Digital Redesign:

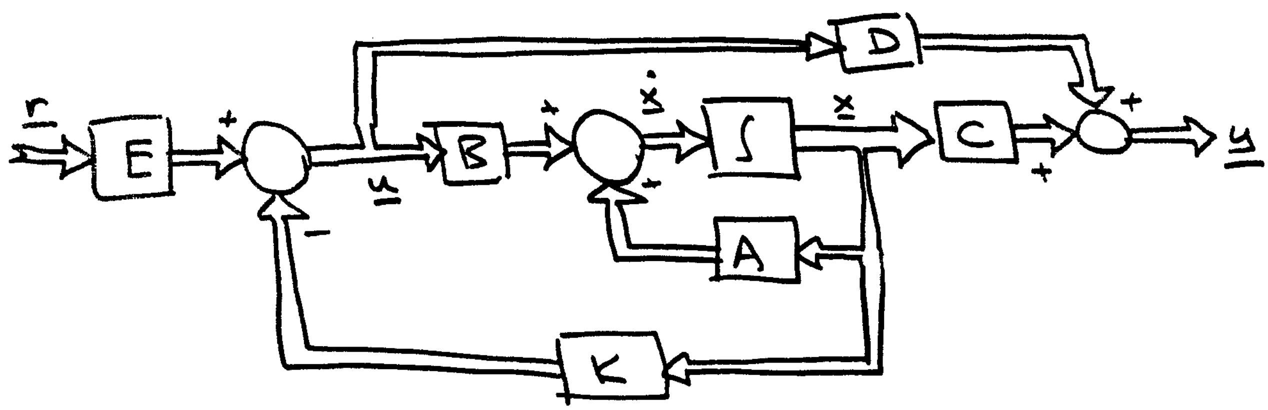
Problem: Given a MIMO system:

$$\left| \begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u} \end{aligned} \right|$$

with a linear state-feedback:

$$\underline{u} = E\underline{r} - K\underline{x}$$

(In ECE 501, you learn how to design such a controller for the case of the MIMO system.)



We want to replace the continuous controller by a digital controller.

- We could go ahead, sample the continuous system:

$$\left| \begin{array}{l} \underline{x}(k+1) = F\underline{x}(k) + G\underline{u}(k) \\ \underline{y}(k) = H\underline{x}(k) + I\underline{u}(k) \end{array} \right|$$

and use the same technique as in the continuous case to come up with new values for the matrices K^* and E^*.

<u>Question</u>: Is it possible to avoid this complete redesign? Is it possible to determine K^* and E^* out of K and E ?

$$\dot{\underline{x}} = A\underline{x} + B(E\underline{r} - K\underline{x})$$

$$\Rightarrow \dot{\underline{x}} = (A - BK)\underline{x} + BE\underline{r}$$

$$\underline{y} = C\underline{x} + D(E\underline{r} - K\underline{x})$$

$$\Rightarrow \underline{y} = (C - DK)\underline{x} + DE\underline{r}$$

The closed-loop (continuous) system can be described by:

$$\left| \begin{array}{l} \underline{\dot{x}} = A_{CL}\,\underline{x} + B_{CL}\,\underline{r} \\ \underline{y} = C_{CL}\,\underline{x} + D_{CL}\,\underline{r} \end{array} \right|$$

where:

$$\left| \begin{array}{l} A_{CL} = A - BK \\ B_{CL} = BE \\ C_{CL} = C - DK \\ D_{CL} = DE \end{array} \right|$$

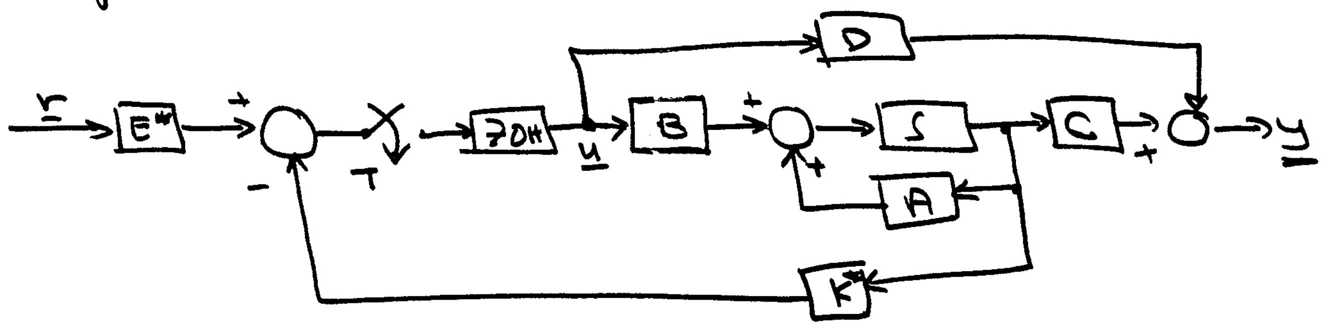
We can sample this system (with ZOH):

$$\left| \begin{array}{l} \underline{x}(k+1) = F\,\underline{x}(k) + G\,\underline{u}(k) \\ \underline{y}(k) = H\,\underline{x}(k) + I\,\underline{u}(k) \end{array} \right|$$

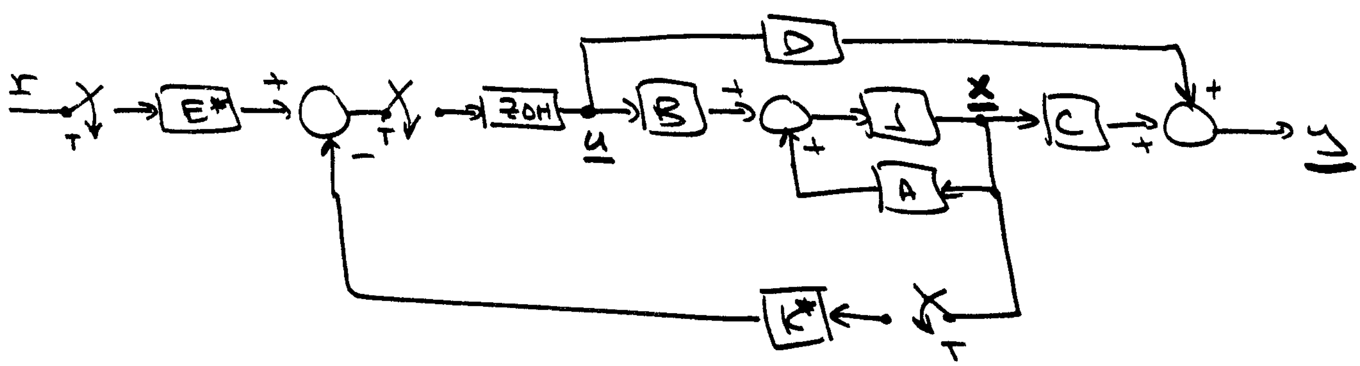
where:

$$\left| \begin{array}{l} F = e^{A_{CL}T} = e^{(A-BK)T} \\ G = F \int_0^T e^{-(A-BK)\sigma}\, d\sigma \cdot BE \\ H = C - DK \\ I = DE \end{array} \right|$$

Now, we look at the sampled-data system:



As E^*, K^* are constant matrices, we can introduce additional samplers at the inputs of E^* and K^* without modifying the system.



Continuous subsystem:

$$\left| \begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} + D\underline{u} \end{aligned} \right|$$

We consider $\underline{x}$ as additional outputs:

$$\underline{y}_n = \begin{bmatrix} \underline{y} \\ \underline{x} \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y}_n = C_n \underline{x} + D_n \underline{u} \end{array} \right|$$

where: $\left| \begin{array}{l} C_n = \begin{bmatrix} C \\ I^{(n)} \end{bmatrix} \\ \\ D_n = \begin{bmatrix} D \\ \phi^{(n)} \end{bmatrix} \end{array} \right|$

Now, we sample the continuous subsystem:

$$\left| \begin{array}{l} \underline{x}(k+1) = \hat{F}\,\underline{x}(k) + \hat{G}\,\underline{u}(k) \\ \underline{y}(k) = \hat{H}\,\underline{x}(k) + \hat{I}\,\underline{u}(k) \end{array} \right|$$

where:

$$\hat{F} = e^{AT}$$

$$\hat{G} = F \cdot \int_0^T e^{-A\sigma} d\sigma \cdot B$$

$$\hat{H} = C_n = \begin{bmatrix} C \\ I \end{bmatrix} \; ; \quad \hat{I} = D_n = \begin{bmatrix} D \\ \phi \end{bmatrix}$$

Discrete Subsystem:

$$\underline{u}(k) = E^* \underline{r}(k) - K^* \underline{x}^*(k)$$
$$= E^* \underline{r}(k) - K^* \underline{y}_2(k)$$

We plug in above:

$$\underline{x}(k+1) = \hat{F}\,\underline{x}(k) + \hat{G}\left(E^* \underline{r}(k) - K^* \underline{y}_2(k)\right)$$

$$\underline{y}_1(k) = C\,\underline{x}(k) + D\left(E^* \underline{r}(k) - K^* \underline{y}_2(k)\right)$$

$$\underline{y}_2(k) = \underline{x}(k)$$

$$\Rightarrow \left|\begin{array}{l} \underline{x}(k+1) = \left[\hat{F} - \hat{G}K^*\right]\underline{x}(k) + \hat{G}E^* \underline{r}(k) \\ \underline{y}(k) = \left[C - DK^*\right]\underline{x}(k) + DE^* \underline{r}(k) \end{array}\right|$$

should look as much as possible like the system without samplers.

Abbreviations:

$$F = e^{(A-BK)T} = \Phi_c(T)$$

$$G = e^{(A-BK)T} \cdot \int_0^T e^{-(A-BK)\sigma} d\sigma \cdot BE = \vartheta_c(T)$$

$$\hat{F} = e^{AT} = \Phi^*(T)$$

$$\hat{G} = e^{AT} \int_0^T e^{-A\sigma} d\sigma \cdot B = \vartheta^*(T)$$

$$\Rightarrow F = \Phi_c(T) \approx \hat{F} - \hat{G}K^* = \Phi^*(T) - \vartheta^*(T) \cdot K^*$$

$$\Rightarrow \vartheta^*(T) \cdot K^* \cong \Phi^*(T) - \Phi_c(T)$$

If $\vartheta^*(T)$ is rectangular (usual case), we need to use a <u>pseudo-inverse</u>:

$$\underbrace{\vartheta^*(T)' \cdot \vartheta^*(T)}_{\text{square and positive definit}} \cdot K^* = \vartheta^*(T)' \left[\Phi^*(T) - \Phi_c(T)\right]$$

$$\Rightarrow K^* = \left[\vartheta^*(T)' \cdot \vartheta^*(T)\right]^{-1} \cdot \vartheta^*(T)' \left[\Phi^*(T) - \Phi_c(T)\right]$$

$$G = \vartheta_c(T) \approx \hat{G} E^* = \vartheta^*(T) \cdot E^*$$

$$\Rightarrow E^* = \left[\vartheta^*(T)' \cdot \vartheta^*(T) \right]^{-1} \cdot \vartheta^*(T)' \cdot \vartheta_c(T)$$

$$\left| \begin{array}{l} H = C - DK \approx C - DK^* \\ I = DE \approx DE^* \end{array} \right|$$

$\Rightarrow$ won't work. We need to modify the system:

Use a different C and D matrix for the sampled system:

$$H = C_c - D_c K \approx C_D - D_D K^*$$

$$I = D_c E \approx D_D E^*$$

$$\Rightarrow D_D = D_c E \cdot E^{*-1}$$

$$\Rightarrow C_D = C_c - D_c K + D_D K^*$$

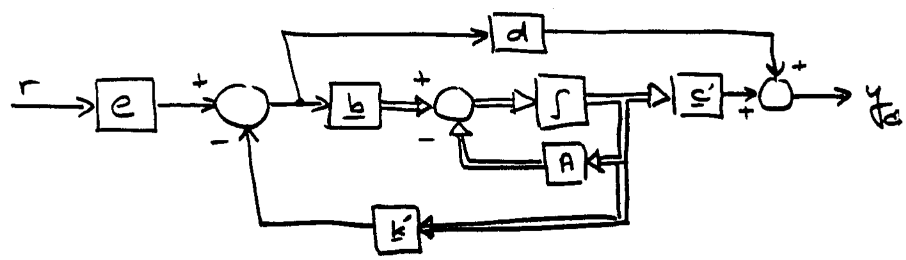
```
%
%  Given an open-loop transfer function:
%
%                                    1
%                      G(s)  =  ----------
%                               (s-1)(s-2)
%
P = 1

P =

     1

Qroot = [ 1 ; 2 ]

Qroot =

     1
     2

Q = poly(Qroot)

Q =

     1    -3     2

%
%  We wish to design a state feedback, such that the closed-loop system has
%  a transfer function of:
%
%                                      2                2
%                      Gtot(s)  =  ------------  =  --------------
%                                  s^2 + 2s + 2     (s+1+j)(s+1-j)
%
Ptot = 2

Ptot =

     2

Qtot = [ 1 2 2 ]

Qtot =

     1     2     2

Qtotroot = roots(Qtot)

Qtotroot =

  -1.0000 + 1.0000i
  -1.0000 - 1.0000i

%
%  Start by converting the open-loop system to the time domain.
%
[A,b,c,d] = tf2ss(P,Q)

A =
```

```
      3     -2
      1      0


b =

      1
      0


c =

      0      1


d =

      0

%
%  Do pole placement now
%
k = place(A,b,Qtotroot)
place: ndigits= 16

k =

      5      0

%
%  Calculate the closed-loop system in the time domain.
%
Acl = A - b*k

Acl =

     -2     -2
      1      0

bcl = b

bcl =

      1
      0

ccl = c - d*k

ccl =

      0      1

dcl = d

dcl =

      0

%
%  Check whether the polas are where we want them to be.
```

$$\left| \begin{aligned} \dot{\underline{x}} &= A\cdot\underline{x} + \underline{b}\cdot u \\ y &= \underline{c}'\cdot\underline{x} + d\cdot u \\ u &= e\cdot r - \underline{k}'\cdot\underline{x} \end{aligned} \right|$$

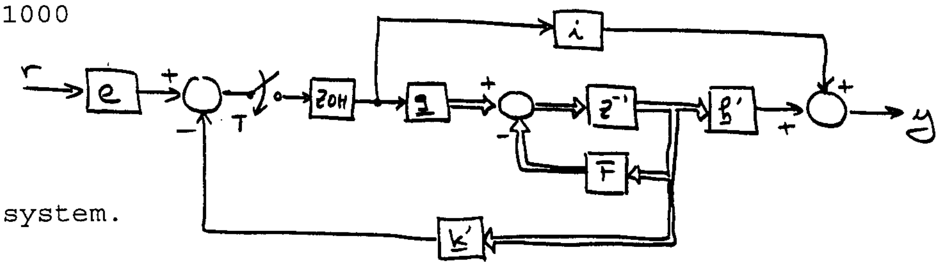
$$\Rightarrow \left| \begin{aligned} \dot{\underline{x}} &= (A - \underline{b}\cdot\underline{k}')\underline{x} + (\underline{b}\cdot e)\, r \\ y &= (\underline{c}' - d\cdot\underline{k}')\underline{x} + (d\cdot e)\, r \end{aligned} \right|$$

```
%
eig(Acl)

ans =

  -1.0000 + 1.0000i
  -1.0000 - 1.0000i

%
%  Calculate the steady-state response to a step input.
%
yss = -ccl/Acl*bcl + dcl

yss =

    0.5000

%
%  Adjust the input level to make the gain equal to 1.
%
e = 1/yss

e =

     2

%
%  Adjust bcl and dcl to reflect e.
%
bcl = b*e

bcl =

     2
     0

dcl = d*e

dcl =

     0

```




```
%
%  Okay, this is the continuous system.  Let us simulate a step response
%  over 10 time units.
%
t = [ 0:0.1:10 ]';
uc = ones(size(t));
x0 = [ 0 ; 0 ];
yc = lsim(Acl,bcl,ccl,dcl,uc,t,x0);
%
%  Plot the continuous-time closed-loop system.
%
plot(t,yc)
grid on
title('Continuous-Time Closed-Loop System')
xlabel('Time')
ylabel('yc')
print -dps redesign_1.ps
%
```

```
%  Let us now sample the open-loop system with a ZOH.  We want to repeat
%  the design with different values of the sampling rate T.
%
TT = [ 0.001 , 0.01 , 0.1 ]

TT =

    0.0010    0.0100    0.1000
```




```
m = 0;
yd = zeros(10001,3);
for T = TT,
  m = m + 1;
  %
  %  Sample the open-loop system.
  %
  [F,g] = c2d(A,b,T);
  h = c;
  i = d;
  %
  %  We compute the closed-loop system.
  %
  Fcl = F - g*k;
  gcl = g*e;
  hcl = h - i*k;
  icl = i*e;
```

$$\underline{x}_{k+1} = F\cdot\underline{x}_k + \underline{g}\cdot u_k$$
$$y_k = \underline{h}'\cdot\underline{x}_k + i\cdot u_k$$
$$u_k = e\cdot r_k - \underline{k}'\cdot\underline{x}_k$$

```
  %
  %  Simulate the discrete-time system.
  %
  ud = ones(10/T+1,1);
  yd(1:10/T+1,m) = dlsim(Fcl,gcl,hcl,icl,ud,x0);
end
yd1 = yd(1:100:10001,1);
yd2 = yd(1:10:1001,2);
yd3 = yd(1:101,3);
y = [ yc , yd1 , yd2 , yd3];
```

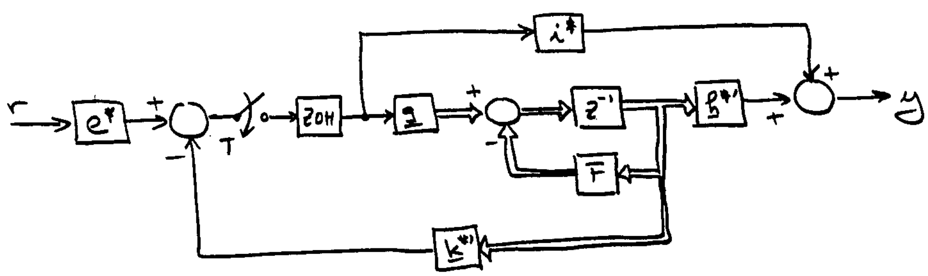
$$\Rightarrow \quad \underline{x}_{k+1} = (F - \underline{g}\cdot\underline{k}')\underline{x}_k + (\underline{g}\cdot e)r$$
$$y = (\underline{h}' - i\cdot\underline{k}')\underline{x}_k + (i\cdot e)r$$

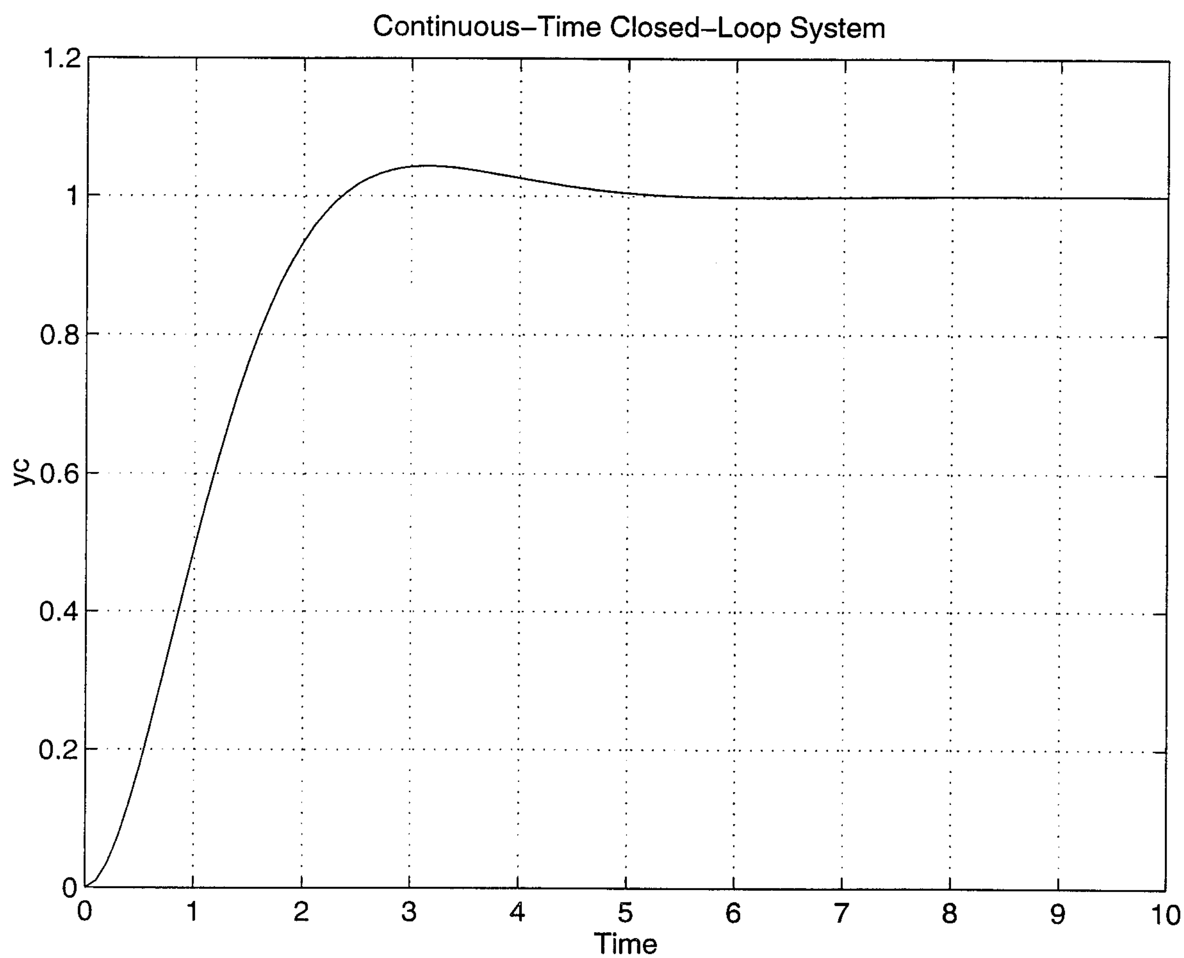
```
%
%  Plot the discrete-time vs. the continuous-time closed-loop system.
%
plot(t,y)
grid on
title('Discrete-Time vs. Continuous-Time Closed-Loop Systems')
xlabel('Time')
ylabel('y')
print -dps redesign_2.ps
%
%  Repeat the design, this time making use of the digital redesign technique
%  to adjust the k and e values.
%
m = 0;
yd = zeros(10001,3);
for T = TT,
  m = m + 1;
  %
  %  Sample the closed-loop and open-loop systems.
  %
  [phic,thetac] = c2d(Acl,bcl,T);
  [phistar,thetastar] = c2d(A,b,T);
  %
  %  Calculate the kstar and estar values, as well as the modified outputs.
```

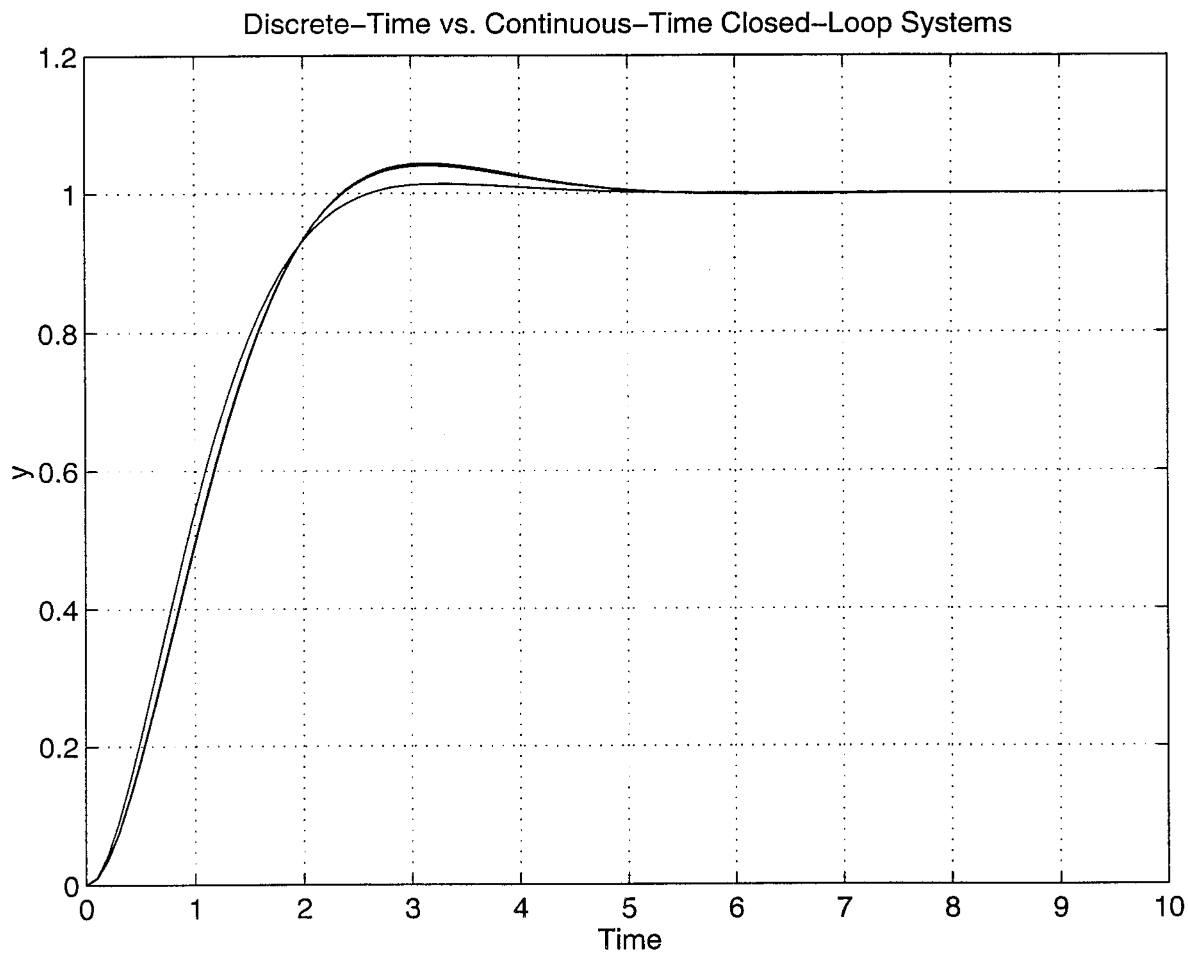
```
  %
  kstar = thetastar\(phistar - phic);
  estar = thetastar\thetac;
  dd = d*e/estar;
  cd = c - d*k + dd*kstar;
  %
  %  Now, sample the open-loop system again.
  %
  [F,g] = c2d(A,b,T);
  h = cd;
  i = dd;
  %
  %  We compute the closed-loop system.
  %
  Fcl = F - g*kstar;
  gcl = g*estar;
  hcl = h - i*kstar;
  icl = i*estar;
  %
  %  Simulate the discrete-time system.
  %
  ud = ones(10/T+1,1);
  yd(1:10/T+1,m) = dlsim(Fcl,gcl,hcl,icl,ud,x0);
end
yd1 = yd(1:100:10001,1);
yd2 = yd(1:10:1001,2);
yd3 = yd(1:101,3);
y = [ yc , yd1 , yd2 , yd3];
%
%  Plot the redesigned discrete-time vs. the continuous-time closed-loop system.
%
plot(t,y)
grid on
title('Redesigned Discrete-Time vs. Continuous-Time Closed-Loop Systems')
xlabel('Time')
ylabel('y')
print -dps redesign_3.ps
%
diary off
```

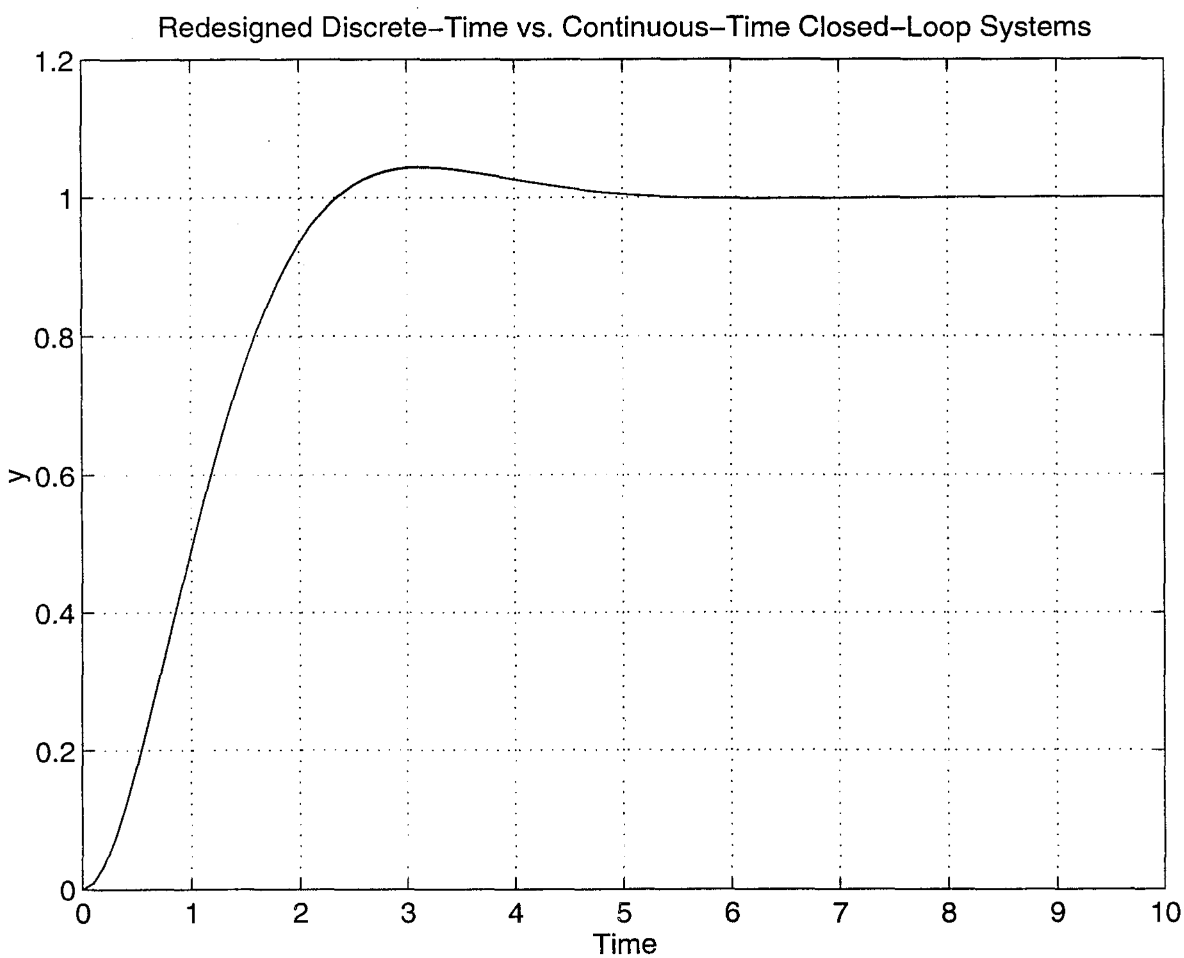
$$\left| \begin{aligned} \underline{x}_{k+1} &= \mathcal{F} \cdot \underline{x}_k + \underline{g} \cdot u_k \\ y_k &= \underline{h}^{*\prime} \cdot \underline{x}_k + i^* \cdot u_k \\ u_k &= e^* \cdot r_k - \underline{k}^{*\prime} \cdot \underline{x}_k \end{aligned} \right|$$

$$\Rightarrow \left| \begin{aligned} \underline{x}_{k+1} &= (F - \underline{g} \cdot \underline{k}^{*\prime})\, \underline{x}_k + (\underline{g} \cdot e^*) r \\ y &= (\underline{h}^{*\prime} - i^* \cdot \underline{k}^{*\prime})\, \underline{x}_k + (i^* \cdot e^*) r \end{aligned} \right|$$




Continuous−Time Closed−Loop System
yc
0
0.2
0.4
0.6
0.8
1
1.2
Time
0 1 2 3 4 5 6 7 8 9 10


Discrete–Time vs. Continuous–Time Closed–Loop Systems
1.2
1
0.8
y 0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8 9 10
Time


Redesigned Discrete-Time vs. Continuous-Time Closed-Loop Systems
y
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
9
10
Time