Controllability & Observability:

Def: A system represented by polynomial matrices is controllable (observable) iff it is equivalent to a state-space representation which is controllable (observable).

without proof:

- A PM-system is controllable iff $\{P(s), Q(s)\}$ are relative left prime.
- A PM-system is observable iff $\{P(s); R(s)\}$ are relative right prime.
- Non-controllable modes are zeros of the determinant of the LCLD of $P(s)$ & $Q(s)$.
- Non-observable modes are zeros of the determinant of the LCRD of $P(s)$ & $R(s)$.

(Proof: cf. Wolovich)

Note: This is particularly true for
$P(s) = (sI - A)$, $Q(s) = B$; $R(s) = C$
$\Longrightarrow$ Another way to determine controllability & observability of a state-space model.

Poles & Zeros:

We want to formally define the terms "pole" and "zero" for MIMO-systems. This helps us to see the relations between the different representations better.

(I) Poles & Zeros of $G(s)$:

$$G(s) = \frac{N(s)}{d(s)} = U_L^{-1}(s) \cdot M(s) \cdot U_R^{-1}(s)$$

(Smith-McMillan form)

Example:

$$G(s) = \underbrace{\begin{bmatrix} \frac{s}{(s+1)^2(s+2)^2} & \frac{s}{(s+2)^2} \\ \frac{-s}{(s+2)^2} & \frac{-s}{(s+2)^2} \end{bmatrix}}_{\text{(strictly proper)}} = \begin{bmatrix} 1 & 0 \\ -(s+1)^2 & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} \frac{s}{(s+1)^2(s+2)^2} & 0 \\ 0 & \frac{s^2}{s+2} \end{bmatrix}}_{\text{(not proper)}} \cdot \begin{bmatrix} 1 & (\\ 0 & \end{bmatrix}$$

Def:
- The poles of $G(s)$ are the roots of $\psi_i(s)$.
- The zeros of $G(s)$ are the roots of $\varepsilon_i(s)$.

In our example:

$$\Rightarrow \left| \begin{array}{lll} 3 \text{ zeros} & \text{at} & s = \phi \\ 2 \text{ poles} & \text{at} & s = -1 \\ 3 \text{ poles} & \text{at} & s = -2 \end{array} \right\} 5^{th} \text{ order}$$

- $d(s) = (s+1)^2 \cdot (s+2)^2$ is only 4^{th} order.
- The full characteristic polynomial is $(s+1)^2 (s+2)^3$.

Note: $G(s)$ may have poles and zeros at the same place.

Example:

$$M(s) = \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \phi \\ \phi & \frac{s(s+1)}{(s+2)} \end{bmatrix}$$

$$\Rightarrow \left| \begin{array}{llll} 2 \text{ zeros} & \text{at} & s = \phi & \\ 1 \text{ zero} & \text{at} & s = -1 & \rbrace \text{ do } \underline{\text{not}} \text{ cancel} \\ 1 \text{ pole} & \text{at} & s = -1 & \\ 2 \text{ poles} & \text{at} & s = -2 & \end{array} \right|$$

(II) Poles & Zeros of State-Space Representatio[n]

Assume: $\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} \end{array} \right|$ (A,B) = controllable, (A,C) = observable

⇒ The poles are the Eigenvalues of A.

⇒ The zeros are the values of s for which the polynomial matrix

$$S_{(s)} = \begin{bmatrix} (sI-A) & B \\ -C & \phi \end{bmatrix}$$

loses its full rank.

Interpretation: Assume s_o is a zero. Then with $\underline{u}(t) = \underline{U}_o e^{s_o t}$, there exists an initial state $\underline{x}_o$ such that $\underline{y}(t) \equiv \phi$; $\forall t > \phi$. This is called transmission blocking.

- These zeros are called the invariant dividers of the system matrix S = roots(det($S_{(s)}$)).

(III) Poles & Zeros of Polynomial Matrices:

(a) <u>Assume</u>: $\{P, Q, R, W\}$ are <u>irreducible</u>
(i.e., the system is completely controllable & observable).

⇒ The poles are the roots of $\det(P(s)) = \phi$.

⇒ The zeros are the invariant dividers of the system matrix:

$$S(s) = \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}$$

(b) <u>Assume</u>: $\{P, Q, R, W\}$ are <u>reducible</u>.

(α) Find the LCLD of $[P(s), Q(s)]$

$$\Rightarrow \left| \begin{array}{l} P(s) = Z_{id}(s) \cdot P_{id}(s) \\ Q(s) = Z_{id}(s) \cdot Q_{id}(s) \end{array} \right|$$

(β) Find the LCRD of $[P_{id}(s); R(s)]$

$$\Rightarrow \left| \begin{array}{l} P_{id}(s) = P_{oi}(s) \cdot Z_{oi}(s) \\ R(s) = R_{oi}(s) \cdot Z_{oi}(s) \end{array} \right|$$

$$\Rightarrow \quad S(s) = \begin{bmatrix} P & Q \\ -R & W \end{bmatrix} = \begin{bmatrix} Z_{id} & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} P_{id} & Q_{id} \\ -R & W \end{bmatrix}$$

$$= \begin{bmatrix} Z_{id} & 0 \\ 0 & I \end{bmatrix} \cdot \underbrace{\begin{bmatrix} P_{oi} & Q_{id} \\ -R_{oi} & W \end{bmatrix}}_{\text{controllable \& observable}} \cdot \begin{bmatrix} Z_{oi} & 0 \\ 0 & I \end{bmatrix}$$

⇒ The <u>poles</u> are the roots of $\det(P_{oi}(s)) = 0$.

⇒ The <u>transmission zeros</u> are the invariant dividers of

$$S_{io} = \begin{bmatrix} P_{oi} & Q_{id} \\ -R_{oi} & W \end{bmatrix}$$

⇒ The <u>input-decoupling zeros</u> are the invariant dividers of $[P(s), Q(s)]$ (uncontrollable modes). They are the roots of $\det(Z_{id}(s)) = 0$.

⇒ The <u>output-decoupling zeros</u> are the invariant dividers of $[P(s); R(s)]$ (unobservable modes).

Controller - canonical Realization:

Let us start from the controller - canonical state-space representation:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{array} \right|$$

We wish to find an equivalent Controller - canonical PM representation

<u>Warning</u>: This is <u>not</u>
$P(s) = sI - A$
$Q(s) = B$
$R(s) = C$
$W(s) = D$

We want: $Q(s) = I^{(m)}$! Using the above approach, the length of the partial state vector (q) would be equal to the system order (n), but we want a solution where q is equal to the number of inputs (m).

Algorithm:

(i) We build an $(m \times n)$ matrix A_m out of the m (ordered) σ_k-rows of A.

(ii) We build an $(m \times m)$ matrix B_m out of the m σ_k-rows of B:

$$B_m = \begin{bmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & \emptyset & & 1 & * \\ & & & & 1 \end{bmatrix}$$

$$\Rightarrow \quad |B_m| = 1$$

(iii) We define:

$$\Sigma(s) = \begin{bmatrix} \begin{matrix} 1 \\ s \\ \vdots \\ s^{d_1-1} \end{matrix} & & & \emptyset \\ & \begin{matrix} 1 \\ s \\ \vdots \\ s^{d_2-1} \end{matrix} & & \\ & & \ddots & \\ \emptyset & & & \begin{matrix} 1 \\ s \\ \vdots \\ s^{d_m-1} \end{matrix} \end{bmatrix}$$

$\Sigma(s)$ is an $(n \times m)$ polynomial matrix

(iv) We define:

$$\delta(s) = \begin{bmatrix} s^{d_1} & & \emptyset \\ & s^{d_2} & \\ \emptyset & & \ddots \; s^{d_m} \end{bmatrix} - A_m \cdot \Sigma(s)$$

$\delta(s)$ is an $(m \times m)$ polynomial matri[x]

(v) Among these quantities, the so-called <u>structure theorem</u> hold[s] (without proof):

$$\boxed{(sI-A)\cdot\Sigma(s) \equiv B \cdot B_m^{-1} \cdot \delta(s)}$$

$$\Rightarrow \quad (sI-A)^{-1} B = \Sigma(s) \cdot \left[B_m^{-1} \cdot \delta(s)\right]^{-}$$

(vi) Remember:

$$G(s) = C(sI-A)^{-1}B + D$$

$$= C \cdot \Sigma(s) \cdot \left[B_m^{-1} \cdot \delta(s)\right]^{-1} + D$$

$$\Rightarrow G(s) = \underbrace{\left[C \cdot \Sigma(s) + D \cdot B_m^{-1} \cdot \delta(s)\right]}_{R(s)} \cdot \underbrace{\left[B_m^{-1} \cdot \delta(s)\right.}_{}$$

$$= \quad \underset{(p \times m)}{R(s)} \quad \cdot \quad \underset{(m \times m)}{P(s)^{-1}}$$

Since: $G(s) = R(s) \cdot P^{-1}(s) \cdot Q(s) + W(s)$

$$\Rightarrow \left| \begin{array}{l} Q(s) = I^{(m)} \\ W(s) = \emptyset^{(p \times m)} \end{array} \right|$$

- $P(s)$ & $R(s)$ have the following properties:

(1) $P(s)$ is column-proper since

$$|\Gamma_c[P(s)]| = |B_u^{-1}| = 1 \neq \emptyset$$

(2) The degree of every column of $R(s)$ is smaller (if $D = \emptyset$) or equal (if $D \neq \emptyset$) to the degree of the same column of $P(s)$:

$$\partial c_j[R(s)] \leq \partial c_j[P(s)] \; ; \forall j = 1, \ldots,$$

Example:

$$\dot{\underline{x}} = \begin{bmatrix} \emptyset & 1 & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & 1 & \emptyset & \emptyset \\ -6 & -11 & -6 & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & 1 \\ \emptyset & \emptyset & \emptyset & -2 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & \emptyset \\ 1 & 2 \\ \emptyset & \emptyset \\ \emptyset & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & \phi & 1 & -1 & 2 \\ 2 & 4 & 1 & \phi & 3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & \phi \\ \phi & 2 \end{bmatrix} \underline{u}$$

is a state-space description in controller-canonical form with

$d_1 = 3$; $d_2 = 2$

$$\Rightarrow A_m = \begin{bmatrix} -6 & -11 & -6 & \phi & \phi \\ \phi & \phi & \phi & -2 & -3 \end{bmatrix} ; \; B_m = \begin{bmatrix} 1 & 2 \\ \phi & 1 \end{bmatrix}$$

$$\Sigma(s) = \begin{bmatrix} 1 & \phi \\ s & \phi \\ s^2 & \phi \\ \phi & 1 \\ \phi & s \end{bmatrix}$$

$$\Rightarrow A_m \cdot \Sigma(s) = \begin{bmatrix} (-6-11s-6s^2) & \phi \\ \phi & (-2-3s) \end{bmatrix}$$

(does not have to be diagonal)

$$\delta(s) = \begin{bmatrix} s^3 & \phi \\ \phi & s^2 \end{bmatrix} - A_m \cdot \Sigma(s)$$

$$= \begin{bmatrix} (s^3+6s^2+11s+6) & \phi \\ \phi & (s^2+3s+2) \end{bmatrix}$$

$$B_m^{-1} = \begin{bmatrix} 1 & -2 \\ \phi & 1 \end{bmatrix}$$

$$C \cdot \Sigma(s) = \begin{bmatrix} (s^2+1) & (2s-1) \\ (s^2+4s+2) & (3s) \end{bmatrix}$$

$$D \cdot B_m^{-1} \cdot \delta(s) = \begin{bmatrix} 1 & \phi \\ \phi & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ \phi & 1 \end{bmatrix} \cdot \delta(s)$$

$$= \begin{bmatrix} 1 & -2 \\ \phi & 2 \end{bmatrix} \cdot \begin{bmatrix} (s^3+6s^2+11s+6) & \phi \\ \phi & (s^2+3s+2) \end{bmatrix}$$

$$= \begin{bmatrix} (s^3+6s^2+11s+6) & (-2s^2-6s-4) \\ \phi & (2s^2+6s+4) \end{bmatrix}$$

$$\Rightarrow R(s) = \begin{bmatrix} (s^3+7s^2+11s+7) & (-2s^2-4s-5) \\ (s^2+4s+2) & (2s^2+6s+4) \end{bmatrix}$$

$$B_m^{-1} \cdot \delta(s) = P(s) = \begin{bmatrix} (s^3+6s^2+11s+6) & (-2s^2-6s-4) \\ \phi & (s^2+3s+2) \end{bmatrix}$$

$$\Rightarrow G(s) = R(s) \cdot P(s)^{-1}$$

is one way to find the transfer function matrix of this MIMO system.

The PM-representation in controller-canonical form is:

$$\left\{ \begin{array}{l} \begin{bmatrix} (s^3+6s^2+11s+6) & (-2s^2-6s-4) \\ \emptyset & (s^2+3s+2) \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (s^3+7s^2+11s+7) & (-2s^2-4s-5) \\ (s^2+4s+2) & (2s^2+6s+4) \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{array} \right.$$

With a small modification, this algorithm can also be used for not completely controllable systems:

$$A = \left[\begin{array}{c|c} A_c & A_{c\bar{c}} \\ \hline \emptyset & A_{\bar{c}} \end{array} \right] \; ; \quad B = \left[\begin{array}{c} B_c \\ \hline \emptyset \end{array} \right]$$

$$C = \left[\begin{array}{c|c} C_c & C_{\bar{c}} \end{array} \right]$$

after input-decoupling, with $\{A_c, B_c\}$ in controller-canonical form

<u>Remember</u>:

$$\left[\begin{array}{c|c} K & L \\ \hline \emptyset & M \end{array} \right]^{-1} = \left[\begin{array}{c|c} K^{-1} & -K^{-1}LM^{-1} \\ \hline \emptyset & M^{-1} \end{array} \right]$$

$$\Rightarrow G(s) = \left[C_c \,\middle|\, C_{\bar{c}} \right] \cdot \left[\begin{array}{c|c} (sI-A_c)^{-1} & -A_{c\bar{c}} \\ \hline \phi & (sI-A_{\bar{c}})^{-1} \end{array} \right] \left[\begin{array}{c} B_c \\ \hline \phi \end{array} \right]$$

$$= \frac{C_c (sI-A_c)^{+} |sI-A_{\bar{c}}| B_c}{|sI-A_c| \cdot |sI-A_{\bar{c}}|} + D$$

(only the controllable part goes into $G(s)$) ... expanded with $|sI-A_{\bar{c}}|$ to get <u>all</u> poles into the representation

Let: $\Sigma_c(s) = \Sigma(s)$ for A_c

$\delta_c(s) = \delta(s)$ for A_c

$\Delta_{\bar{c}}(s) = |sI-A_{\bar{c}}|$

$$\Rightarrow G(s) = \underbrace{\left[C_c \cdot \Sigma_c(s) + D \cdot B_{cm}^{-1} \cdot \delta_c(s) \right] \cdot \Delta_{\bar{c}}(s)}_{R(s)}$$

$$\cdot \left[\underbrace{B_{cm}^{-1} \cdot \delta_c(s) \cdot \Delta_{\bar{c}}(s)}_{P(s)} \right]^{-1}$$

gives rise to a PM-representation that contains the uncontrollable modes.

Observer-canonical Realization:

Let us start from the observer-canonical state-space representation:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} + D\underline{u} \end{array} \right|$$

We want: $R(s) = I^{(p)}$.

We can either use the duality principle:

$$\begin{array}{ccc} S_{OCF} & \xrightarrow{\text{duality}} & S_{CCF} \\ \vdots\downarrow & & \downarrow \text{structure theor} \\ PM_{OCF} & \xleftarrow{\text{duality}} & PM_{CCF} \end{array}$$

or come up with an equivalent form of the structure theorem.

Algorithm:

(i) Get A_p out of the p T_k-columns of A $\quad (A_p \in \mathbb{R}^{n \times p})$

(ii) Get C_p out of the T_k-columns of C

$$C_p = \begin{bmatrix} 1 & \emptyset & \emptyset & \emptyset \\ * & 1 & \emptyset & \emptyset \\ * & * & 1 & \emptyset \\ * & * & * & 1 \end{bmatrix} ; \quad |C_p| = 1$$

(iii) $$\Sigma(s) = \begin{bmatrix} 1 \; s \cdots s^{e_1 - 1} & & \emptyset \\ & 1 \; s \cdots s^{e_2 - 1} & \\ \emptyset & & \ddots \\ & & & 1 \; s \cdots \end{bmatrix}$$

is a $(p \times n)$ polynomial matrix

(iv) $$\delta(s) = \begin{bmatrix} s^{e_1} & & \emptyset \\ & s^{e_2} & \\ \emptyset & & \ddots & \\ & & & s^{e_p} \end{bmatrix} - A_p \cdot \Sigma(s)$$

is a $(p \times p)$ diagonal polynomial matrix.

(v) Use the <u>structure theorem</u>:

$$\boxed{\Sigma(s) \cdot (sI - A) \equiv \delta(s) \cdot C_p^{-1} \cdot C}$$

$$\Rightarrow \quad C(sI-A)^{-1} = \left[\delta(s) \cdot C_p^{-1}\right]^{-1} \cdot \Sigma(s)$$

$$\Rightarrow G(s) = \left[\delta(s) \cdot C_p^{-1}\right]^{-1} \cdot \Sigma(s) \cdot B + D$$

$$= \underbrace{\left[\delta(s) \cdot C_p^{-1}\right]}_{P(s)}{}^{-1} \cdot \underbrace{\left[\Sigma(s) \cdot B + \delta(s) \cdot C_p^{-1} \cdot D\right]}_{Q(s)}$$

For not totally observable systems:

$$G(s) = \Big[\underbrace{\Delta_{\bar{o}}(s)\cdot\delta_o(s)\cdot C_{op}^{-1}}_{P(s)}\Big]^{-1}$$

$$\cdot\; \underbrace{\Delta_{\bar{o}}(s)\cdot\left[\Sigma_o(s)\cdot B_o + \delta_o(s)\cdot C_{op}^{-1}\cdot D\right]}_{Q(s)}$$

$$\Rightarrow G(s) = P^{-1}(s)\cdot Q(s)$$

$$\Rightarrow \left|\begin{array}{l} R(s) = I^{(p)} \\ W(s) = \phi^{(p\times m)} \end{array}\right|$$

is an observer-canonical realization.