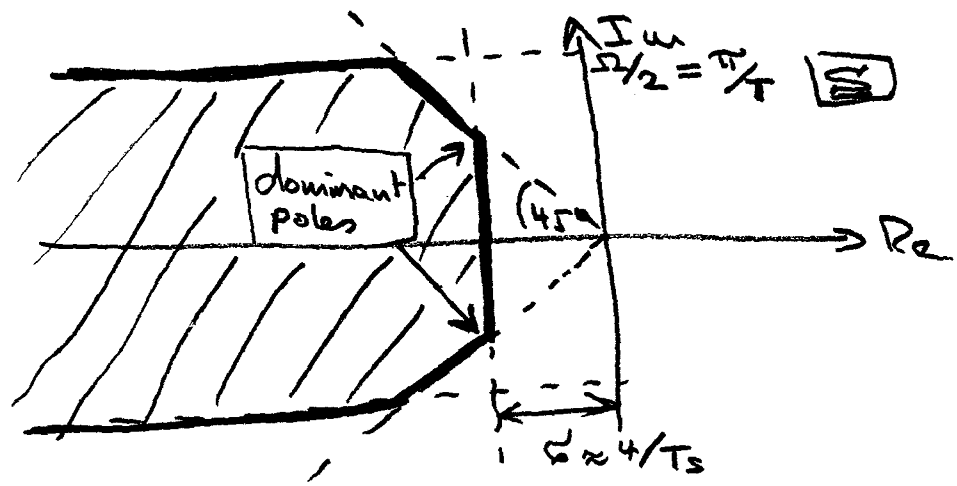
Design Recipe:

(1) Choose your controller poles such that they are inside:



→ don't put several poles too close to each other, as this increases the sensitivity.

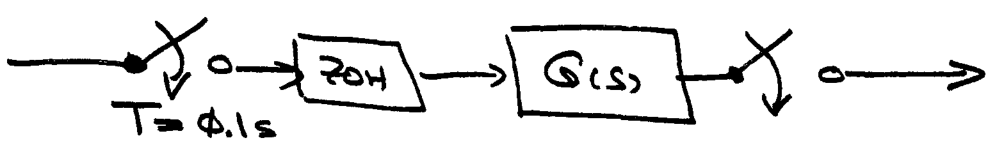
→ don't put any poles out too far to the left as this increases the gain factors ($\underline{k}'$-vector). ⇒ Try to keep all $|k_i| \leq 100$.

→ Design the observer-poles slightly faster than the controller poles. (Multiple poles are again no good, but observer poles <u>may</u> coincide with controller poles.) (without proof!)

<u>Example</u>: Given:

$$G(s) = \frac{4}{(s+1)(s+2)^2}$$

Design an output feedback for the system:



such that the settling time is ≤ 2 sec.; $\approx 5\%$ overshoot; no steady-state error to step input.

$$G(s) = \frac{4}{(s+1)(s+2)^2} = \frac{4}{s^3+5s^2+8s+4}$$

$$\Rightarrow \left| \begin{aligned} \underline{\dot{x}} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -8 & -5 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 4 & 0 & 0 \end{bmatrix} \underline{x} \end{aligned} \right|$$

is a state-space representation.
We discretize it using:

$$F = e^{AT} \quad ; \quad \underline{g} = \left[e^{AT} - I \right] A^{-1} \cdot \underline{b}$$

$$\underline{h}' \equiv \underline{c}'$$

$$\Rightarrow \left| \begin{aligned} \underline{x}(k+1) &= \begin{bmatrix} 0.9994 & 0.0988 & 0.0042 \\ -0.0169 & 0.9655 & 0.0776 \\ -0.3106 & -0.6381 & 0.5773 \end{bmatrix} \underline{x}(k) + \begin{bmatrix} 0.0001 \\ 0.0042 \\ 0.0776 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 4 & 0 & 0 \end{bmatrix} \underline{x}(k) \end{aligned} \right|$$

is the discretized representation.

$\Rightarrow$ The z-Transform is:

$$G(z) = \frac{0.0006z^2 + 0.0021z + 0.0005}{z^3 - 2.5423z^2 + 2.152z - 0.6065}$$

$\Rightarrow$

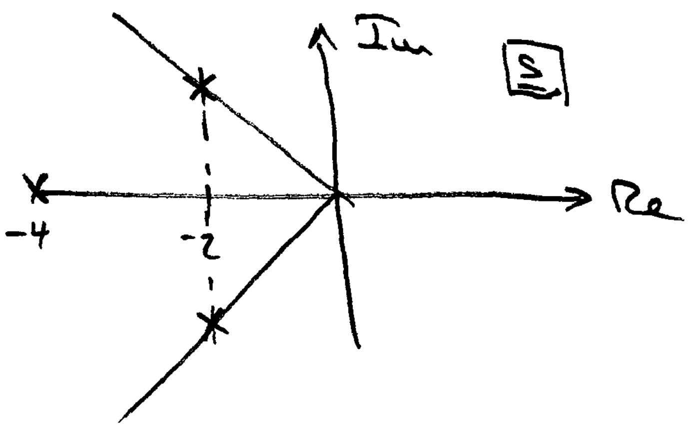
$$\underline{n}(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.6065 & -2.152 & 2.5423 \end{bmatrix} \underline{n}(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0.0005 & 0.0021 & 0.0006 \end{bmatrix} \underline{n}(k)$$

is the controller-canonical representation.

Controller-Design:

We choose the three poles at:



$\sigma = \frac{4}{T_S} = 2$

$$\Rightarrow \left| \begin{array}{l} \lambda_{1c} = -2+2j \\ \lambda_{2c} = -2-2j \\ \lambda_{3c} = -4 \end{array} \right| \Rightarrow \left| \begin{array}{l} \lambda_{1d} = e^{\lambda_{1c}T} = 0.8024 + 0.1627j \\ \lambda_{2d} = 0.8024 - 0.1627j \\ \lambda_{3d} = 0.6703 \end{array} \right.$$

$$\Rightarrow Q_{CL}(z) = (z-\lambda_{1d})(z-\lambda_{2d})(z-\lambda_{3d})$$

$$= z^3 - 2.2751\, z^2 + 1.7461\, z - 0.4493$$

$$a_0 = -0.6065 \quad ; \quad a_0 + k_1 = -0.4493 \Rightarrow k_1 = 0.1572$$

$$a_1 = 2.152 \quad ; \quad a_1 + k_2 = 1.7461 \Rightarrow k_2 = -0.4059$$

$$a_2 = -2.5423 \quad ; \quad a_2 + k_3 = -2.2751 \Rightarrow k_3 = 0.2672$$

$$\Rightarrow \underline{k}' = \begin{bmatrix} 0.1572 & -0.4059 & 0.2672 \end{bmatrix}$$

Now, we design the observer. For this purpose, we need to transform into observer-canonical form:

$$Q_o = \begin{bmatrix} \underline{h}' \\ \underline{h}'F \\ \underline{h}'F^2 \end{bmatrix} = \begin{bmatrix} 0.0005 & 0.0021 & 0.0096 \\ 0.0004 & -0.0008 & 0.0036 \\ 0.0022 & -0.0073 & 0.0083 \end{bmatrix}$$

$$\Rightarrow Q_o^{-1} = \begin{bmatrix} 1060.5 & -1168.8 & 429.3 \\ 260.4 & 136.8 & -77.5 \\ -47.0 & 427.2 & -60.4 \end{bmatrix} \Rightarrow \underline{q} = \begin{bmatrix} 429.2534 \\ -77.5431 \\ -60.3872 \end{bmatrix}$$

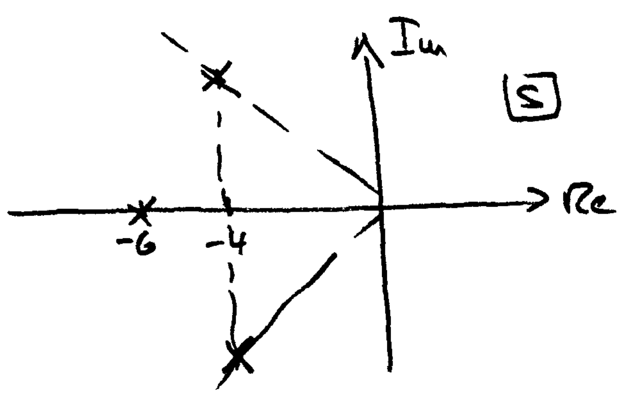
$$\Rightarrow T^{-1} = \left[\underline{q} \;\vdots\; F\cdot\underline{q} \;\vdots\; F^2\cdot\underline{q} \right] = \begin{bmatrix} 429.2534 & -77.5431 & -60.3872 \\ -77.5431 & -60.3872 & 273.7023 \\ -60.3872 & 273.7023 & 778.7575 \end{bmatrix}$$

We can stop here as T^{-1} is the only thing that we need. We can write down the observer-canonical representation at once:

$$\underline{\eta}(k+1) = \begin{bmatrix} 0 & 0 & 0.6065 \\ 1 & 0 & -2.1520 \\ 0 & 1 & 2.5423 \end{bmatrix} \underline{\eta}(k) + \begin{bmatrix} 0.0005 \\ 0.0021 \\ 0.0006 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \underline{\eta}(k)$$

Let us place the observer poles as follows:



$$\begin{aligned} \lambda_{1c} &= -4+4j \\ \lambda_{2c} &= -4-4j \\ \lambda_{3c} &= -6 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \lambda_{1d} &= 0.6174 + 0.261j \\ \lambda_{2d} &= 0.6174 - 0.261j \\ \lambda_{3d} &= 0.5488 \end{aligned}$$

$$\Rightarrow Q_{CL}(z) = z^3 - 1.7836z^2 + 1.127z - 0.2466$$

$$a_0 = -0.6065 \; ; \quad a_0 + \hat{l}_1 = -0.2466 \; ; \Rightarrow \hat{l}_1 = 0.3599$$
$$a_1 = 2.152 \; ; \quad a_1 + \hat{l}_2 = 1.127 \; ; \Rightarrow \hat{l}_2 = -1.0249$$
$$a_2 = -2.5423 \; ; \quad a_2 + \hat{l}_3 = -1.7836 \; ; \Rightarrow \hat{l}_3 = 0.7587$$

$$\Rightarrow \underline{\hat{l}} = \begin{bmatrix} 0.3599 \\ -1.0249 \\ 0.7587 \end{bmatrix} \Rightarrow \underline{l} = T^{-1} \cdot \underline{\hat{l}} = \begin{bmatrix} 188.1661 \\ 241.6348 \\ 288.5538 \end{bmatrix}$$

The gain factors are too large. However, this is not because the poles are too far left, it is because the elements of the $\underline{h}'$ - vector are extremely small.

Solution: Find another base representation that balances the $\underline{\hat{k}}$- and $\underline{\hat{l}}$-vectors such that:

$$|\hat{k}_i| = |\hat{l}_i| \; .$$

- We take the transformation:

$$\underline{\mu} = \tilde{T} \cdot \underline{x}$$

In these new state-variables, we require a different feedback vector:

$$u = r - \underline{k}' \cdot \underline{x}$$

$$\underline{x} = \tilde{T}^{-1} \cdot \underline{\mu}$$

$$\Rightarrow u = r - \underline{k}' \tilde{T}^{-1} \cdot \underline{\mu} = r - \hat{\underline{k}}' \cdot \underline{\mu}$$

$$\Rightarrow \hat{\underline{k}}' = \underline{k}' \cdot \tilde{T}^{-1}$$

From before, we know that:

$$\hat{\underline{\ell}} = \tilde{T} \cdot \underline{\ell}$$

We choose $\tilde{T}$ to be diagonal:

$$\tilde{T} = \begin{bmatrix} t_1 & & & \emptyset \\ & t_2 & & \\ & & \ddots & \\ \emptyset & & & t_n \end{bmatrix}$$

$$\Rightarrow |\hat{k}_i| = \left| k_i \cdot \left(\frac{1}{t_i}\right) \right| \overset{!}{\equiv} |\hat{\ell}_i| = |t_i \cdot \ell_i|$$

For our example:

$$T = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} (\frac{1}{t_1}) & 0 & 0 \\ 0 & (\frac{1}{t_2}) & 0 \\ 0 & 0 & (\frac{1}{t_3}) \end{bmatrix}$$

$$|\hat{k}_1| = |k_1 \cdot (\tfrac{1}{t_1})| = |\hat{\ell}_1| = |t_1 \cdot \ell_1|$$

$$\Longrightarrow \quad t_1 = \sqrt{|k_1 / \ell_1|}$$

$\underline{\text{etc.}}$: $$t_i = \sqrt{|k_i / \ell_i|}$$

$$\Rightarrow T = \begin{bmatrix} 0.0289 & 0 & 0 \\ 0 & 0.041 & 0 \\ 0 & 0 & 0.0304 \end{bmatrix}$$

$\Rightarrow$ Base representation:

$$\underline{\mu}(k+1) = \begin{bmatrix} 0 & 0.7052 & 0 \\ 0 & 0 & 1.347 \\ 0.6385 & -1.5976 & 2.5423 \end{bmatrix} \underline{\mu}(k) + \begin{bmatrix} 0 \\ 0 \\ 0.0304 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0.0159 & 0.0507 & 0.0193 \end{bmatrix} \underline{\mu}(k)$$

$$\Rightarrow \quad t_i^2 = |k_i / \ell_i|$$

$$\Rightarrow \quad \underline{\underline{t_i = \sqrt{|k_i/\ell_i|}}}$$

For our example:

$$\tilde{T} = \begin{bmatrix} 0.0289 & 0 & 0 \\ 0 & 0.041 & 0 \\ 0 & 0 & 0.0304 \end{bmatrix}$$

Using this for a similarity transformation leads to:

$$\underline{\mu}(k+1) = \begin{bmatrix} 0 & 0.7052 & 0 \\ 0 & 0 & 1.347 \\ 0.6385 & -1.5976 & 2.5423 \end{bmatrix} \underline{\mu}(k) + \begin{bmatrix} 0 \\ 0 \\ 0.0304 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0.0159 & 0.0507 & 0.0193 \end{bmatrix} \underline{\mu}(k)$$

to be our new base representation. This representation is <u>balanced</u> with respect to the gain factors.

$$\Rightarrow \underline{\hat{k}}' = \underline{k}' \cdot \tilde{T}^{-1} = \begin{bmatrix} 5.4388 & -9.9034 & 8.7841 \end{bmatrix}$$

$$\underline{\hat{\ell}} = \tilde{T} \cdot \underline{\ell} = \begin{bmatrix} 5.4388 \\ 9.9034 \\ 8.7841 \end{bmatrix}$$

is perfectly within acceptable limits.

Warning: If either one of the k_i's or ℓ_i's is $= \emptyset$, the procedure does not work, and we need to solve a more general problem (using a non-diagonal $\tilde{T}$-matrix) where:

$$\| \underline{\hat{k}}' \|_\infty \cdot \| \underline{\hat{\ell}} \|_\infty \equiv \| \underline{k}' \tilde{T}^{-1} \|_\infty \cdot \| \tilde{T} \cdot \underline{\ell} \|_\infty \stackrel{!}{=} \min.$$