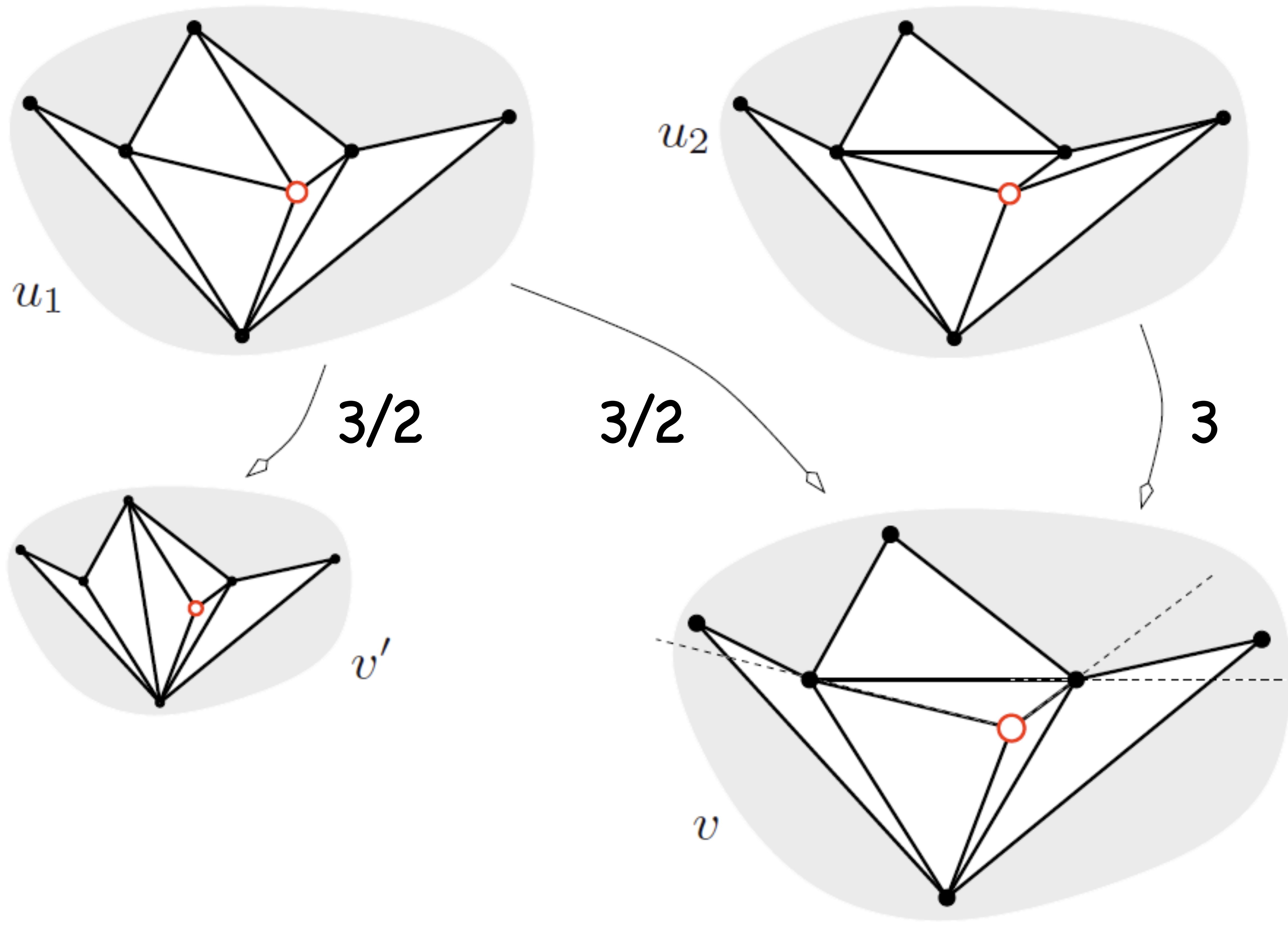
Proof uses (dis)charging among vertices as in 4-color theorem, except that we have to charge here across all triangulations of P^+.

First, every vertex starts with a charge of 1.

Second, we distribute within each triangulation so that a vertex of degree i has charge $\leq(7-i)$.

Finally, every vertex uniformly distributes its charge to all degree 3 vertices it can be flipped down to.

Show: No degree 3 vertex gets charge exceeding 30.


u_1
u_2
3/2
3/2
3
v'
v

Lemma: If $\hat{v}_3 \geq \delta|P|$ for all P with P^+ in general position, then $tr^+(n) \leq (1/\delta)^n$ for all n.

Proof.

$$\delta n \cdot \mathrm{tr}^+(P) \leq \hat{v}_3(P) \cdot \mathrm{tr}^+(P) = \sum_{q \in P} \mathrm{tr}^+(P \setminus \{q\}) \leq n \cdot \mathrm{tr}^+(n-1)$$

$$\Rightarrow \quad \mathrm{tr}^+(n) \leq \frac{1}{\delta} \cdot \mathrm{tr}^+(n-1)$$

Open Problem

$\hat{v}_3$ versus Number of Triangulations

δ^* := supremum over all δ such that $\hat{v}_3 \geq \delta|P|$ for all sufficiently large sets P (P^+ in gen. pos.).

c^* := infimum over all c such that $tr^+(n) \leq c^n$ for all sufficiently large n.

We know that $c^* \leq (1/\delta^*)$.

$$c^* = (1/\delta^*)\ ?$$

III. Algorithmic Counting

1. Counting Triangulations by Enumeration

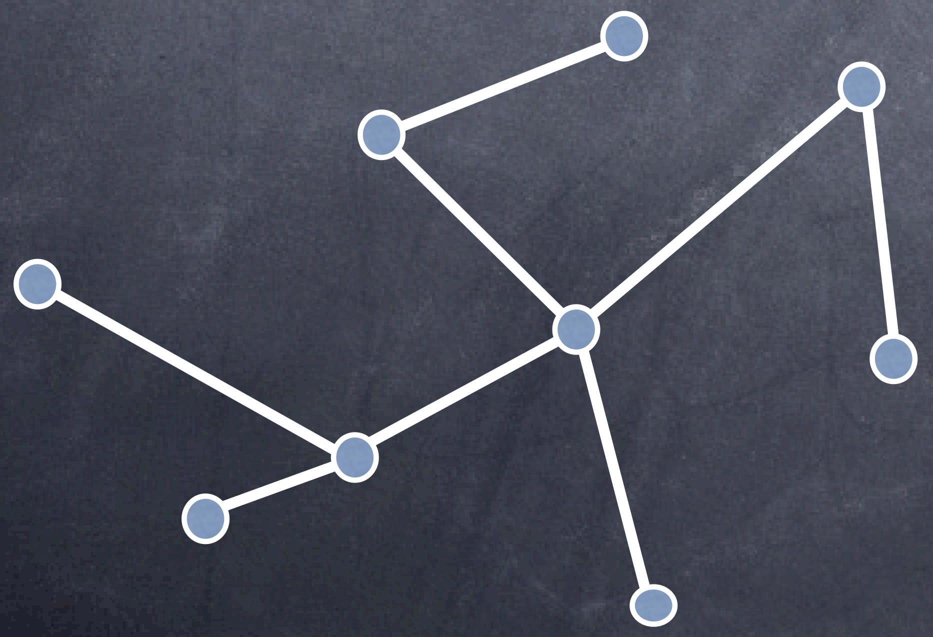
Counting by Enumeration

The number tr(P) of triangulations of an n-element point set P can be computed in time

$$O(tr(P) \cdot poly(n))$$

by enumerating all of them.

[Avis,Fukuda`96]

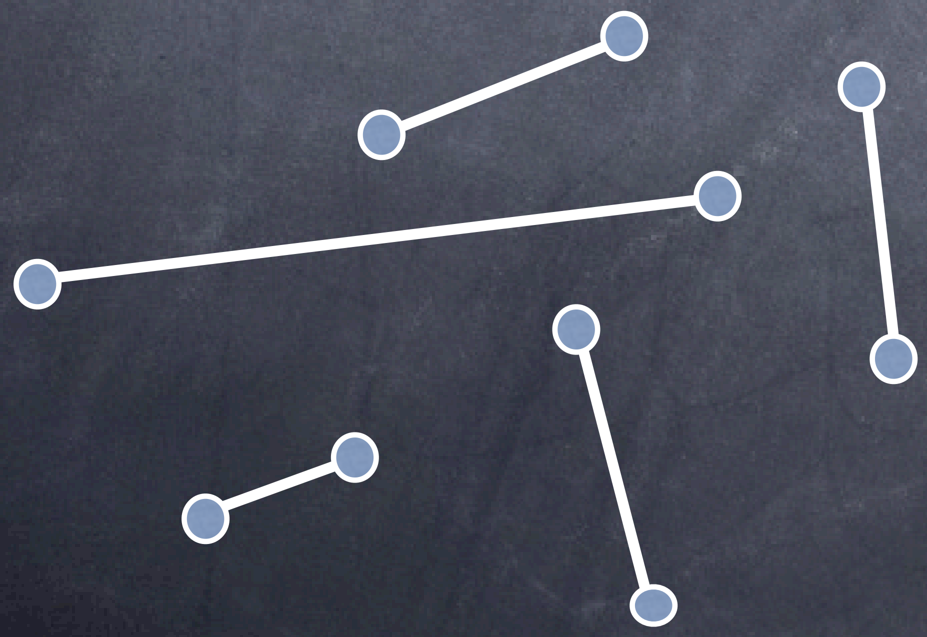


… same for spanning trees (in time $O(st(P) \cdot poly(n))$).

Open Problem

Enumerating Crossing-Free Perfect Matchings

Can we enumerate all crossing-free perfect matchings of an n-point set in time O(pm(P) poly(n))?



Is possible for set of all maximal crossing-free matchings.

Status of Algorithmic Counting

No #P results known.

No polynomial counting results known.

(Except for counting stacked triangulations via dynamic programming.)

III. Algorithmic Counting

2. All Crossing-Free Graphs with Exponential Speed-up

Need: (i) More extremal counting and (ii) constrained Delaunay triangulations.

Exponential Speed-up

For a set P of n points we can compute pg(P) in time $O(0.36^n \, pg(P))$.

[Razen,W.`08]
[Katoh,Tanigawa`08]

I.e. exponentially faster than the number computed.

All Crossing-Free Graphs versus Triangulations

obvious estimates:

$$\mathrm{tr}(P) \le \mathrm{pg}(P) \le 2^{3n-6}\,\mathrm{tr}(P) \le 8^n\,\mathrm{tr}(P)$$

can be improved to:

needs general position

$$2.82^n\,\mathrm{tr}(P) \le \mathrm{pg}(P) \le 7.98^n\,\mathrm{tr}(P)$$

[Razen,Snoeyink,W.`08]

Open Problem

All Crossing-Free Graphs versus Triangulations

Is pg(P)/tr(P) minimized for point sets in convex position?

Note: $pg(P)/tr(P) \geq 2.82^n$ is known and $pg(G_n)/tr(G_n) \approx_n 2.914...^n$

[Flajolet,Noy`99]

Basic Idea

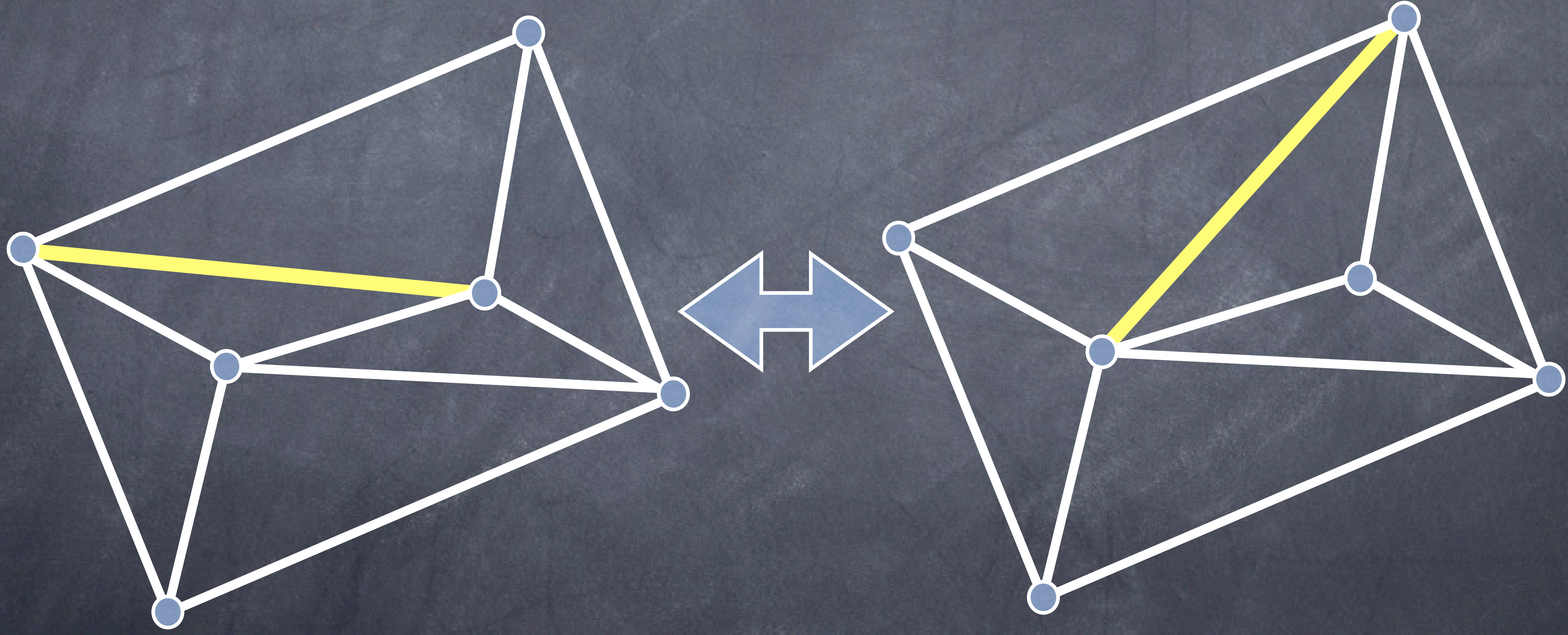
$$2.82^n \, tr(P) \le pg(P)$$

$$\Rightarrow tr(P) \le 0.36^n \, pg(P)$$

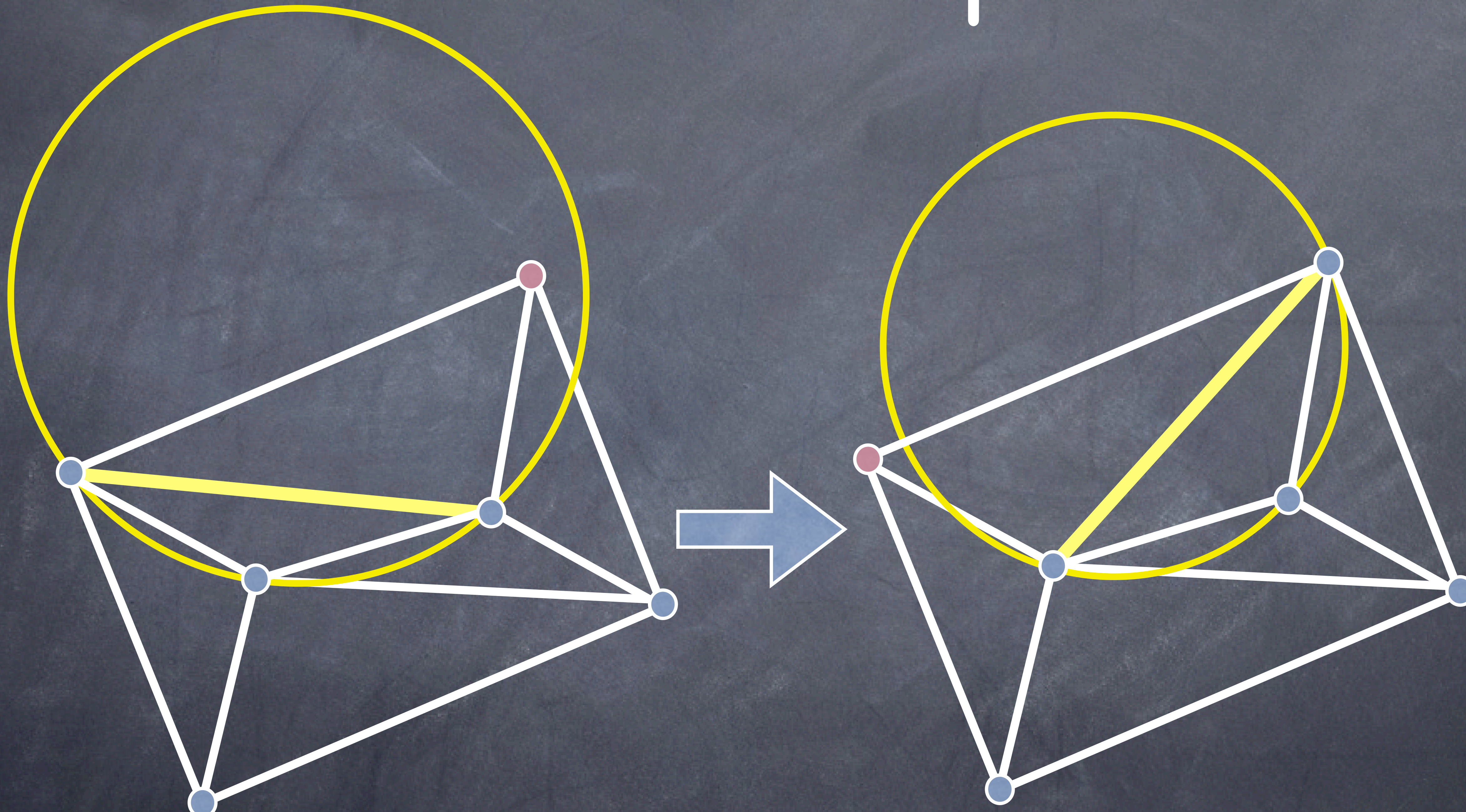
and

pg(P) can be computed in time $O(tr(P)\, poly(n))$.

Back to Flips



Lawson Flips



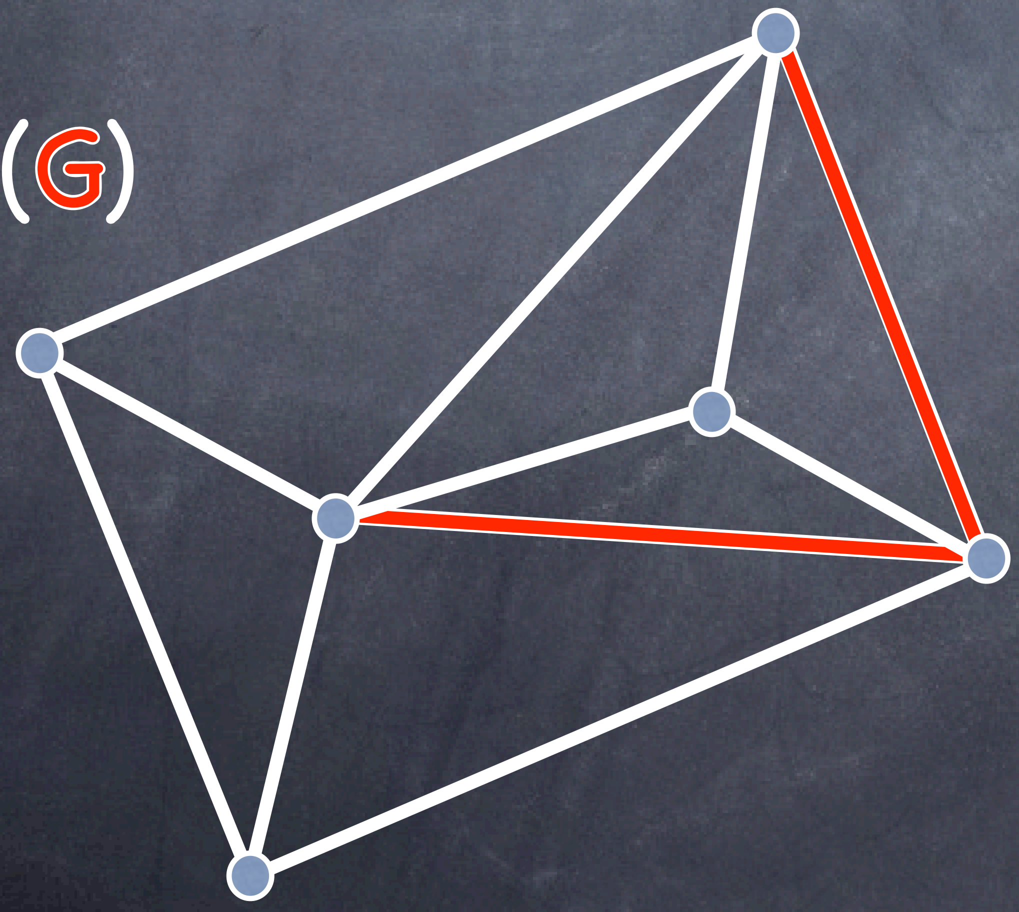
Applying Lawson Flips ...

assume general position

Eventually gives the Delaunay Triangulation.

If edges G are constrained as unflippable, eventually gives Constrained Delaunay Triangulation CD(G)

CD(G)

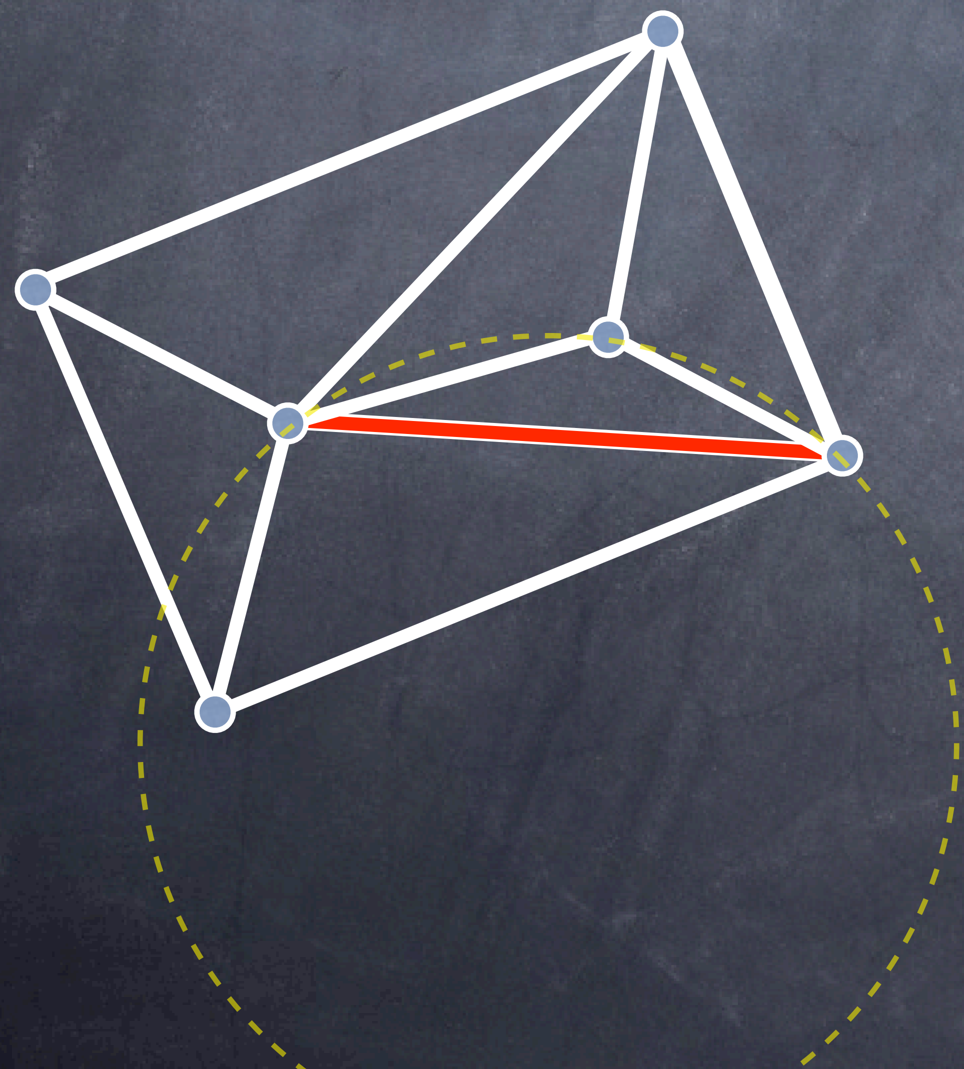


CD(G) does not depend on starting triangulation T⊇G and choice of flips!

$$pg(P) = \sum_T 2^{m-|L(T)|}$$

Consider the map $G \mapsto CD(G)$

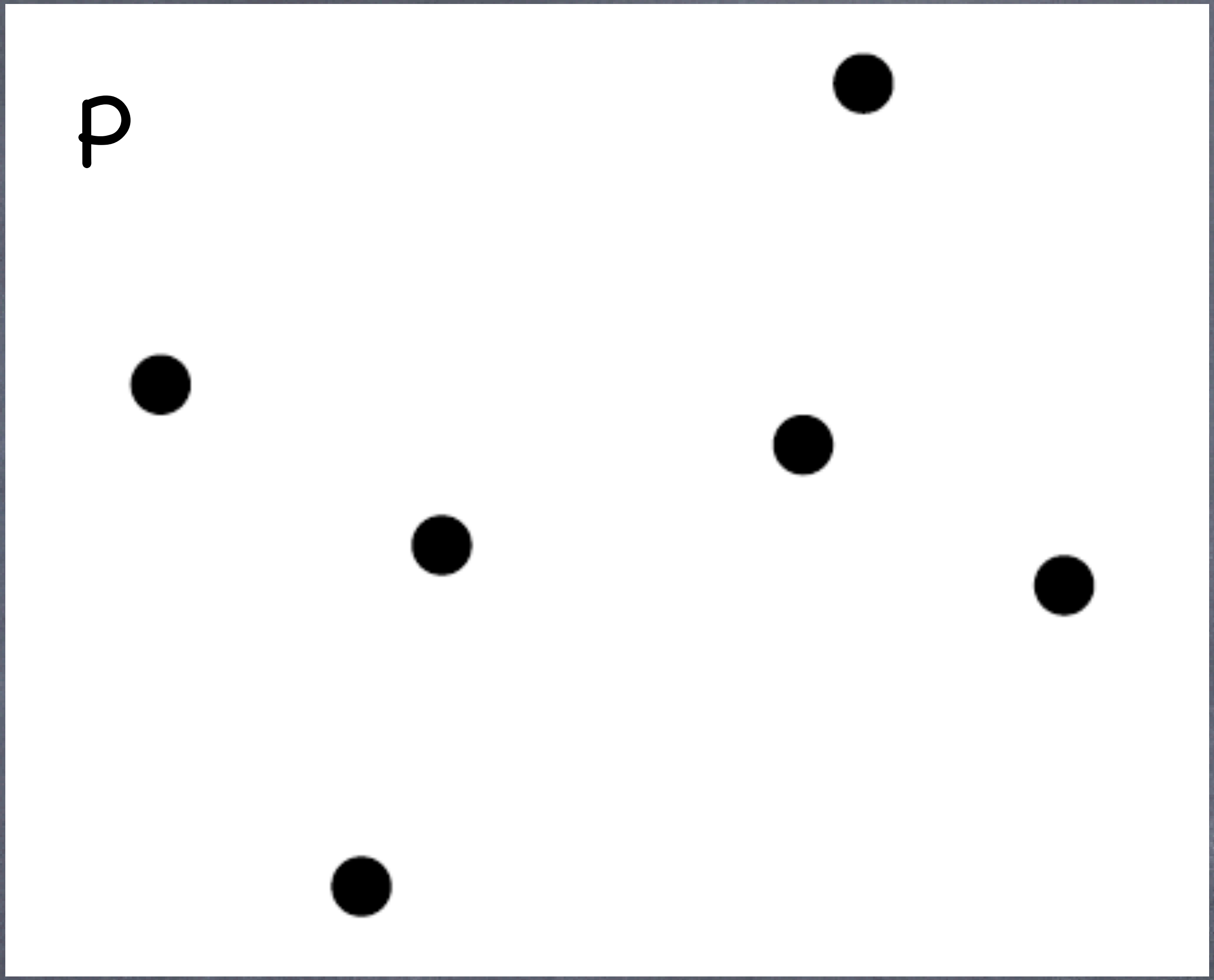
Then, for triangulation T, $|CD^{-1}(T)| = 2^{m-|L(T)|}$



m ... number of edges

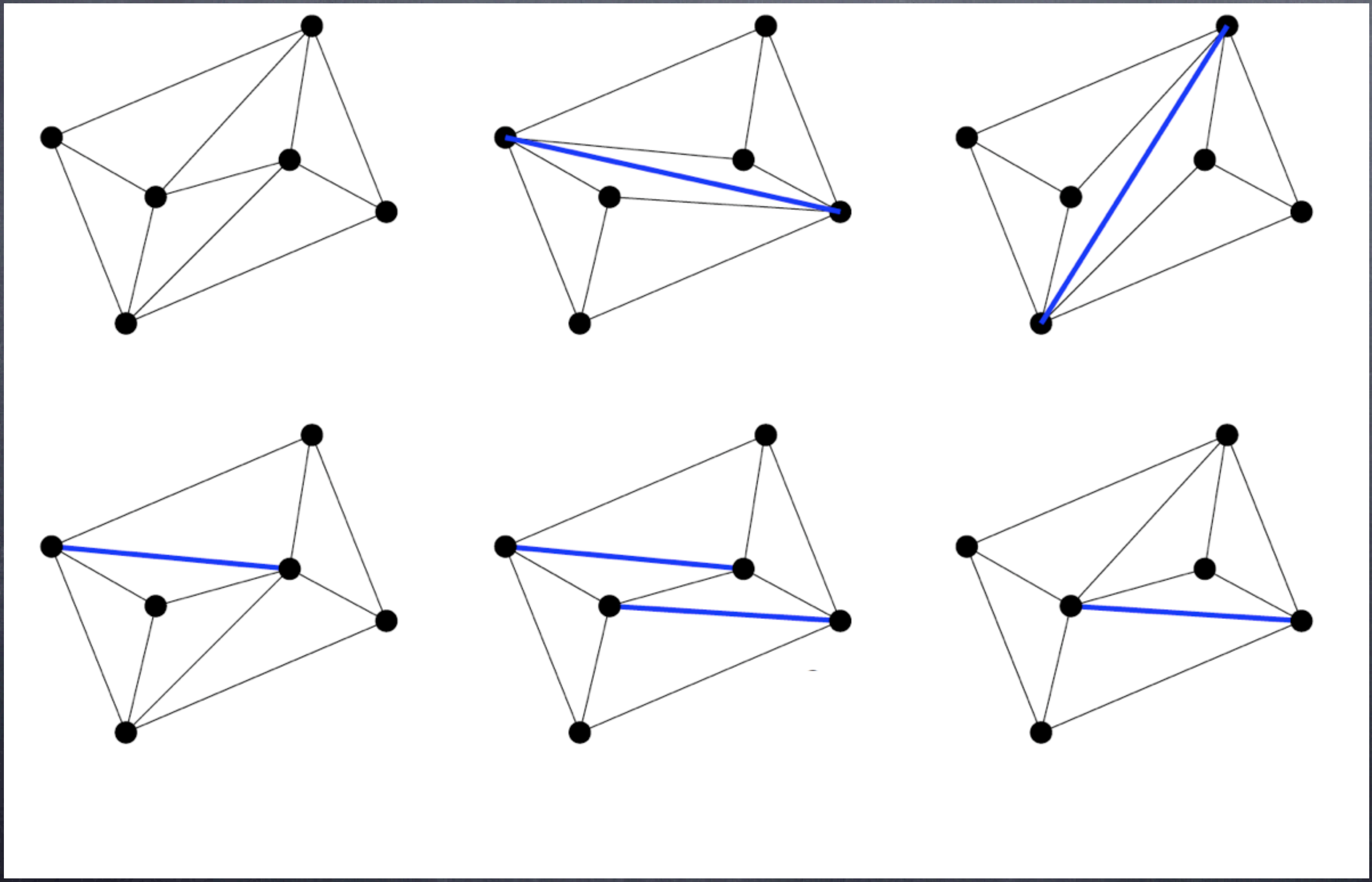
L(T) ... candidates for Lawson flips

$G \mapsto T$ iff $L(T) \subseteq G \subseteq T$

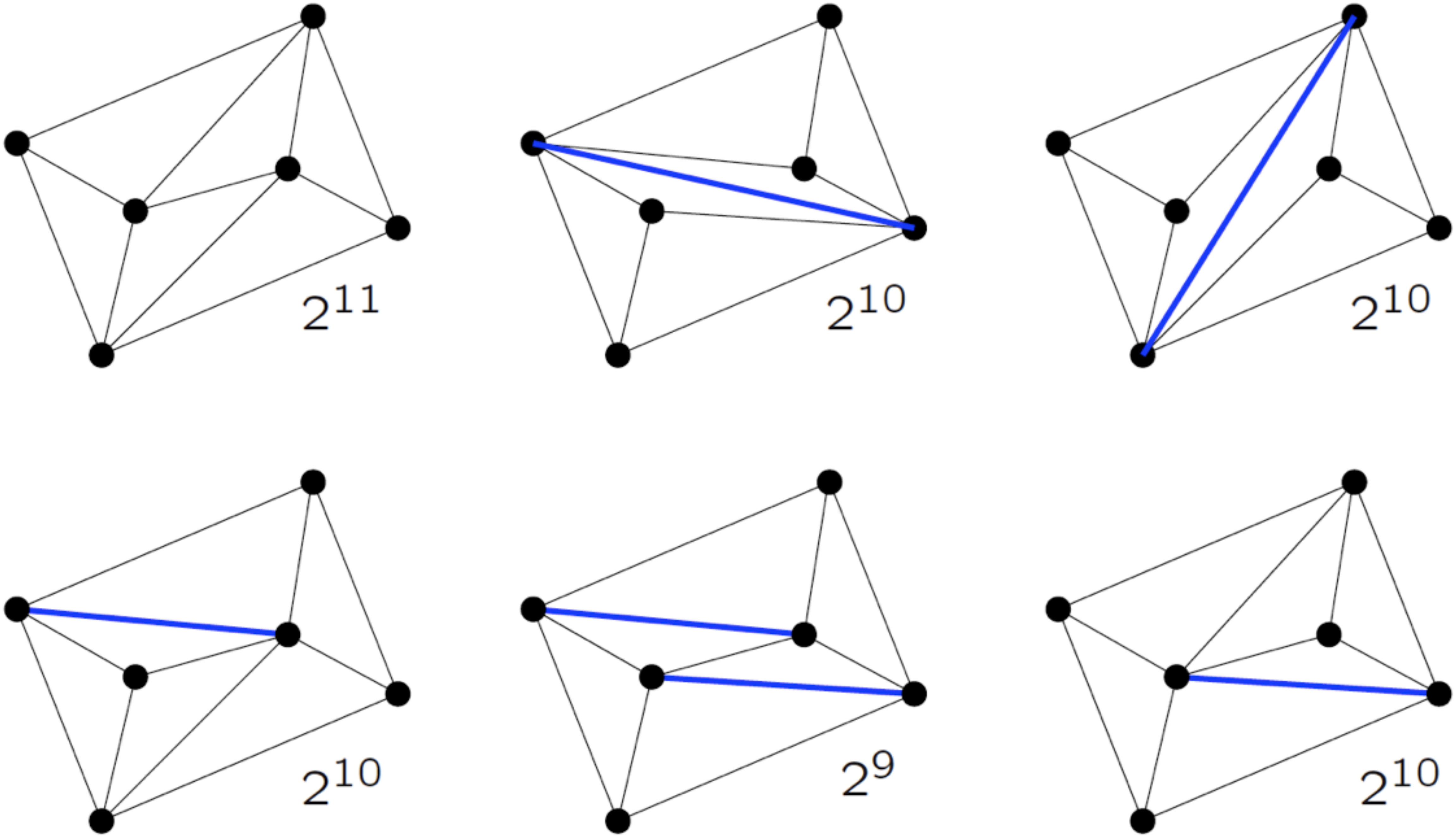


What is the number, pg(P), of all crossing-free graphs on P?

Consider all triangulations and mark candidate edges for Lawson flips.



… and add up these numbers.



pg(P) = $2^{11} + 2^{10} + 2^{10} + 2^{10} + 2^{9} + 2^{10} = 6656$

Open Problem

Always Many Crossing-Free Spanning Trees?

Is there a constant $c>1$ such that $st(P) \geq c^n\, tr(P)$ for every large enough n-point set.

st(P) number of crossing-free spanning trees

(Would imply counting of crossing-free spanning trees with exponential speed-up.)

Open Problem

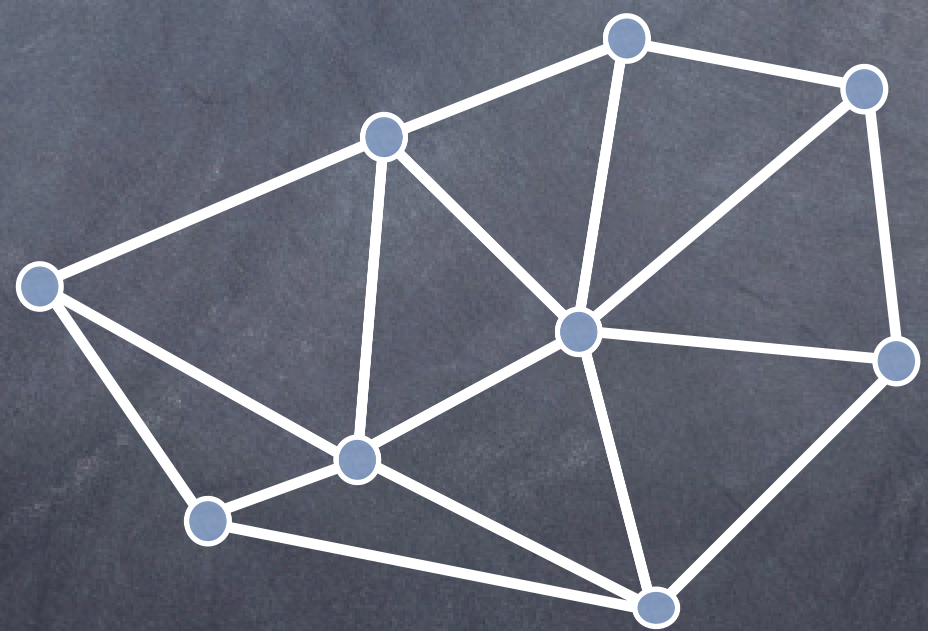
Triangulations with exponential speed-up

Can we compute tr(P) with exponential speed-up, i.e. in time $O(c^{|P|}\, tr(P))$ for a constant $c<1$?

Open Problem

Flip-Markov Chain

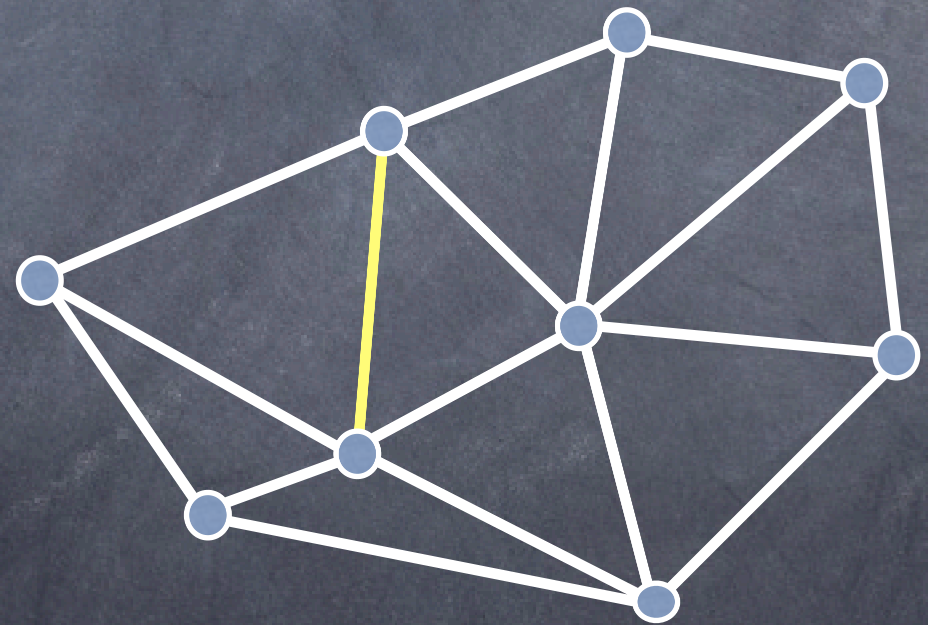
What is the mixing rate of the Flip-Markov Chain on an arbitrary n-point set?



Open Problem

Flip-Markov Chain

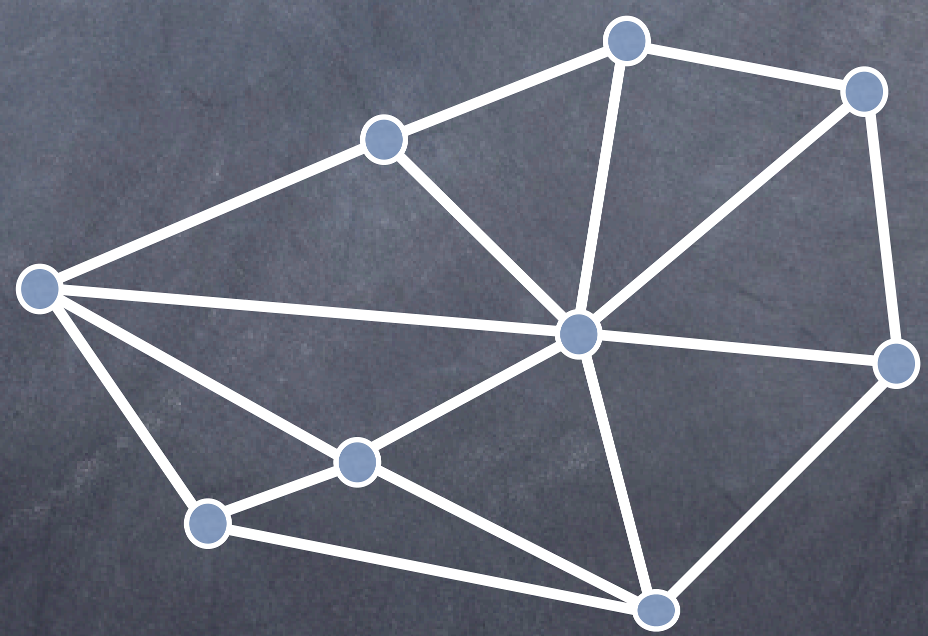
What is the mixing rate of the Flip-Markov Chain on an arbitrary n-point set?



Open Problem

Flip-Markov Chain

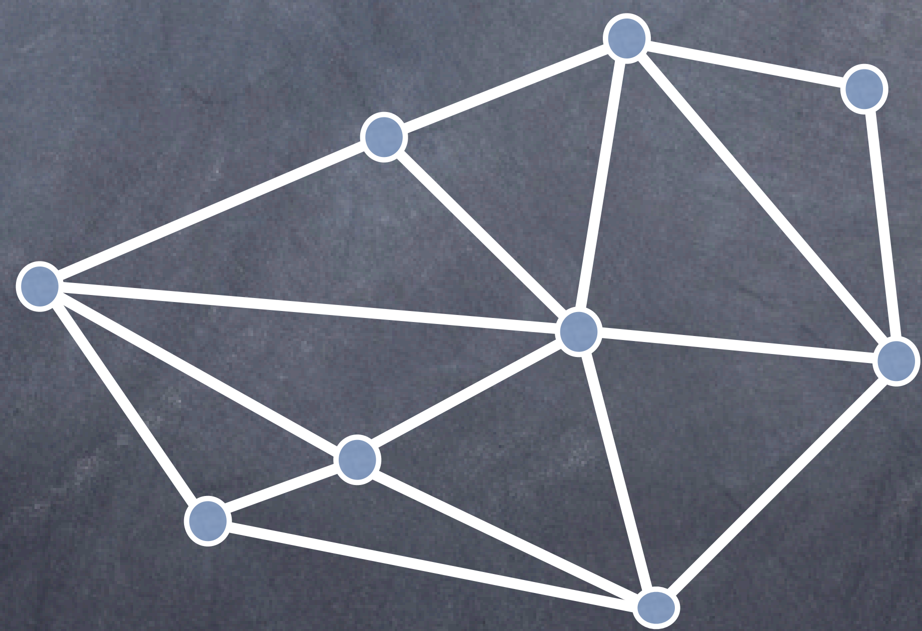
What is the mixing rate of the Flip-Markov Chain on an arbitrary n-point set?



Open Problem

Flip-Markov Chain

What is the mixing rate of the Flip-Markov Chain on an arbitrary n-point set?



and so on

Open Problem

Flip-Markov Chain

uses
$\hat{v}_3/n=\Omega(1)$

What is the mixing rate of the Flip-Markov Chain on an arbitrary n-point set?

Polynomial mixing rate would give polynomial approximate counting of triangulations.

What is the mixing rate of the Flip-Markov Chain on the $(n\times n)$-lattice?

(Known to be polynomial for points in convex position.)

[Molley,Reed,Steiger`98] [McShine,Tetali`98]