Toeplitz & Hankel Matrices

Lemma: Polynomial multiplications can be expressed as multiplications of Toeplitz matrices with vectors.

Example: $a(s) = -3 + 6s + 5s^2$
$b(s) = -8 + 7s$

$\Rightarrow c(s) = a(s) \cdot b(s) \equiv b(s) \cdot a(s)$
$= 24 - 69s + 2s^2 + 35s^3$

This can be written as:

$$\underline{c}_s = \mathcal{T}_\ell(\underline{a}_s) \cdot \underline{b}_s \equiv \mathcal{T}_\ell(\underline{b}_s) \cdot \underline{a}_s$$

$$\begin{bmatrix} -3 & \emptyset \\ 6 & -3 \\ 5 & 6 \\ \emptyset & 5 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 7 \end{bmatrix} = \begin{bmatrix} 24 \\ -69 \\ 2 \\ 35 \end{bmatrix}$$

or:
$$\begin{bmatrix} -8 & 0 & 0 \\ 7 & -8 & 0 \\ 0 & 7 & -8 \\ 0 & 0 & 7 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 24 \\ -69 \\ 2 \\ 35 \end{bmatrix}$$

q.e.d.

Of course, if we reverse the order of the coefficients (start with the highest order coefficients), it works just as well, e.g.

$$\begin{bmatrix} 7 & 0 & 0 \\ -8 & 7 & 0 \\ 0 & -8 & 7 \\ 0 & 0 & -8 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 35 \\ 2 \\ -69 \\ 24 \end{bmatrix}$$

If only one of the polynomials is reversed, the Toeplitz-matrix turns into a Hankel-matrix:

$$\mathcal{H}_{up}\{\underline{a}_s\} \cdot \mathcal{R}\{\underline{b}_s\} = \underline{c}_s$$

$$\begin{bmatrix} \phi & \phi & -8 \\ \phi & -8 & 7 \\ -8 & 7 & \phi \\ 7 & \phi & \phi \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 24 \\ -69 \\ 2 \\ 35 \end{bmatrix}$$

- As polynomial multiplications are so common in frequency domain operations, Toeplitz- and Hankel-matrices (their equivalent time-domain operators) must be equally common.

- The reversal-operator correspon[ds] to replacing $s \rightarrow (\frac{1}{s})$ in the frequency domain:

$$p(s) = 7 + 21s + 13s^2 + s^3$$
$$\equiv s^3\left(7s^{-3} + 21s^{-2} + 13s^{-1} + 1\right)$$
$$\equiv s^3\left[1 + 13\left(\tfrac{1}{s}\right) + 21\left(\tfrac{1}{s}\right)^2 + 7\left(\tfrac{1}{s}\right)^3\right]$$

$\Rightarrow$ The coefficients have been reversed.

- This gives rise to yet another interpretation of the Markov parameters. From p. 81, we remember that:

$$\underline{p}_s = \mathcal{H}_{up}(\underline{a}_s) \cdot \underline{\beta}$$

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} a_1 & \cdots & a_{n-1} & 1 \\ \vdots & & \iddots & \\ a_{n-1} & \iddots & & \emptyset \\ 1 & & & \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

This can be reinterpreted in the light of our new discovery:

$$\begin{array}{c} \\ n \\ \\ \\ \\ n \\ \\ \end{array} \left[\begin{array}{cccc} & & & a_0 \\ & \phi & \iddots & a_1 \\ & \iddots & \iddots & \vdots \\ a_0 & a_1 & \cdots & a_{n-1} \\ \hline a_1 & \cdots & a_{n-1} & 1 \\ \vdots & \iddots & \iddots & \\ a_{n-1} & \iddots & \phi & \\ 1 & & & \end{array}\right] \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \left[\begin{array}{c} \gamma_n \\ \vdots \\ \gamma_2 \\ \gamma_1 \\ \hline b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{array}\right]$$

Interpretation in the frequency domain:

$$q(s) \cdot \left[s^n \cdot \beta\left(\tfrac{1}{s}\right) \right] =$$

$$\gamma_n + \gamma_{n-1} s + \cdots + \gamma_2 s^{n-2} + \gamma_1 s^{n-1}$$
$$+ b_0 s^n + b_1 s^{n+1} + \cdots + b_{n-1} s^{2n-1}$$

$$\Rightarrow \quad q(s)\cdot\beta\left(\tfrac{1}{s}\right) = \gamma\left(\tfrac{1}{s}\right) + p(s)$$

$$\Rightarrow \quad \beta\left(\tfrac{1}{s}\right) = G(s) + \frac{\gamma\left(\tfrac{1}{s}\right)}{q(s)}$$

where:

$$\begin{bmatrix} & & & & a_0 \\ & \emptyset & & ⋰ & a_1 \\ & & ⋰ & ⋰ & \vdots \\ & ⋰ & ⋰ & & \vdots \\ a_0 & a_1 & \cdots & & a_{n-1} \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} \gamma_n \\ \gamma_{n-1} \\ \vdots \\ \vdots \\ \gamma_1 \end{bmatrix}$$

or:

$$\begin{bmatrix} a_0 & a_1 & \cdots & \cdots & a_{n-1} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & \emptyset & & \ddots & a_1 \\ & & & & a_0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \vdots \\ \gamma_n \end{bmatrix}$$

$$\Rightarrow \quad \underline{\gamma} = T_{up}\{\underline{q}_s\}\cdot\underline{\beta}$$

Eigenvalues:

Def: An eigenvalue of a square matrix is a scalar λ for which a nontrivial solution ($\underline{x} \neq \emptyset$) of the equation

$$A\underline{x} = \lambda\underline{x}$$

exists.

$$\Leftrightarrow \quad \underbrace{(\lambda I - A)}_{M}\underline{x} = \underbrace{\emptyset}_{\underline{b}}$$

If M is regular:

$$\Rightarrow \quad \underline{x} = M^{-1} \cdot \underline{b} \equiv \emptyset$$

$\Rightarrow$ only trivial solution exists.

$\Rightarrow$ A non-trivial solution λ exists only if M is singular

$$\Leftrightarrow \quad \text{Rank}(\lambda I - A) < n$$

$$\Leftrightarrow \quad \det(\lambda I - A) = \emptyset$$

As $$G(s) = \underline{c}'(sI-A)^{-1}\underline{b} + d$$
$$= \frac{\underline{c}'(sI-A)^{+}\underline{b} + d\,|(sI-A)|}{|(sI-A)|}$$

$$\Leftrightarrow \left|\begin{array}{l} q(s) = \det(sI-A) \\ p(s) = \underline{c}'\,\mathrm{adj}(sI-A)\underline{b} + d\cdot\det(sI \end{array}\right.$$

$\Rightarrow$
- The roots of $q(s)$ are identical with the eigen= values of A. The polynomial

$$\det(\lambda I - A)$$

is called the <u>characteristic polynomial</u> of the system matrix A.

- One (good!) way to compute the roots of a polynomial is to find a matrix which has the same characteristic polynomial, and solve for its eigenvalues, e.g.

$$q(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_{n-1} s^{n-1}$$

$$\Rightarrow A = \begin{bmatrix} & 1 & & \emptyset \\ & & \ddots & \\ \emptyset & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}$$

is such a matrix.

$$\Rightarrow \boxed{\mathrm{Roots}\{q(s)\} \equiv \mathrm{Eig}\{\mathcal{C}_{lo}\{q_s\}\}}$$

This is the reason why the above matrix is often called <u>companion matrix</u>.

- As the transfer function is unique $\Rightarrow$ eigenvalues are insensitive to similarity transformations.

Direct Proof:

$$\det(\lambda I - \hat{A}) = \det(\lambda I - TAT^{-1})$$

$$= \det(\lambda TT^{-1} - TAT^{-1})$$

$$= \det\left[T(\lambda I - A)T^{-1}\right]$$

$$= \det(T) \cdot \det(\lambda I - A) \cdot \det(T^{-1})$$

$$= \det(T) \cdot \det(T^{-1}) \cdot \det(\lambda I - A)$$

$$= \det(T \cdot T^{-1}) \cdot \det(\lambda I - A)$$

$$= \det(I) \cdot \det(\lambda I - A)$$

$$= \det(\lambda I - A) \qquad \text{q.e.d.}$$

Eigenvectors:

Def: Eigenvectors $\underline{v}_i$ of eigenvalues λ_i are solutions to the equation:

$$A \cdot \underline{v}_i = \lambda_i \underline{v}_i$$

or:

$$(\lambda_i I - A) \cdot \underline{v}_i = \phi$$

$$\text{Rank}(\lambda_i I - A) = \rho_i < n$$

$\Rightarrow$ There exist as many linearly independent eigenvectors as the rank deficiency (nullity) of $(\lambda_i I - A)$ indicates:

$$\Leftrightarrow \quad \boxed{\#\underline{v}_i \equiv \nu_i = n - \rho_i}$$

- We notice that eigenvectors are <u>not</u> totally determined.

<u>Proof</u>:
$$(\lambda_i I - A)\cdot \underline{v}_i = \emptyset$$

$$\Rightarrow \alpha \cdot (\lambda_i I - A)\cdot \underline{v}_i = \emptyset \; ; \forall \alpha_*$$

$$\Rightarrow (\lambda_i I - A)\cdot \underbrace{(\alpha \cdot \underline{v}_i)}_{\text{this is also an eigenvector.}} = \emptyset$$

$\Rightarrow$ Eigenvectors are only determined up to their lengths. Can be <u>normalized</u>.

Example:

$$A = \begin{bmatrix} \phi & 1 & \phi \\ \phi & -1 & 1 \\ \phi & \phi & -2 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & \phi \\ \phi & (\lambda+1) & -1 \\ \phi & \phi & (\lambda+2) \end{vmatrix}$$

$$= \lambda(\lambda+1)(\lambda+2)$$

$$\Rightarrow \underline{\underline{\lambda_1 = \phi}} \; ; \; \underline{\underline{\lambda_2 = -1}} \; ; \; \underline{\underline{\lambda_3 = -2}}$$

$$(\lambda_i I - A)\,\underline{v}_i = \phi$$

$$\begin{bmatrix} \lambda_i & -1 & \phi \\ \phi & (\lambda_i+1) & -1 \\ \phi & \phi & (\lambda_i+2) \end{bmatrix} \cdot \begin{bmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{bmatrix} = \phi$$

$$\Rightarrow \left| \begin{array}{l} \lambda_i v_{1i} - v_{2i} = \phi \\ (\lambda_i+1) v_{2i} - v_{3i} = \phi \\ (\lambda_i+2) v_{3i} = \phi \end{array} \right|$$

$\lambda_1 = 0$:
$$\left| \begin{array}{c} -v_{21} = 0 \\ v_{21} - v_{31} = 0 \\ 2v_{31} = 0 \end{array} \right|$$

$$\Rightarrow v_{21} = v_{31} = 0$$

We can choose v_{11} freely, e.g.

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad |\underline{v}_1| = 1$$

$\lambda_2 = -1$:
$$\left| \begin{array}{c} -v_{12} - v_{22} = 0 \\ -v_{32} = 0 \\ v_{32} = 0 \end{array} \right|$$

$$\Rightarrow v_{32} = 0 \quad ; \quad v_{22} = -v_{12}$$

e.g.
$$\underline{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Normalization: $|\underline{v}_2| = \sqrt{1+1} = \sqrt{2}$

$$\Rightarrow \quad \underline{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0.707 \\ -0.707 \\ 0 \end{bmatrix}$$

$\lambda_3 = -2$:

$$\left| \begin{array}{c} -2v_{13} - v_{23} = 0 \\ -v_{23} - v_{33} = 0 \\ 0 = 0 \end{array} \right|$$

e.g. $\quad \underline{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

$$|\underline{v}_3| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\Rightarrow \quad \underline{v}_3 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

without proof:

- Distinct eigenvalues have always exactly one eigenvector associated with them.

Multiple Eigenvalues:

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \det(\lambda I - A) = \begin{vmatrix} (\lambda-1) & 0 & 0 \\ -1 & (\lambda-1) & -1 \\ 1 & 0 & \lambda \end{vmatrix}$$

$$= \lambda(\lambda-1)^2$$

$$\Rightarrow \underline{\underline{\lambda_1 = 0}} \quad ; \quad \underline{\underline{\lambda_2 = \lambda_3 = 1}}$$

$$(\lambda_i I - A)\underline{v}_i = 0$$

$$\begin{bmatrix} (\lambda_i-1) & 0 & 0 \\ -1 & (\lambda_i-1) & -1 \\ 1 & 0 & \lambda_i \end{bmatrix} \cdot \begin{bmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{bmatrix} = 0$$

$$\Rightarrow \left| \begin{array}{l} (\lambda_i - 1) v_{1i} = 0 \\ -v_{1i} + (\lambda_i - 1) v_{2i} - v_{3i} = 0 \\ v_{1i} + \lambda_i v_{3i} = 0 \end{array} \right.$$

$\underset{\sim\sim}{\lambda_1 = 0}$: $\quad v_{11} = 0 \quad ; \quad v_{21} = -v_{31}$

e.g. $\quad \underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$|\underline{v}_1| = \sqrt{2} \Rightarrow \underline{v}_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\underset{\sim\sim}{\lambda_2 = \lambda_3 = 1}$: $\quad v_{32} = -v_{12}$

e.g. $\quad \underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; \quad \underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

normalized:

$$\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; \quad \underline{v}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$\underline{v}_2$ and $\underline{v}_3$ are linearly independent. $\Longleftrightarrow$

$$\text{Rank}(\lambda_2 I - A) = \text{Rank} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} = .$$

- We notice that, in this example, the multiple eigenvalue led to multiple eigenvectors.

Example:

$$A = \begin{bmatrix} 1 & 1 \\ \phi & 1 \end{bmatrix} \Rightarrow \det(\lambda I - A) = \begin{vmatrix} (\lambda-1) & - \\ \phi & (\lambda- \end{vmatrix}$$

$$= (\lambda-1)^2$$

$$\Rightarrow \underline{\underline{\lambda_1 = \lambda_2 = 1}}$$

$$\begin{bmatrix} (\lambda_i-1) & -1 \\ \phi & (\lambda_i-1) \end{bmatrix} \cdot \begin{bmatrix} v_{1i} \\ v_{2i} \end{bmatrix} = \phi$$

$$\Rightarrow \left| \begin{matrix} (\lambda_i-1)\, v_{1i} - v_{2i} = \phi \\ (\lambda_i-1)\, v_{2i} = \phi \end{matrix} \right|$$

$\lambda_1 = 1$: $\quad v_{21} = \phi$

$$\Rightarrow \quad \underline{v}_1 = \begin{bmatrix} 1 \\ \phi \end{bmatrix}$$

- We observe that multiple eigenvalues need not lead to multiple eigenvectors:

$$\boxed{\begin{aligned} & m_i = \text{mult}\{\lambda_i\} \\ \Rightarrow\ & \nu_i \leq m_i \end{aligned}}$$

- Similarly to the above given definition for <u>right eigenvectors</u>, we can create a definition for <u>left eigen=vectors</u>:

$$\underline{w}_i' A = \lambda_i \underline{w}_i'$$

<u>or</u>: $$\underline{w}_i' (\lambda_i I - A) = \phi$$

$$\vdots$$