Controllability:

Def: A system is called controllable iff it is possible to find an input $u(t)$ such that the system is brought to the origin $\underline{x}(t) = \emptyset$ in a finite amount of time from arbitrary initial conditions.

- We will start by constructing a recipe by which some types of systems can be brought from arbitrary initial conditions to zero in even zero time.

- We notice that for $u(t) \equiv \emptyset$, $\underline{x}(t)$ is a set of continuous and continuously differentiable functions of time:

$$\underline{x}(t) = e^{At}\,\underline{x}_0 \quad .$$

Thus, as it is our aim to immediately bring the system to the origin, obviously, the zero-inp response does not contribute :

$$\left\{ \begin{array}{l} u(t) = \emptyset \\ \underline{x}(t=\emptyset-) = \underline{x}_0 \end{array} \right\} \Longrightarrow \underline{x}(t=\emptyset+) \equiv \underline{x}(t=\emptyset-$$

- Let us thus see what we can do with the zero-state response:

$$\underline{x}(t) = \int_{0-}^{t} e^{A(t-\tau)} \underline{b}\, u(\tau)\, d\tau$$

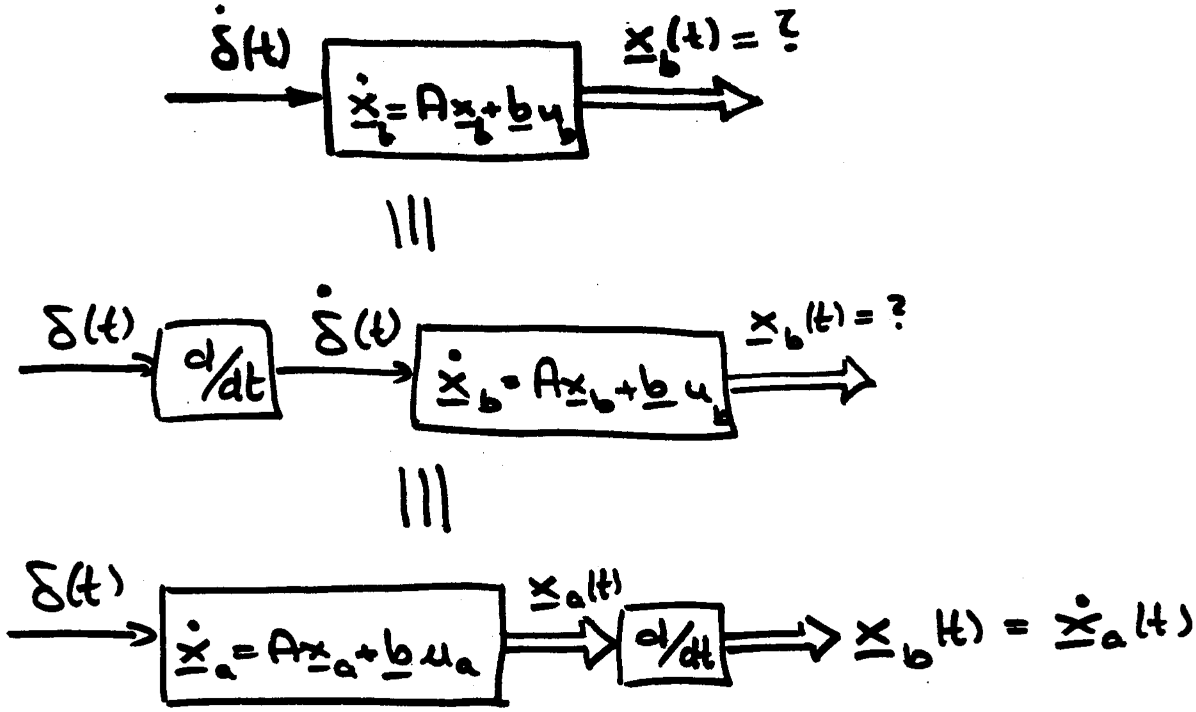
$$\equiv e^{At} \int_{0-}^{t} e^{-A\tau} \underline{b}\, u(\tau)\, d\tau$$

Let us first apply: $u_a(t) = \delta(t)$

$$\Rightarrow \underline{x}_a(t=\emptyset+) = \underline{x}_a(t=\emptyset-) + e^{A\emptyset} \underbrace{\int_{0-}^{0+} e^{-A\tau} \underline{b}\, \delta(\tau)\, d}_{e^{-A\emptyset}\underline{b}}$$

$$\Rightarrow \underline{x}_a(t=\emptyset+) = \underline{x}_a(t=\emptyset-) + \underline{b}$$

- Let us now see what happens if $u_b(t) = \dot{\delta}(t)$.



due to linearity of system.

$\Rightarrow$ The new state vector will be the derivative of the previous one, thus:

$$\underline{x}_b(t) = \dot{\underline{x}}_a(t) = Ae^{At}\int_{0^-}^{t} e^{-A\tau}\underline{b}\,u(\tau)\,d\tau + \underbrace{e^{At}\cdot e^{-At}}_{I^{(n)}}\cdot\underline{b}\cdot u(t)$$

$$\Rightarrow \underline{x}_b(t) = A\underline{x}_a(t) + \underline{b}u(t)$$

$$\Rightarrow \underline{x}_b(t=\phi+) = \underline{x}_b(t=\phi-) + A\underbrace{e^{A\phi}e^{-A\phi}}_{I^{(n)}}\underline{b} + \underbrace{\underline{b}\dot{\delta}(t=\phi}_{\phi}$$

$$\Rightarrow \underline{x}_b(t=\phi+) = \underline{x}_b(t=\phi-) + A\underline{b}$$

<u>etc.</u>

$$u_k(t) = \delta^{(k)}(t)$$

$$\Rightarrow \underline{x}_k(t=\phi+) = \underline{x}_k(t=\phi-) + A^k\underline{b}$$

We now use the superposition principle:

$$u(t) = k_0\delta(t) + k_1\dot{\delta}(t) + \cdots + k_{n-1}\delta^{(n-1)}(t)$$

$$\Rightarrow \underline{x}(t=\phi+) = \underline{x}(t=\phi-) + k_0\underline{b} + k_1A\underline{b} + \cdots + k_{n-1}A$$

$$\equiv \underline{x}(t=\phi-) + [\underline{b}, A\underline{b}, \ldots, A^{n-1}\underline{b}] \cdot \begin{bmatrix} k_0 \\ k_1 \\ \vdots \\ k_{n} \end{bmatrix}$$

$$\Rightarrow \underline{x}(t=\phi+) = \underline{x}(t=\phi-) + Q_c \cdot \underline{k}$$

As $\underline{x}(t=\phi+) \overset{!}{=} \phi \Rightarrow$ we need to choose the k_i such that:

$$\underline{x}(t=\phi-) = -Q_c \cdot \underline{k}$$

For arbitrary initial conditions, this can be accomplished iff Q_c spans the total n-dimensional space $\iff$ $\underline{\underline{\text{Rank}(Q_c) \equiv n}}$

or: $\underline{\underline{Q_c \text{ is nonsingular.}}}$

- We now realize that the Q_c-matrix tells us how well we can reach a particular state from the input.

The controllability-canonical form

It may make sense to see a representation where

$$Q_c \equiv I^{(n)} .$$

In such a representation, the $\underline{k}$-vector is immediately applied to the state-vector, that is: each state can be reached equally well. The **sensitivitie[s]** of reaching the states from the inputs are perfectly **balanc[ed]**

- Let us see what happens to Q_c in a similarity transformation:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}u \\ y = \underline{c}'\underline{x} + du \end{array} \right| \xrightarrow{T} \left| \begin{array}{l} \dot{\underline{\xi}} = \hat{A}\underline{\xi} + \hat{\underline{b}}u \\ y = \hat{\underline{c}}'\underline{\xi} + \hat{d}u \end{array} \right.$$

$$Q_c = [\underline{b}, A\underline{b}, \ldots, A^{n-1}\underline{b}] \qquad \hat{Q}_c = [\hat{\underline{b}}, \hat{A}\hat{\underline{b}}, \ldots$$

$$\Rightarrow \hat{Q}_c = [T\underline{b}, (TAT^{-1})T\underline{b}, (TAT^{-1})(TAT^{-1})T\underline{b}, \ldots$$

$$= [T\underline{b}, TA\underline{b}, TA^2\underline{b}, \ldots, TA^{n-1}\underline{b}]$$

$$\Rightarrow \underline{\underline{\hat{Q}_c = T \cdot Q_c}}$$

__but__: $\hat{Q}_c \overset{!}{=} I^{(n)} \Longleftrightarrow \underline{\underline{T = Q_c^{-1}}}$

If we apply a similarity transformation with $T = Q_c^{-1}$, we obtain a new representation which has $\hat{Q}_c = I^{(n)}$. This is called the controllability-canonical form.

- It turns out that:

$$\left|\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & \emptyset & -a_1 \\ & 1 & \ddots & & \vdots \\ & \emptyset & \ddots & 0 & \vdots \\ & & & 1 & -a_{n-1} \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ \emptyset \\ \vdots \\ \emptyset \end{bmatrix} u \\ y &= [\beta_1\ \beta_2 \cdots \beta_n]\, \underline{x} + [d]\, u \end{aligned}\right|$$

is this controllability-canonical form. Thus:

$$A_{\check{CCF}} \equiv A_{OCF} \equiv A'_{CCF}$$

$$\underline{b}_{\check{CCF}} \equiv \mathcal{R}\{\underline{b}_{CCF}\}$$: the reverse vector of $\underline{b}_{CC}$

$$\underline{c}'_{\check{CCF}} \equiv \underline{\beta}'$$: the Markov Vector.

Proof: $\underline{\underline{Q_{c_{\check{CCF}}} \equiv I^{(n)}}}$

(can be verified by inspection.)

but: $\underline{\underline{\underline{\beta}'}} \equiv \underline{c}' \cdot Q_c \equiv \underline{\underline{c}'}$ // q.e.d

Observability:

Def: A system is called observable iff all initial conditions can be reconstructed in a finite amount of time from measurements of inputs and outputs alone.

- Given the system:

$$\left| \begin{array}{l} \dot{\underline{x}} = A\underline{x} + \underline{b}u \\ y = \underline{c}'\underline{x} + du \end{array} \right|$$

we can (at least theoretically) construct the Nordsieck-vector of $y(t)$:

$$y(t) = \underline{c}'\underline{x}(t) + du(t)$$

$$\Rightarrow y(t=0+) = \underline{c}'\underline{x}(t=0+) + du(t=$$

$$\dot{y}(t) = \underline{c}'\dot{\underline{x}}(t) + d\dot{u}(t)$$

$$= \underline{c}'A\underline{x}(t) + \underline{c}'\underline{b}\,u(t) + d\,\dot{u}(t)$$

$$\Rightarrow \dot{y}(t=\phi+) = \underline{c}'A\underline{x}(t=\phi+) + \underline{c}'\underline{b}\,u(t=\phi+) + d\,\dot{u}(t=\phi+)$$

$$\ddot{y}(t) = \underline{c}'A\dot{\underline{x}}(t) + \underline{c}'\underline{b}\,\dot{u}(t) + d\,\ddot{u}(t)$$

$$= \underline{c}'A^2\underline{x}(t) + \underline{c}'A\underline{b}\,u(t) + \underline{c}'\underline{b}\,\dot{u}(t) + d\,\ddot{u}(t$$

etc.

Thus, let

$$\mathcal{N}\{y\} = \begin{bmatrix} y(t=\phi+) \\ \dot{y}(t=\phi+) \\ \vdots \\ y^{(n-1)}(t=\phi+) \end{bmatrix} ; \quad \mathcal{N}\{u\} = \begin{bmatrix} u(t=\phi \\ \dot{u}(t=\phi \\ \vdots \\ u^{(n-1)}(t= \end{bmatrix}$$

be the output and input Nordsieck vectors. Then:

$$\mathcal{N}\{y\} = \begin{bmatrix} \underline{c}' \\ \underline{c}'A \\ \vdots \\ \underline{c}'A^{n-1} \end{bmatrix} \underline{x}(t=\phi+) + \begin{bmatrix} d & \phi & \cdots & \cdots & \phi \\ \underline{c}'\underline{b} & d & \phi & \cdots & \phi \\ \underline{c}'A\underline{b} & \underline{c}'\underline{b} & d & \phi \cdots & \phi \\ \vdots & \ddots & \ddots & \ddots & \phi \\ \vdots & & \ddots & \ddots & d \\ \cdots & & & \underline{c}'A\underline{b} & \underline{c}'\underline{b} \end{bmatrix}$$

$$\Rightarrow \mathcal{N}\{y\} = Q_o \cdot \underline{x}(t=\phi+) + \mathcal{T}_\ell\{[d;\, \overbrace{Q_o\underline{b}}^{B}]\} \cdot \mathcal{N}\{u\}$$

$\Rightarrow$ Out of measurements of inputs and outputs, we can reconstruct $\underline{x}(t=\phi+)$ iff $\underline{\underline{\text{Rank}(Q_o) = n}}$, that is: $\underline{\underline{Q_o \text{ is nonsingular.}}}$

- Let us now apply $u(t) \equiv \phi$.

$$\Rightarrow \mathcal{N}\{y\} = Q_o \cdot \underline{x}(t=\phi+)$$

The Q_o-matrix tells us how well we can observe a particular state from the output.

The observability-canonical Form

It may make sense to seek a representation where

$$Q_o = I^{(n)}$$

Then, the sensitivities of observation of all states will be perfectly balanced.

- Let us see what happens to Q_0 in a similarity transformation:

$$\hat{Q}_0 = \begin{bmatrix} \hat{\underline{c}}' \\ \hat{\underline{c}}'\hat{A} \\ \vdots \\ \hat{\underline{c}}'\hat{A}^{n-1} \end{bmatrix} = \begin{bmatrix} \underline{c}'T^{-1} \\ \underline{c}'T^{-1}\cdot(TAT^{-1}) \\ \underline{c}'T^{-1}\cdot(TAT^{-1})(TAT^{-1}) \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \underline{c}'T^{-1} \\ \underline{c}'AT^{-1} \\ \vdots \\ \underline{c}'A^{n-1}T^{-1} \end{bmatrix} = Q_0 \cdot T^{-1}$$

$$\Rightarrow \quad \hat{Q}_0 = Q_0 \cdot T^{-1}$$

but: $\hat{Q}_0 \equiv I^{(n)} \Longleftrightarrow Q_0 = T$

If we apply a similarity transformation with $T = Q_0$, we obtain a new representation which has $Q_0 = I^{(n)}$. This is called the observability-canonical form.

- It turns out that:

$$\dot{\underline{x}} = \begin{bmatrix} & 1 & & \emptyset \\ & & \ddots & \\ \emptyset & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \underline{x} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} u$$

$$y = [1 \ \emptyset \ \cdots \ \emptyset] \, \underline{x} + [d] u$$

is the observability-canonical form

$\Longrightarrow$ Observability-canonical form and controllability-canonical form are <u>dual</u> to each other.