Problem: In the past, we had seen that two systems which differ in their transfer functions are <u>different</u> systems thus transfer functions <u>are</u> unique. Let us prove that two state-space representations which are similar to each other result in the same transfer function.

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}u \\ y = \underline{c}'\underline{x} + du \end{array} \right| \Longleftrightarrow G(s) = \underline{c}'(sI-A)^{-1}\underline{b} + d$$

$$\Updownarrow \text{similar}$$

$$\left| \begin{array}{l} \dot{\underline{\xi}} = TAT^{-1}\underline{\xi} + T\underline{b}u \\ y = \underline{c}'T^{-1}\underline{\xi} + du \end{array} \right| \Longleftrightarrow \hat{G}(s) = ?$$

$$\hat{G}(s) = \hat{\underline{c}}'(sI-\hat{A})^{-1}\hat{\underline{b}} + \hat{d}$$

$$= \underline{c}'T^{-1}(sI - TAT^{-1})^{-1}T\underline{b} + d$$

$$= \underline{c}'T^{-1}\left(TsIT^{-1} - TAT^{-1}\right)^{-1}T\underline{b} + d$$

$$= \underline{c}'T^{-1}\left[T(sI-A)T^{-1}\right]^{-1}T\underline{b} + d$$

$$= \underline{c}'T^{-1}T(sI-A)^{-1}T^{-1}T\underline{b} + d$$

$$= \underline{c}'(sI-A)^{-1}\underline{b} + d$$

$$= G(s) \qquad \text{q.e.d.}$$

<u>Thus</u>:

$$\boxed{\text{State-space Representation}} \xrightarrow{\text{unique}} \boxed{\text{Transfer Function}}$$

$$\boxed{\text{Transfer Function}} \xrightarrow[\text{unique}]{\text{not}} \boxed{\text{State-Space Representation}}$$

Example:

$$\left| \begin{array}{l} \dot{\underline{x}} = \begin{bmatrix} -56 & -15 & 30 \\ 30 & 9 & -16 \\ -90 & -25 & 48 \end{bmatrix} \underline{x} + \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} u \\ \\ y = \begin{bmatrix} -7 & -1 & 4 \end{bmatrix} \underline{x} \end{array} \right|$$

Find the transfer function!

$$Q_c = \left[\underline{b} \,,\, A\underline{b} \,,\, A^2\underline{b} \right]$$

$$= \begin{bmatrix} -2 & 7 & -17 \\ 1 & -3 & 7 \\ -3 & 11 & -27 \end{bmatrix}$$

$\Rightarrow \det(Q_c) \equiv 0$

$\Rightarrow Q_c^{-1}$ does not exist !!!

Something went wrong. Let u try the other technique.

$$(sI-A) = \begin{bmatrix} (s+56) & 15 & -30 \\ -30 & (s-9) & 16 \\ 90 & 25 & (s-48) \end{bmatrix}$$

$$\Rightarrow \det(sI-A) = s^3 - s^2 - 10s - 8$$

$$\operatorname{adj}(sI-A) = \begin{bmatrix} (s^2-57s+32) & (-15s-30) & (30s- \ldots \\ 30s & (s^2+8s+12) & (-16s+ \ldots \\ (-90s+60) & (-25s-50) & (s^2+47s \ldots \end{bmatrix}$$

$$\Rightarrow (sI-A)^{+}\underline{b} = \begin{bmatrix} (-2s^2+9s-4) \\ (s^2-4s) \\ (-3s^2+14s-8) \end{bmatrix}$$

$$\Rightarrow \underline{c}'(sI-A)^{+}\underline{b} = (s^2-3s-4)$$

$$\Rightarrow G(s) = \frac{s^2-3s-4}{s^3-s^2-10s-8}$$

We factorize:

$$\underline{\underline{G(s)}} = \frac{(s-4)(s+1)}{(s-4)(s+1)(s+2)} = \underline{\underline{\frac{\ldots}{s\ldots}}}$$

$\Rightarrow$ Pole/zero cancellations.

$$G(s) = \frac{1}{s+2}$$

Thus, another state-space representation would be:

$$\left| \begin{array}{l} \dot{\xi} = -2\xi + u \\ y = \xi \end{array} \right|$$

Obviously, ξ is <u>not</u> simila to $\underline{x}$ (even not dimensionwise

- This system is obviously with respect to input/output behavior a first-order system. However, there exist some internal states that can either not be reached from the input (uncontrollable stat or that cannot be seen fro the output (unobservable states

Recipe: We must find a transformation which makes the uncontrollable states as such visible.

Algorithm:

(1) Take as many columns of Q_c as are linearly independent, and extend by anything that gives a full rank:

$$\tilde{Q}_c = \left[Q_{c_{\text{l.ind.}}} \;\vdots\; \text{extension} \right]$$

(2) Use:

$$T = \tilde{Q}_c^{-1}$$

for a similarity transformation

Example continued:

$$Q_c = \begin{bmatrix} -2 & 7 & -17 \\ 1 & -3 & 7 \\ -3 & 11 & -27 \end{bmatrix}$$

lin. indep.

$$\Rightarrow \quad \tilde{Q}_c = \left[\begin{array}{cc:c} -2 & 7 & * \\ 1 & -3 & * \\ -3 & 11 & * \end{array}\right]$$

e.g. $$\tilde{Q}_c = \left[\begin{array}{cc:c} -2 & 7 & \phi \\ 1 & -3 & \phi \\ -3 & 11 & -1 \end{array}\right]$$

$$\Rightarrow \det(\tilde{Q}_c) = 1$$

$$\Rightarrow \operatorname{inv}(\tilde{Q}_c) = \operatorname{adj}(\tilde{Q}_c) = \underbrace{\begin{bmatrix} 3 & 7 & \phi \\ 1 & 2 & \phi \\ 2 & 1 & -1 \end{bmatrix}}_{T}$$

$$\Rightarrow \hat{A} = T A T^{-1} = \begin{bmatrix} \phi & -2 & 22 \\ 1 & -3 & 2 \\ \phi & \phi & 4 \end{bmatrix}$$

$$\underline{\hat{b}} = T \cdot \underline{b} = \begin{bmatrix} 1 \\ \phi \\ \phi \end{bmatrix} \qquad \underline{\hat{c}}' = \underline{c}' T^{-1} = [1 \;\; -2 \;\; -1$$

$$\Rightarrow \quad \dot{\underline{\eta}} = \left[\begin{array}{cc|c} 0 & -2 & 22 \\ 1 & -3 & 2 \\ \hline 0 & 0 & 4 \end{array}\right] \underline{\eta} + \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cc|c} 1 & -2 & -4 \end{array}\right] \underline{\eta}$$

$$\Rightarrow \quad \underline{\underline{\dot{\eta}_3 = 4\,\eta_3}}$$

can obviously not be reached from the input. There exists an uncontrollable mode at

$$\underline{\underline{\lambda = 4}}\,.$$

This mode is even instable

$\Rightarrow$ The smallest disturbance will send some internal variables off into saturation.

$\Rightarrow$ Bad system.

In general: This transformation generates:

$$\underline{\dot{y}} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline \emptyset & A_{22} \end{array}\right] \underline{y} + \left[\begin{array}{c} \underline{b}_1 \\ \hline \emptyset \end{array}\right] u$$

all these modes are uncontrollable. A_{22} contains the uncontrollable eigenvalues.

The controllable subsystem is:

$$\left| \begin{array}{l} \underline{\dot{x}}_c = A_{11} \cdot \underline{x}_c + \underline{b}_1 \cdot u \\ y = \underline{c}_1' \cdot \underline{x}_c + d \cdot u \end{array} \right|$$

Example continued:

$$\left| \begin{array}{l} \underline{\dot{x}}_c = \begin{bmatrix} \emptyset & -2 \\ 1 & -3 \end{bmatrix} \underline{x}_c + \begin{bmatrix} 1 \\ \emptyset \end{bmatrix} u \\ \\ y = [1 \quad -2] \underline{x}_c \end{array} \right|$$

We must now also get rid of those modes which are unobservable.

Algorithm:

(1) Build the observability matrix:

$$Q_o = [\underline{c}' ; \underline{c}'A ; \dots ; \underline{c}'A^{n-}$$

(2) If $\det(Q_o) \equiv \phi \Longrightarrow$ there exist unobservable modes.

(3) Take as many rows a are linearly independent, an extend from below with something that gives a full rank

$$\tilde{Q}_o = [Q_{o_{\text{lin. indep.}}} ; *]$$

(4) Use $\tilde{Q}_0$ as T for a similarity transformation.

<u>Example continued:</u>

$$Q_0 = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \Rightarrow \det(Q_0) = 0$$

$$\Rightarrow \tilde{Q}_0 = \underbrace{\begin{bmatrix} 1 & -2 \\ \hdashline 0 & 1 \end{bmatrix}}_{T} \Rightarrow \det(\tilde{Q}_0) = 1$$

$$\Rightarrow \hat{A} = T \cdot \overset{*}{A}_c / T = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\hat{\underline{b}} = T \cdot \overset{*}{\underline{b}}_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{\underline{c}}' = \overset{*}{\underline{c}}_c{}' / T = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\Rightarrow \dot{\underline{v}} = \left[\begin{array}{c:c} -2 & 0 \\ \hdashline 1 & -1 \end{array}\right] \underline{v} + \begin{bmatrix} 1 \\ \hdashline 0 \end{bmatrix} u$$

$$y = \left[\begin{array}{c:c} 1 & 0 \end{array}\right] \underline{v}$$

The second state-variable does not directly go into the output neither does it influence any of the other states

$\Rightarrow$ ν_2 is unobservable.

<u>In general</u>: This transformation gives a representation:

$$\left|\begin{array}{l} \dot{\underline{\nu}} = \left[\begin{array}{c:c} A_{11} & \emptyset \\ \hdashline A_{21} & A_{22} \end{array}\right] \underline{\nu} + \left[\begin{array}{c} \underline{b}_1 \\ \hdashline \underline{b}_2 \end{array}\right] u \\ y = \left[\begin{array}{c:c} \underline{c}_1' & \underbrace{\emptyset}_{\substack{\text{all these} \\ \text{states are} \\ \text{unobservable}}} \end{array}\right] \underline{\nu} + [d] u \end{array}\right|$$

A_{22} contains the unobservable eigenvalues.

$$\Rightarrow \left|\begin{array}{l} \dot{\underline{\nu}}_1 = A_{11} \cdot \underline{\nu}_1 + \underline{b}_1 \cdot u \\ y = \underline{c}_1' \cdot \underline{\nu}_1 + d \cdot u \end{array}\right|$$

is the observable subsystem.

Example continued:

There is a stable unobservable eigenvalue at

$$\underline{\underline{\lambda = -1}}$$

The remaining subsystem is:

$$\left| \begin{array}{l} \dot{x}_1 = -2x_1 + u \\ y = x_1 \end{array} \right|$$

$$\Longrightarrow \quad \underline{\underline{G(s) = \frac{1}{s+2}}}$$

From the last example, a number of observations can be made:

(1) While we have shown earlier that similar state-space representations always show the same transfer function, it is incorrect to conclude that all possible representations of a system with the same transfer function are similar to each other. (The third order representation is <u>not</u> related by a similarity transformation to the first order representation.)

(2) The presence of uncontrollable or unobservable modes leads always to pole/zero-cancellation in the transfer function. It is correct (at least for SISO systems!) that pole/zero-cancellations in the transfer function always indicate the presence of either uncontrollable

(3) Among the state-space representatio[ns] there exists one important subclass namely containing all those representations in which al[l] uncontrollable and unobservable mo[des] have been eliminated. These are the representations of minimal degre[e] and they are often referred to as <u>minimal realizations</u>.

(4) All minimal realizations of a system are related by similarity transformations.

(5) Iff the Q_c-matrix of a system is non-singular ($\det(Q_c) \neq \emptyset$), there do not exist any uncontrollab[le] modes.

(6) Iff the Q_o-matrix of a system is non-singular, there are no unobservable modes.

(7) A controller-canonical realization of a system exists iff the syste[m] is fully controllable.

(8) The number of linearly independent columns of the Q_c-matrix determ the number of controllable modes. This is also called the <u>Rank</u> of the Q_c-matrix. The Rank of a matrix can e.g. be determined by the following algorithm:

> Try to invert Q_c by a <u>Gaussian elimination</u>. The number of <u>pivot steps</u> that can be performed before the remaining subsystem is all zero is equal to the rank of the matrix.

We shall often write:

$\text{ord}(Q_c)$ ∴ dimension of a square matrix Q_c

$\rho_c(Q_c)$ ∴ Rank of the Q_c-matrix

$\nu_c(Q_c)$ ∴ <u>Rank deficiency</u> or <u>Nullity</u> of the Q_c-matrix.

Obviously: $$\underset{\substack{\uparrow \\ \text{column}}}{\rho_c}(Q_c) + \nu_c(Q_c) \equiv \text{ord}(Q_c)$$

(9) If in a representation there exist a number of rows of $[A, \underline{b}]$ which are zero except of the diagonal block of A $\Longrightarrow$ all these states are uncontrollable

$$\dot{\underline{x}} = \begin{bmatrix} A_1 & * & A_2 \\ \phi & * & \phi \\ A_3 & * & A_4 \end{bmatrix} \underline{x} + \begin{bmatrix} \underline{b}_1 \\ \phi \\ \underline{b}_2 \end{bmatrix} u$$

$$y = [\, \underline{c}_1' \;\vdots\; * \;\vdots\; \underline{c}_2' \,] \underline{x} + [d]\, u$$

More Controllable subsystem (if no other states are uncontrollable):

$$\dot{\underline{\xi}} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \underline{\xi} + \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \end{bmatrix} u$$

$$y = [\, \underline{c}_1' \;\vdots\; \underline{c}_2' \,] \underline{\xi} + [d]\, u$$

This can of course be a little generalized:

$$\dot{\underline{x}} = \begin{bmatrix} -1 & 10 & 25 & -18 & -13 & 2 \\ \phi & -2 & \phi & \phi & 1 & \phi \\ 23 & 14 & 14 & -1 & \phi & -1 \\ 2 & 22 & 18 & -18 & \phi & -5 \\ \phi & 2 & \phi & \phi & -8 & \phi \\ 18 & \phi & 23 & 27 & 5 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} \phi \\ \phi \\ 1 \\ -1 \\ \phi \\ 1 \end{bmatrix} u$$

$$y = [\, 1 \;\; 2 \;\; 3 \;\; \phi \;\; 4 \;\; \phi \,] \underline{x} + [27]\, u$$

$$\left| \begin{aligned} \dot{x}_2 &= -2x_2 + x_5 \\ \dot{x}_5 &= 2x_2 - 8x_5 \end{aligned} \right| \Longrightarrow \text{uncontrollab}$$

$\begin{bmatrix} -2 & 1 \\ 2 & -8 \end{bmatrix}$ is such a "diagonal block

$\Rightarrow$ States 2 and 5 are uncontrol

We can cancel the 2nd and 5th row and column

$$\Rightarrow \left| \begin{array}{l} \dot{\underline{x}}_c = \begin{bmatrix} -1 & 25 & -18 & 2 \\ 23 & 14 & -1 & -1 \\ 2 & 28 & -10 & -5 \\ 18 & 23 & 27 & -1 \end{bmatrix} \underline{x}_c + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 3 & 0 & 0 \end{bmatrix} \underline{x}_c + [27] u \end{array} \right|$$

is the controllable subsystem (unless there exist further uncontrollable modes that are yet unvisible as such).

(10) One convenient way to get rid of <u>all</u> uncontrollable modes is the <u>input-decoupling</u> algorithm shown before which transforms the system into:

$$\left| \begin{array}{l} \dot{\underline{\xi}} = \left[\begin{array}{c|c} A_c & A_{12} \\ \hline 0 & A_{\bar{c}} \end{array} \right] \underline{\xi} + \left[\begin{array}{c} b_c \\ \hline 0 \end{array} \right] u \\ y = \left[\underline{c}_c' \;\vdots\; \underline{c}_{\bar{c}}' \right] \underline{\xi} + [d] u \end{array} \right|$$

A_c contains the controllable modes
$A_{\bar{c}}$ contains the uncontrollable mod

$$\Rightarrow \quad \left| \begin{array}{l} \dot{\underline{x}}_c = [A_c]\,\underline{x}_c + [\underline{b}_c]\,u \\ y = [\underline{c}_c']\,\underline{x}_c + [d]\,u \end{array} \right|$$

is the controllable subsystem which has <u>no</u> uncontrollable modes left.

(ii) The number of linearly independent rows of the Q_o-matrix determines the number of observable modes, thus:

$$\rho_r(Q_o) \therefore \ \# \text{ observable modes}$$
$$\nu_r(Q_o) \therefore \ \# \text{ unobservable modes.}$$

row

[However, in a square matrix:
$\rho_r(\cdot) \equiv \rho_c(\cdot)$.].

(12) If in a representation there exist a number of columns of $[A \,;\, \underline{c}']$ which are zero except for the "diagonal" block of $A \Rightarrow$ all these states are unobservable

$$\dot{\underline{x}} = \left[\begin{array}{c|c|c} A_1 & \phi & A_2 \\ \hline * & * & * \\ \hline A_3 & \phi & A_4 \end{array}\right] \underline{x} + \left[\begin{array}{c} \underline{b}_1 \\ \hline * \\ \hline \underline{b}_2 \end{array}\right] u$$

$$y = \left[\begin{array}{c|c|c} \underline{c}_1' & \phi & \underline{c}_2' \end{array}\right] \underline{x} + [d]\, u$$

$\Rightarrow$ Reduced subsystem with these unobservable modes removed:

$$\dot{\underline{\xi}} = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \underline{\xi} + \left[\begin{array}{c} \underline{b}_1 \\ \hline \underline{b}_2 \end{array}\right] u$$

$$y = \left[\begin{array}{c|c} \underline{c}_1' & \underline{c}_2' \end{array}\right] \underline{\xi} + [d]\, u$$

(13) One convenient way to get rid of <u>all</u> unobservable modes is the <u>output-decoupling</u> algorithm shown before which transforms the system into:

$$\left| \begin{array}{l} \dot{\underline{\xi}} = \left[\begin{array}{c:c} A_o & \phi \\ \hdashline A_{21} & A_{\bar{o}} \end{array}\right] \underline{\xi} + \left[\begin{array}{c} \underline{b}_o \\ \hdashline \underline{b}_2 \end{array}\right] u \\ y = \left[\begin{array}{c:c} \underline{c}_o' & \phi \end{array}\right] \underline{\xi} + [d] u \end{array} \right|$$

A_o contains the observable modes, and
$A_{\bar{o}}$ contains the unobservable modes

$$\Rightarrow \quad \begin{array}{l} \dot{\underline{x}}_o = [A_o] \underline{x}_o + [\underline{b}_o] u \\ y = [\underline{c}_o'] \underline{x}_o + [d_r] u \end{array}$$

is the observable subsystem which has __no__ unobservable modes left.

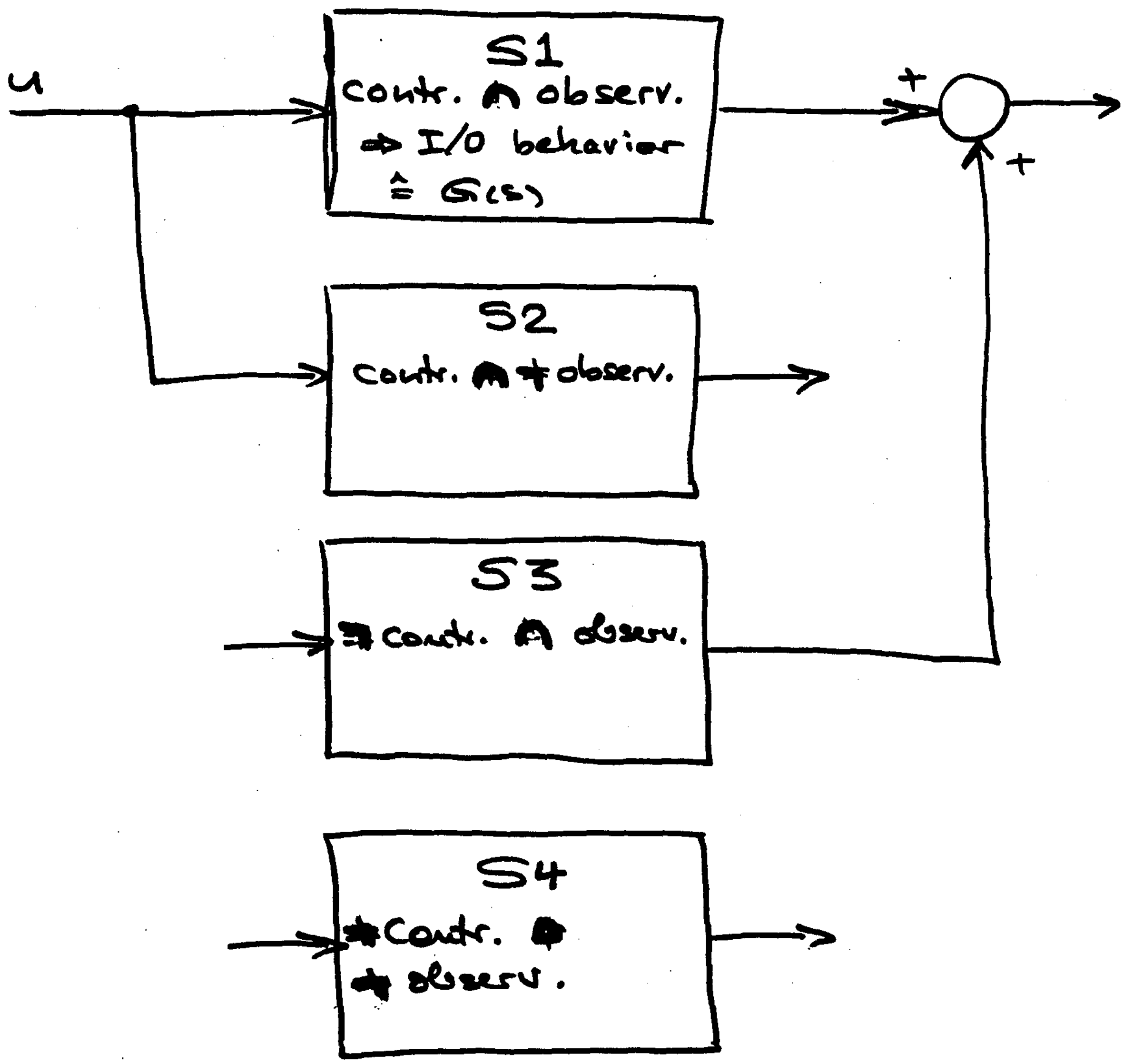
(14) Uncontrollable / unobservable modes are not visible from the outside. They do not influence the input/output behavior (that is why they don't show in the transfer function). They influence internal state behavior only

(15) Uncontrollable / unobservable modes are normally harmless (just wasteful). However, this is true only if they are all <u>stable</u>:

- If there exist instable uncontrollable/unobservable modes, the smallest disturbance will drive some internal state variables into <u>saturation</u>. Saturation is a non-linear element. The system will at this moment cease to behave like a linear system, an the formerly uncontrollable/unobserval modes become suddenly visible and will make the performance of the system unusable.
- A system with instable uncontrolle unobservable modes is to be <u>trashed</u> There is nothing that we can do to make this system work correctly.

The Kalman Decomposition

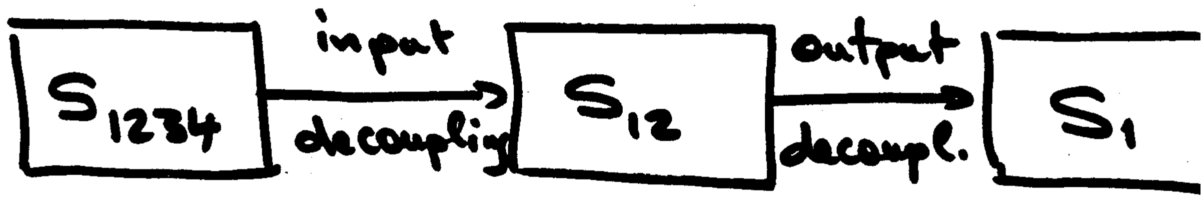
Any system can be decomposed in the following manner:



whereby any and all of these subsystems may be present / missing.

- The transfer function contains the subsystem S_1 only, all others cancel out (pole/zero cancellation). The same is true for any <u>minimal realization</u>.

- Given a particular state-space representation, a minimal realization can be found by input/output decoupling:



<u>or</u>: