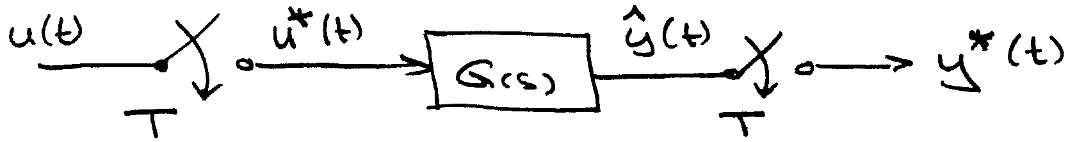
Modified z-Transform

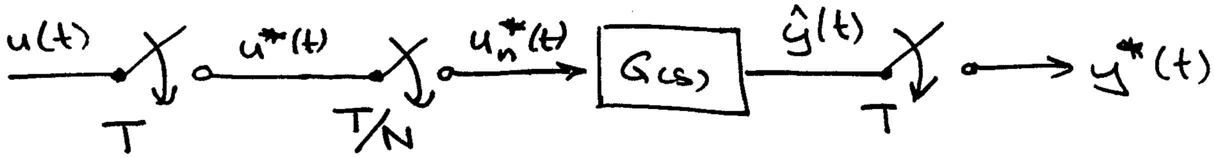
(a) Submultiple-Sampling:

The regular z-Transform will give us only information about the continuous variables in our sampled-data system at the sampling points.

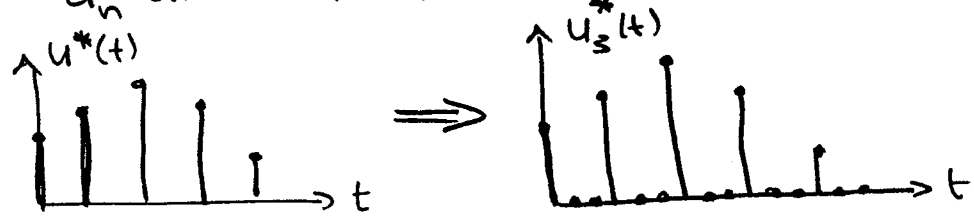
Question: How can we learn something about the time in between sampling points?



$\equiv$



as $u_n^*(t) \equiv u^*(t)$



$$u^*(t) = \sum_{k=0}^{\infty} u(kT)\cdot\delta(t-kT) \equiv u_n^*(t) = \sum_{k=0}^{\infty} u^*\left(\frac{kT}{N}\right)\cdot\delta\left(t-\frac{kT}{N}\right)$$

$$\Rightarrow \quad \hat{y}(t) = \sum_{k=0}^{\infty} u(kT)\cdot g(t-kT)$$

where: $g(t) = \mathcal{L}^{-1}\{G(s)\}$

but: $\hat{y}(t) = \sum_{k=0} u^*\left(\frac{kT}{N}\right)\cdot g\left(t-\frac{kT}{N}\right)$

evaluated at $t = \frac{mT}{N}$ $\Longrightarrow$

$$\hat{y}\left(\frac{mT}{N}\right) = \sum_{k=0}^{\infty} u(kT)\cdot g\left(\frac{mT}{N}-kT\right) \equiv \sum_{k=0} u^*\left(\frac{kT}{N}\right)\cdot g\left(\frac{mT}{N}-\frac{kT}{N}\right)$$

$$\Rightarrow \mathcal{Y}_N(z) = \sum_{m=0}^{\infty} \hat{y}\left(\frac{mT}{N}\right)\cdot z^{-m/N}$$

$$\Rightarrow \mathcal{Y}_N(z) = \sum_{m=0}^{\infty}\left\{\sum_{k=0}^{\infty} u(kT)\cdot g\left(\frac{mT}{N}-kT\right)\right\} z^{-m/N}$$

Substitution: $\quad \nu = m - k\cdot N \iff m = \nu + k\cdot N$

$$\Rightarrow \left(\frac{mT}{N}-kT\right) \equiv \frac{m-kN}{N}T \equiv \frac{\nu T}{N}$$

$$\left(-{}^{m}/_{N}\right) \equiv -\frac{\nu+kN}{N} = -\frac{\nu}{N} - k$$

m	ν
0	$-k\cdot N$
∞	∞

$$\Rightarrow Y_N(z) = \sum_{k=0}^{\infty} u(kT) \cdot \left\{ \sum_{m=0}^{\infty} g\left(\frac{mT}{N} - kT\right) \cdot z^{-m/n} \right\}$$

$$\equiv \sum_{k=0}^{\infty} u(kT) \cdot \left\{ \sum_{\nu=-kN}^{\infty} g\left(\frac{\nu T}{N}\right) \cdot \underbrace{z^{\left(-\frac{\nu}{N}-k\right)}}_{z^{-\nu/N} \cdot z^{-k}} \right\}$$

$$\equiv \sum_{k=0}^{\infty} u(kT) \cdot z^{-k} \left\{ \sum_{\nu=-kN}^{\infty} g\left(\frac{\nu T}{N}\right) z^{-\nu/N} \right\}$$

However: for $\nu < 0$, we have to evaluate

$g(t^*)$, $t^* < 0 \equiv 0$

$\Rightarrow$ We can replace the lower boundary by 0

$$\Rightarrow Y_N(z) = \underbrace{\left\{ \sum_{k=0}^{\infty} u(kT) \cdot z^{-k} \right\}}_{\mathcal{U}(z)} \cdot \left\{ \sum_{\nu=0}^{\infty} g\left(\frac{\nu}{N} \cdot T\right) \cdot z^{-\nu/N} \right\}$$

It makes sense to define:

$$\mathcal{Y}_N(z) = \mathcal{G}_N(z) \cdot \mathcal{U}(z)$$

$$\Longleftrightarrow \boxed{\mathcal{G}_N(z) = \sum_{k=0}^{\infty} g\left(\frac{k}{N} \cdot T\right) \cdot z^{-k/N}}$$

$$\Longleftrightarrow \boxed{\mathcal{G}_N(z) \equiv \mathcal{G}(z) \Big|_{\substack{z = z^{1/N} \\ T = T/N}}}$$

Example: $G(s) = \frac{1}{s+1} \longleftrightarrow \mathcal{G}(z) = \frac{z}{z - e^{-T}}$

$$\Rightarrow \mathcal{G}_3(z) = \frac{z^{1/3}}{z^{1/3} - e^{-T/3}}$$

Assume: $T = 1$; $u(t) = \varepsilon(t)$

$$\Rightarrow U(s) = \frac{1}{s} \Rightarrow \mathcal{U}(z) = \frac{z}{z-1}$$

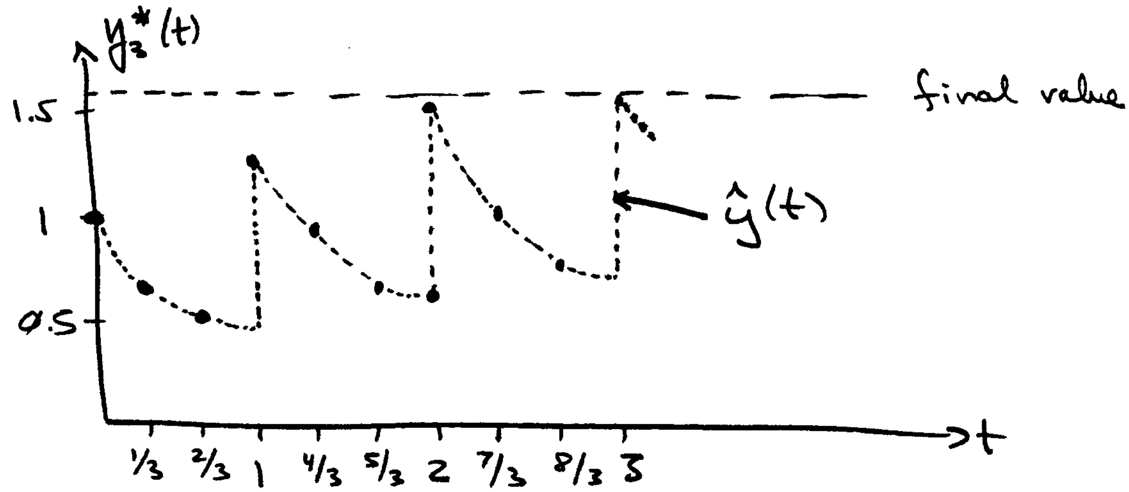
$$\Rightarrow \mathcal{Y}_3(z) = \mathcal{G}_3(z) \cdot \mathcal{U}(z) = \frac{z^{1/3}}{z^{1/3} - e^{-1/3}} \cdot \frac{z}{z-1}$$

Substitute: $z_3 = z^{1/3} \longleftrightarrow z = z_3^{3}$

$$\Rightarrow \quad y_3(z_3) = \frac{z_3}{z_3 - 0.717} \cdot \frac{z_3^{3}}{z_3^{3} - 1} = \frac{z_3^{4}}{(z_3 - 0.717)(z_3^{3} - 1)}$$

Use any method to expand into an infinite power series in z_3^{-1}:

$$\Rightarrow \quad y_3(z_3) \equiv 1 + 0.717\, z_3^{-1} + 0.513\, z_3^{-2} + 1.368\, z_3^{-3} + 0.98\, z_3^{-4} + \dots$$



$\Longrightarrow$ If we had looked only at the values $y^*(t)$, we would have "interpolated" quite differently.

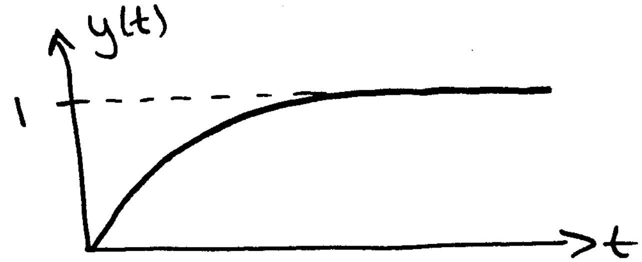
$\Longrightarrow$ $\hat{y}(t)$ is <u>not</u> a very smooth function.

$$Y(s) = G(s) \cdot U(s) = \frac{1}{s(s+1)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= \varepsilon(t) - e^{-t}$$

$$\underline{\underline{\lim_{t\to\infty} y(t)}} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{1}{s+1} = \underline{\underline{1}}$$



$$\mathcal{Y}(z) = \mathcal{G}(z) \cdot \mathcal{U}(z) = \frac{z}{z - e^{-1}} \cdot \frac{z}{z-1}$$

$$\Rightarrow y^*(t) = \mathcal{Z}^{-1}\left\{\mathcal{Y}(z)\right\}$$

$$\underline{\underline{\lim_{t\to\infty} y^*(t)}} = \lim_{z\to 1} (1 - z^{-1})\, \mathcal{Y}(z)$$

$$= \lim_{z\to 1} \frac{z-1}{z} \cdot \frac{z}{z - e^{-1}} \cdot \frac{z}{z-1} = \lim_{z\to 1} \frac{z}{z - e^{-1}} = \frac{1}{1 - e^{-1}}$$

$$= \frac{e}{e-1} = \frac{2.7181}{1.7181} = \underline{1.582}$$

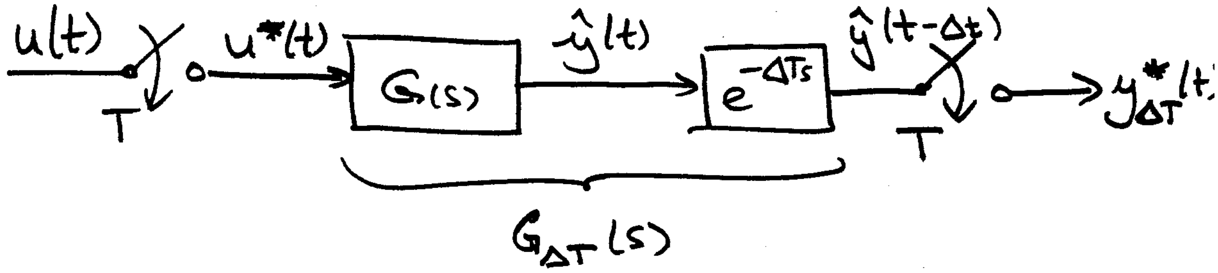
$$\Rightarrow \quad y^*(t) \neq \sum_{k=0}^{\infty} y(kT)\,\delta(t-kT) \qquad !!!$$

- To get a smoother relation between $y(t)$ and $y^*(t)$, we definitely need a ZOH following the sampler.

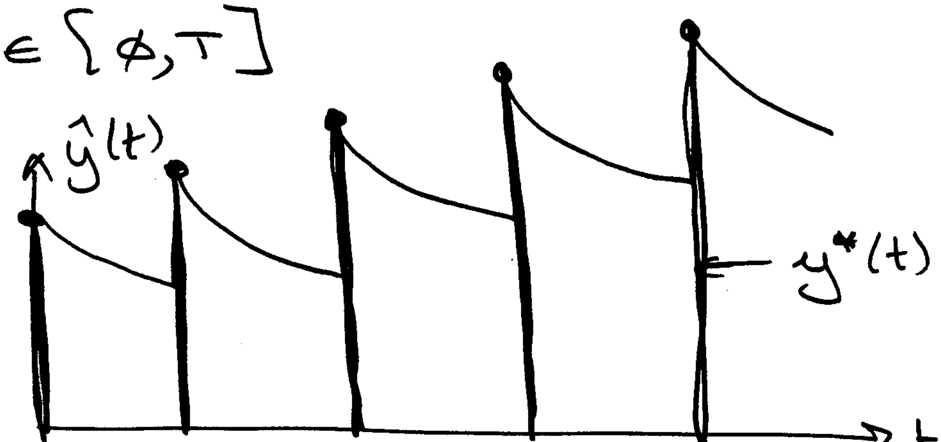
- "Natural" interpolation between sampling points to "guess" the continuous function from which these sampling points stem is a dangerous business.

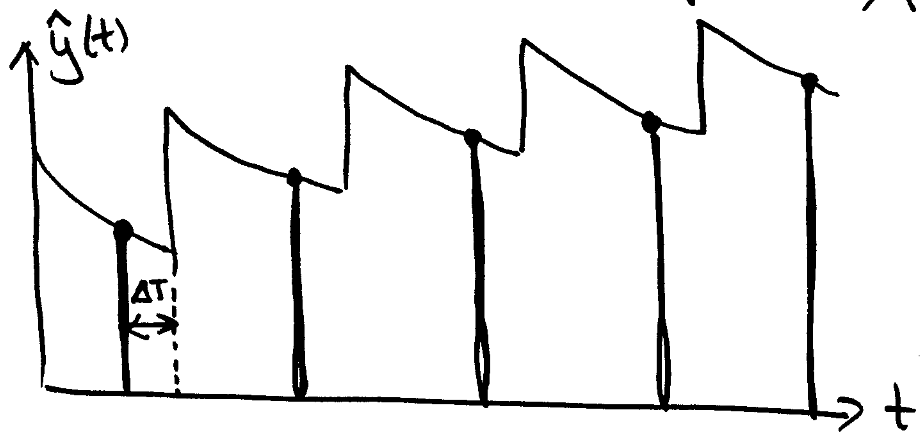
(6) Modified z-Transform:

Another way to achieve the same goal is to add a delay to the continuous system:



$\Delta T \in [\phi, T]$





$$G_{\Delta T}(s) \longrightarrow \mathcal{G}_{\Delta T}(z)$$

$$Y_{\Delta T}(z) = \mathcal{G}_{\Delta T}(z) \cdot U(z)$$

$$\Rightarrow y^*_{\Delta T}(t) = \mathcal{Z}^{-1}\{Y_{\Delta T}(z)\}$$

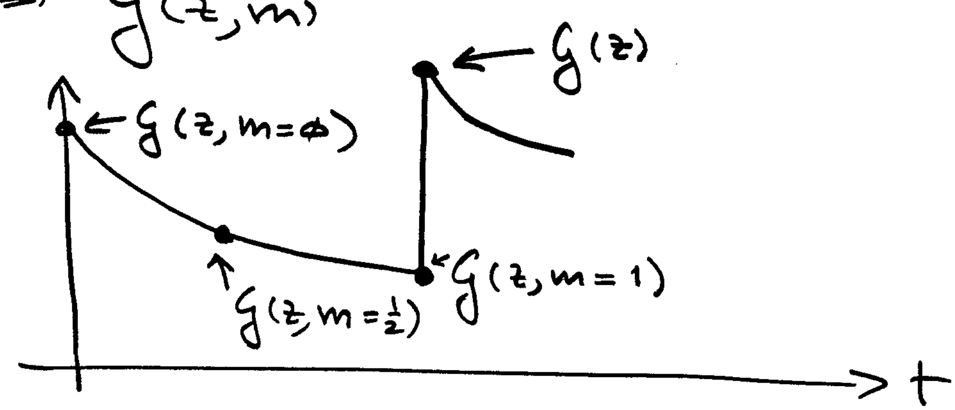
Let us call:

$$\Delta = \frac{\Delta T}{T} \quad ; \quad m = 1 - \Delta$$

Obviously: $\Delta \in [\phi, 1]$; $m \in [\phi, 1]$

Then, $G_{\Delta T}(z)$ can also be written as a z-Transform in the two variables z and m

$\Rightarrow$ $G(z, m)$



Obviously:

$$G(z, m=\phi) \equiv z^{-1} \cdot G(z)$$